Aplikace matematiky

František Štulajter

Robustness of the best linear unbiased estimator and predictor in linear regression models

Aplikace matematiky, Vol. 35 (1990), No. 2, 162–168

Persistent URL: `http://dml.cz/dmlcz/104398`

35 (1990) APLIKACE MATEMATIKY No. 2, 162—168

ROBUSTNESS OF THE BEST LINEAR UNBIASED ESTIMATOR AND PREDICTOR IN LINEAR REGRESSION MODELS

FRANTIŠEK ŠTULAJTER

(Received January 30, 1989)

Summary. If is shown that in linear regression models we do not make a great mistake if we substitute some sufficiently precise approximations for the unknown covariance matrix and covariance vector in the expressions for computation of the best linear unbiased estimator and predictor.

Keywords: linear regression model, mean integrated square error, the best linear unbiased estimator and predictor, robustness.

AMS Classification: 62J05, 62M20.

1. INTRODUCTION

The theory of linear estimators and predictors in linear regression models belongs to the classical methods of mathematical statistics. The best linear unbiased estimator BLUE of an unknown mean value, as well as the best linear unbiased predictor BLUP of an unknown random variable generally depend on the covariance matrix of the observed random vector and on the vector of covariances between the observed vector and the predicted random variable.

We will show that the BLUE and the BLUP do not change too much if we use some sufficiently precise approximations of the generally unknown real covariance matrix and of the vector of covariances in the expressions according to which they are to be computed. This property of the BLUE and the BLUP is called robustness.

2. ROBUSTNESS OF THE BLUE

Let us assume that the observed random $n \times 1$ vector $\mathbf{X}$ follows the linear regression model

$$\mathbf{X} = \mathbf{F}\beta + \varepsilon ,$$

where $\mathsf{E}[\varepsilon] = 0$, $\mathsf{E}[\varepsilon\varepsilon'] = \Sigma$, Σ is regular, $\mathbf{F}$ is an $n \times k$ known matrix of the full rank. Let $\mathbf{m}^* = \mathbf{F}\beta^* = \mathbf{F}(\mathbf{F}'\Sigma^{-1}\mathbf{F})^{-1}\,\mathbf{F}'\Sigma^{-1}\mathbf{X} = \mathbf{P}^*\mathbf{X}$ be the BLUE of an unknown mean value $\boldsymbol{m} = \boldsymbol{F\beta}$ of $\boldsymbol{X}$. It is shown in [4] that $\boldsymbol{m}^*$ minimizes the mean square integrated error $MISE$ in the class of linear unbiased estimators of $\boldsymbol{m}$ in any (reproducing kernel) Hilbert space $H(S)$. The $MISE$ of $\boldsymbol{m}^*$ is defined by

$$MISE_\Sigma[\boldsymbol{m}^*] = \mathsf{E}_\Sigma[\|\boldsymbol{m}^* - \boldsymbol{m}\|^2_{H(S)}],$$

where $H(S)$ is the usual E^n with the inner product $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{H(S)} = \sum_{i,j=1}^n a_i b_j (S^{-1})_{ij}$ and thus $\|\boldsymbol{a}\|^2_{H(S)} = \sum_{i,j=1}^n a_i a_j (S^{-1})_{ij} = \langle \boldsymbol{a}, \boldsymbol{a}\rangle_{H(S)}$ for any $\boldsymbol{a} \in E^n$ and any positive definite $n \times n$ symmetric matrix $\boldsymbol{S}$.

Next, it is shown in [4] that for any linear unbiased estimator $\boldsymbol{m}^\sim = \boldsymbol{P}^\sim \boldsymbol{X}$ of $\boldsymbol{m}$ we have

$$\mathsf{E}_\Sigma[\|\boldsymbol{m}^\sim - \boldsymbol{m}\|^2_{H(S)}] = \operatorname{tr}(\boldsymbol{P}^\sim \Sigma \boldsymbol{P}^{\sim\prime} \boldsymbol{S}^{-1}),$$

where tr denotes the trace of a matrix.

Now let $\boldsymbol{m}^*$ be the BLUE of $\boldsymbol{m}$ and let $\tilde{\Sigma}$ be some approximation (estimate) of the generally unknown actual covariance matrix Σ. Let us denote by

$$\boldsymbol{m}^\sim = \boldsymbol{P}^\sim \boldsymbol{X} = \boldsymbol{F}(\boldsymbol{F}'\tilde{\Sigma}^{-1}\boldsymbol{F})^{-1} \boldsymbol{F}'\tilde{\Sigma}^{-1}\boldsymbol{X}$$

the estimator of $\boldsymbol{m}$ based on $\tilde{\Sigma}$ (we assume that $\tilde{\Sigma}$ is a regular $n \times n$ matrix).

Then we can write

$$\begin{aligned}
&|\mathsf{E}_\Sigma[\|\boldsymbol{m}^\sim - \boldsymbol{m}\|^2_{H(S)}] - \mathsf{E}_\Sigma[\|\boldsymbol{m}^* - \boldsymbol{m}\|^2_{H(S)}]| = \\
&\leqq |\operatorname{tr}(\boldsymbol{P}^\sim \Sigma \boldsymbol{P}^{\sim\prime}\boldsymbol{S}^{-1}) - \operatorname{tr}(\boldsymbol{P}^*\Sigma\boldsymbol{P}^{*\prime}\boldsymbol{S}^{-1})| \leqq \\
&= \|\boldsymbol{P}^\sim \Sigma \boldsymbol{P}^{\sim\prime} - \boldsymbol{P}^*\Sigma\boldsymbol{P}^{*\prime}\| \,.\, \|\boldsymbol{S}^{-1}\| = \\
&= \|(\boldsymbol{P}^\sim - \boldsymbol{P}^*)\,\Sigma(\boldsymbol{P}^\sim - \boldsymbol{P}^*)' + \boldsymbol{P}^*\Sigma(\boldsymbol{P}^\sim - \boldsymbol{P}^{*\prime}) + \\
&+ (\boldsymbol{P}^\sim - \boldsymbol{P}^*)\,\Sigma\boldsymbol{P}^{*\prime}\| \,.\, \|\boldsymbol{S}^{-1}\| \leqq \\
&\leqq \|\boldsymbol{P}^\sim - \boldsymbol{P}^*\| \,.\, (\|\boldsymbol{P}^\sim - \boldsymbol{P}^*\| + 2\|\boldsymbol{P}^*\|) \,.\, \|\Sigma\| \,.\, \|\boldsymbol{S}^{-1}\|,
\end{aligned}$$

where we have used the facts that $\operatorname{tr}(\boldsymbol{AB}') = (\boldsymbol{A}, \boldsymbol{B})$ is an inner product in the (Hilbert) space of $n \times n$ matrices and $\|\boldsymbol{A}\|^2 = \operatorname{tr}(\boldsymbol{AA}') = \sum_{i,j=1}^n A^2_{i,j}$ is the square of the (Euclidean) norm of a matrix $\boldsymbol{A}$, for which the inequality $\|\boldsymbol{AB}\| \leqq \|\boldsymbol{A}\| \,.\, \|\boldsymbol{B}\|$ holds for any matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

We will show now that $\|\boldsymbol{P}^\sim - \boldsymbol{P}^*\|$ is small if $\|\Sigma - \tilde{\Sigma}\|$ is small and hence, according to the above derived inequality, the $MISE_\Sigma[\boldsymbol{m}^\sim]$ does not differ too much from the $MISE_\Sigma[\boldsymbol{m}^*]$ of the best linear unbiased estimator $\boldsymbol{m}^*$ of $\boldsymbol{m}$. Thus, from the point of $MISE$, the actual unknown covariance matrix Σ can be repleaced by its good approximation $\tilde{\Sigma}$ when computing a good estimator of $\boldsymbol{m}$.

A basis for deriving this result is the following lemma, the proof of which is given in [3].

Lemma 2.1. *Let* $\mathbf{F}$ *be an* $n \times k$ *matrix with rank* $(\mathbf{F}) = k$ *and let* $\Delta\Sigma = \Sigma - \tilde{\Sigma}$, *where* Σ *and* $\tilde{\Sigma}$ *are regular matrices. Then we have*

$$(\mathbf{F}'\Sigma^{-1}\mathbf{F})^{-1} = (\mathbf{F}'\tilde{\Sigma}^{-1}\mathbf{F})^{-1} + \\ + (\mathbf{F}'\tilde{\Sigma}^{-1}\mathbf{F})^{-1}\,\mathbf{F}'\tilde{\Sigma}^{-1}\Delta\Sigma(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\,\tilde{\Sigma}^{-1}\mathbf{F}(\mathbf{F}'\tilde{\Sigma}^{-1}\mathbf{F})^{-1}$$

if $\|\Delta\Sigma\|$ *is sufficiently small. Here* $\mathbf{P}_2^{\sim} = \tilde{\Sigma}^{-1} - \tilde{\Sigma}^{-1}\mathbf{F}(\mathbf{F}'\tilde{\Sigma}^{-1}\mathbf{F})^{-1}\,\mathbf{F}'\tilde{\Sigma}^{-1}$.

Proof. It follows directly from the proof of Theorem 2.1 in [3].

Corollary 1. *Let* $\mathbf{P}^* = \mathbf{F}(\mathbf{F}'\Sigma^{-1}\mathbf{F})^{-1}\,\mathbf{F}'\Sigma^{-1}$ *and* $\mathbf{P}^{\sim} = \mathbf{F}(\mathbf{F}'\tilde{\Sigma}^{-1}\mathbf{F})^{-1}\,\mathbf{F}'\tilde{\Sigma}^{-1}$. *Then* $\mathbf{P}^* = \mathbf{P}^{\sim} - \mathbf{P}^{\sim}\Delta\Sigma\Sigma^{-1} + \mathbf{P}^{\sim}\Delta\Sigma(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\,\mathbf{P}^{\sim\prime}\Sigma^{-1}$.

Proof. From Lemma 2.1 we immediately get the equality

$$\mathbf{P}^* = \mathbf{F}(\mathbf{F}'\tilde{\Sigma}^{-1}\mathbf{F}^{-1})\,\mathbf{F}'\Sigma^{-1} + \mathbf{P}^{\sim}\Delta\Sigma(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\,\mathbf{P}^{\sim\prime}\Sigma^{-1}\,.$$

The proof is completed by using the equality

$$\Sigma^{-1} = \tilde{\Sigma}^{-1} - \tilde{\Sigma}^{-1}\Delta\Sigma\Sigma^{-1}\,.$$

Now it will not be difficult to prove the following theorem.

Theorem 2.1. *For any positive definite symmetric matrices* $\mathbf{S}$ *and* Σ *we have*

$$\lim_{\|\Sigma - \tilde{\Sigma}\| \to 0} \left|E_\Sigma[\|\mathbf{m}^{\sim} - \mathbf{m}\|^2_{H(S)}] - E_\Sigma[\|\mathbf{m}^* - \mathbf{m}\|^2_{H(S)}]\right| = 0\,.$$

Proof. Using Corollary 1 and the basic properties of a norm we can write the inequality

$$\|\mathbf{P}^{\sim} - \mathbf{P}^*\| \leqq (\|\mathbf{P}^{\sim}\| + \|\mathbf{P}^{\sim}\|^2\,.\,\|(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\|)\,.\,\|\Sigma^{-1}\|\,.\,\|\tilde{\Sigma} - \Sigma\|$$

and, as follows from the inequality derived before Lemma 2.1, the theorem will be proved by showing that

$$\lim_{\|\Delta\Sigma\| \to 0} \|(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\| < \infty\,.$$

However since $\|\mathbf{A}\| \leqq n\,.\,\|\mathbf{A}\|_0$ and $\|\mathbf{A}\|_0 \leqq \|\mathbf{A}\|$, where $\|\mathbf{A}\|_0$ denotes the operator norm of an $n \times n$ matrix $\mathbf{A}$ defined by

$$\|\mathbf{A}\|_0 = \inf\{c\colon \|\mathbf{Ax}\| \leqq c\|\mathbf{x}\| \textit{ for all } \mathbf{x} \in E^n\}$$

see [6], we can write the inequality

$$\|(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\| \leqq n\|(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\|_0\,.$$

Using $\|\Delta\Sigma\|_0 \leqq \|\Delta\Sigma\|$ and Theorem 8.7.3 in [6], we get the inequality

$$\lim_{\|\Delta\Sigma\| \to 0} \|(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\| \leqq n$$

and the theorem is proved.

Corollary 2. *Let* $\boldsymbol{m}^{\sim} = \boldsymbol{P}^{\sim}\boldsymbol{X}$ *and* $\boldsymbol{m}^* = \boldsymbol{P}^*\boldsymbol{X}$. *Then we have*

$$\lim_{\|\Sigma - \tilde{\Sigma}\| \to 0} \mathsf{E}_{\Sigma}\left[\left\|\boldsymbol{m}^* - \boldsymbol{m}^{\sim}\right\|^2_{H(S)}\right] = 0 \,.$$

Proof. Since

$$\mathsf{E}_{\Sigma}\left[\left\|\boldsymbol{m}^* - \boldsymbol{m}^{\sim}\right\|^2_{H(S)}\right] = \mathsf{E}_{\Sigma}\left[\left\|\left(\boldsymbol{P}^* - \boldsymbol{P}^{\sim}\right)\boldsymbol{X}\right\|^2_{H(S)} = \right.$$
$$= \operatorname{tr}\left(\left(\boldsymbol{P}^* - \boldsymbol{P}^{\sim}\right)\Sigma\left(\boldsymbol{P}^{*\prime} - \boldsymbol{P}^{\sim\prime}\right)\boldsymbol{S}^{-1}\right) \leqq \left\|\boldsymbol{P}^* - \boldsymbol{P}^{\sim}\right\|^2 . \left\|\boldsymbol{S}^{-1}\right\| . \left\|\Sigma\right\| ,$$

the proof follows from the proof of Theorem 2.1.

3. ROBUSTNESS OF THE BLUP

Let us formulate now the problem of a linear prediction in a linear regression model. Let $\boldsymbol{X}$ be an $n \times 1$ observed vector with possible mean values

$$\boldsymbol{m} = \mathsf{E}_{\beta}[\boldsymbol{X}] = \boldsymbol{F}\boldsymbol{\beta} \,, \quad \boldsymbol{\beta} \in E^k \,,$$

where $\boldsymbol{F}$ is a given $n \times k$ matrix $\operatorname{rank}(\boldsymbol{F}) = k$, and let Σ be the regular covariance matrix of the vector $\boldsymbol{X}$. Next, let U be a random variable with mean values

$$\mathsf{E}_{\beta}[U] = \boldsymbol{f}'\boldsymbol{\beta} \,, \quad \boldsymbol{\beta} \in E^k$$

where $\boldsymbol{f}$ is a known $k \times 1$ vector, and let $\boldsymbol{r}$ denotes the vector of covariances between the random variable U and the random variables X_i: $i = 1, \ldots, n$ – components of the vector $\boldsymbol{X}$.

Then it is well known that the random variable U^* given by

$$U^* = \boldsymbol{f}'\boldsymbol{\beta}^* + \boldsymbol{r}'\Sigma^{-1}(\boldsymbol{X} - \boldsymbol{F}\boldsymbol{\beta}^*) \,,$$

where

$$\boldsymbol{\beta}^* = \left(\boldsymbol{F}'\Sigma^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}'\Sigma^{-1}\boldsymbol{X} \,,$$

is the best linear unbiased predictor of U based on $\boldsymbol{X}$.

This means that

$$\mathsf{E}_{\beta}[U^*] = E_{\beta}[U] \quad \text{for every} \quad \boldsymbol{\beta} \in E^k \quad \text{and}$$
$$\mathsf{E}_{(\beta,\Sigma)}[U^* - U]^2 \leqq \mathsf{E}_{(\beta,\Sigma)}[\tilde{U} - U]^2$$

for all $\boldsymbol{\beta}$, Σ and for any linear unbiased predictor $\tilde{U}$ of U.

We will show now that the predictor U^* is robust with respect to small changes of the covariance vector $\boldsymbol{r}$ and the covariance matrix Σ. More precisely, let Σ and $\boldsymbol{r}$ be the true values of the covariance matrix and the covariance vector and let $\tilde{\Sigma}$ and $\boldsymbol{r}^{\sim}$ be their approximations, then we will show that

$$\lim_{\|\Delta \boldsymbol{r}\| \to 0, \|\Delta\Sigma\| \to 0} \mathsf{E}_{\Sigma}[\tilde{U} - U^*]^2 = 0 \,,$$

where s

$$\tilde{U} = \boldsymbol{f}'\tilde{\boldsymbol{\beta}} + \boldsymbol{r}^{\sim\prime}\tilde{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{F}\tilde{\boldsymbol{\beta}}) \,, \quad \tilde{\boldsymbol{\beta}} = \left(\boldsymbol{F}'\tilde{\Sigma}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}'\tilde{\Sigma}^{-1}\boldsymbol{X} \,,$$

Σ and $\tilde{\Sigma}$ are regular, $\Sigma - \tilde{\Sigma} = \Delta\Sigma$ and $\boldsymbol{r} - \boldsymbol{r}^{\sim} = \Delta\boldsymbol{r}$.

We can write

$$U^* = (\boldsymbol{f}, \boldsymbol{\beta}^*)_{E^k} + (\Sigma^{-1}\boldsymbol{r}, \boldsymbol{MX})_{E^n} \quad \text{and}$$
$$\tilde{U} = (\boldsymbol{f}, \tilde{\boldsymbol{\beta}})_{E^k} + (\tilde{\Sigma}^{-1}\boldsymbol{r}^{\sim}, \boldsymbol{M}^{\sim}\boldsymbol{X})_{E^n} ,$$

where $(\cdot, \cdot)_{E^n} = (\cdot, \cdot)$ is the usual inner product in E^n,

$$\boldsymbol{M} = \boldsymbol{I} - \boldsymbol{F}(\boldsymbol{F}'\Sigma^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}'\Sigma^{-1} = \boldsymbol{I} - \boldsymbol{P}^* \quad \text{and}$$
$$\boldsymbol{M}^{\sim} = \boldsymbol{I} - \boldsymbol{F}(\boldsymbol{F}'\tilde{\Sigma}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}'\tilde{\Sigma}^{-1} = \boldsymbol{I} - \boldsymbol{P}^{\sim} .$$

Next, we have

$$\begin{aligned}
\mathsf{E}_\Sigma[\tilde{U} - U^*]^2 &= \mathsf{E}_\Sigma[(\boldsymbol{f}, \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) + (\boldsymbol{M}'\Sigma^{-1}\boldsymbol{r} - \boldsymbol{M}^{\sim\prime}\tilde{\Sigma}^{-1}\boldsymbol{r}^{\sim}, \boldsymbol{X})]^2 = \\
&= \mathsf{E}_\Sigma[(\boldsymbol{f}, \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) + ((\boldsymbol{M}'\Sigma^{-1} - \boldsymbol{M}^{\sim\prime}\tilde{\Sigma}^{-1})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}'\Sigma^{-1}\Delta\boldsymbol{r}, \boldsymbol{X})]^2 = \\
&= \mathsf{E}_\Sigma[(\boldsymbol{f}, \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) + ((\Sigma^{-1}\boldsymbol{M} - \tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \Sigma^{-1}\boldsymbol{M}\Delta\boldsymbol{r}, \boldsymbol{X})]^2 = \\
&= \mathsf{E}_\Sigma[(\boldsymbol{f}, \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) + (\Sigma^{-1}[(\boldsymbol{M} - \Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}], \boldsymbol{X})]^2 = \\
&= \mathsf{E}_\Sigma[(\Sigma^{-1}(\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime} - \boldsymbol{P}_1')\,\boldsymbol{f}, \boldsymbol{X}) + \\
&\quad + (\Sigma^{-1}[(\boldsymbol{M} - \Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}], \boldsymbol{X})]^2 ,
\end{aligned}$$

where

$$\boldsymbol{P}_1 = (\boldsymbol{F}'\Sigma^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}' \quad \text{and} \quad \boldsymbol{P}_1^{\sim} = (\boldsymbol{F}'\tilde{\Sigma}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}' .$$

Let $\langle \cdot, \cdot \rangle_{H(\Sigma)}$ be the inner product in E^n defined in part 2. Then we can write

$$\begin{aligned}
\mathsf{E}_\Sigma[\tilde{U} - U^*]^2 &= \|(\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime} - \boldsymbol{P}_1')\,\boldsymbol{f}\|^2_{H(\Sigma)} + \\
&+ \|(\boldsymbol{M} - \Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}\|^2_{H(\Sigma)} + \\
&+ 2\langle(\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime} - \boldsymbol{P}_1')\,\boldsymbol{f}, (\boldsymbol{M} - \Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}\rangle_{H(\Sigma)} .
\end{aligned}$$

Using the expression $\Sigma = \tilde{\Sigma} + \Delta\Sigma$ we get

$$\begin{aligned}
\mathsf{E}_\Sigma[\tilde{U} - U^*]^2 &= \|(\boldsymbol{P}_1' - \boldsymbol{P}_1^{\sim\prime} - \Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime})\,\boldsymbol{f}\|^2_{H(\Sigma)} + \\
&+ \|(\boldsymbol{M} - \boldsymbol{M}^{\sim} - \Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}\|^2_{H(\Sigma)} - \\
&- 2\langle(\boldsymbol{P}_1' - \boldsymbol{P}_1^{\sim\prime} - \Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime})\,\boldsymbol{f}, \\
&(\boldsymbol{M} - \boldsymbol{M}^{\sim} - \Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}\rangle_{H(\Sigma)} .
\end{aligned}$$

Now, the inequality

$$|\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\langle\boldsymbol{a}, \boldsymbol{b}\rangle| \leqq (\|\boldsymbol{a}\| + \|\boldsymbol{b}\|)^2 ,$$

which holds for any elements $\boldsymbol{a}$, $\boldsymbol{b}$ of any Hilbert space, implies

$$\begin{aligned}
(\mathsf{E}_\Sigma[\tilde{U} - U^*]^2)^{1/2} &\leqq \|(\boldsymbol{P}_1' - \boldsymbol{P}_1^{\sim\prime} - \Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime})\,\boldsymbol{f}\|_{H(\Sigma)} + \\
&+ \|(\boldsymbol{M} - \boldsymbol{M}^{\sim} - \Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim})\,\boldsymbol{r}^{\sim} + \boldsymbol{M}\Delta\boldsymbol{r}\|_{H(\Sigma)} \leqq \\
&\leqq \|(\boldsymbol{P}_1' - \boldsymbol{P}_1^{\sim\prime})\,\boldsymbol{f}\|_{H(\Sigma)} + \|\Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{P}_1^{\sim\prime}\boldsymbol{f}\|_{H(\Sigma)} + \\
&+ \|(\boldsymbol{P}^* - \boldsymbol{P}^{\sim})\boldsymbol{r}^{\sim}\|_{H(\Sigma)} + \|\Delta\Sigma\tilde{\Sigma}^{-1}\boldsymbol{M}^{\sim}\boldsymbol{r}^{\sim}\|_{H(\Sigma)} + \|\boldsymbol{M}\Delta\boldsymbol{r}\|_{H(\Sigma)} .
\end{aligned}$$

Since

$$\|\mathbf{A}\mathbf{h}\|_{H(\Sigma)} \leqq \|\Sigma^{-1}\|^{1/2} \|\mathbf{A}\| \|\mathbf{h}\|$$

for any vector $\mathbf{h} \in E^n$ and any $n \times n$ matrix $\mathbf{A}$, we can see that all members, except the first, on the right hand side of the above inequality for the mean square error of prediction, converge to zero if $\|\Delta\Sigma\| \to 0$, $\|\Delta\mathbf{r}\| \to 0$ (see also Theorem 2.1).

For the first term we get from Lemma 2.1 the equality

$$\mathbf{P}_1' - \mathbf{P}_1^{\sim\prime} = \mathbf{P}^{\sim}\Delta\Sigma(\mathbf{I} + \mathbf{P}_2^{\sim}\Delta\Sigma)^{-1}\, \tilde{\Sigma}^{-1}\mathbf{P}_1^{\sim\prime},$$

from which it can be seen, as follows from the proof of Theorem 2.1, that

$$\lim_{\|\Delta\Sigma\| \to 0} \|\mathbf{P}_1' - \mathbf{P}_1^{\sim\prime}\| = 0 \quad \text{and thus} \quad \lim_{\|\Delta\Sigma\| \to 0} \|(\mathbf{P}_1' - \mathbf{P}_1^{\sim\prime})\, \mathbf{f}\|_{H(\Sigma)} = 0\,.$$

We can formulate these results as a theorem.

Theorem 3.1. *If Σ and $\tilde{\Sigma}$ are regular matrices, then*

$$\lim_{\|\Delta\mathbf{r}\| \to 0,\, \|\Delta\Sigma\| \to 0} E_\Sigma[\tilde{U} - U^*]^2 = 0\,.$$

Proof. It was done before the statement of the theorem.

References

[1] *E. Parzen:* Time series analysis papers. Holden - Day, San Francisco 1967.

[2] *C. R. Rao:* Linear statistical inference and its applications. Wiley, New-York 1965.

[3] *O. N. Strand:* Coefficient errors caused by using the wrong covariance matrix in the general linear regression model. Ann. Stat. (2), 1974, 935—949.

[4] *F. Štulajter:* Estimators with minimal mean integrated square error in regression models. Submitted to Statistics.

[5] *F. Štulajter:* Estimation in random processes. SNTL - Alfa, Bratislava (to appear in 1989).

[6] *B. Z. Vulich:* An introduction to functional analysis (Russian). Nauka, Moscow 1967.

Súhrn

ROBUSTNOSŤ NAJLEPŠIEHO LINEÁRNEHO ODHADU A PREDIKTORA V LINEÁRNYCH REGRESNÝCH MODELOCH

FRANTIŠEK ŠTULAJTER

V článku je dokázané, že v lineárnom regresnom modeli sa nedopustíme veľkej chyby, ak neznámu kovariančnú maticu a neznámy kovariančný vektor nahradíme pri výpočte optimálneho lineárneho odhadu a prediktora ich dostatočne presnými aproximáciami.

Резюме

УСТОЙЧИВОСТЬ ОПТИМАЛЬНОЙ ЛИНЕЙНОЙ ОЦЕНКИ И ПРЕДИКЦИИ В ЛИНЕЙНОЙ РЕГРЕССИОННОЙ МОДЕЛИ

FRANTIŠEK ŠTULAJTER

В статье показано, что мы не сделаем серезной ошибки если мы в выражениях для вычисления оптимальной линейной оценки и предикции поставии какие нибудь достаточно точные аппроксимации для неизвестной ковариационной матрици и ковариационного вектора.

Author's address: Doc. RNDr. *František Štulajter,* CSc., Katedra teórie pravedepod. a mat. štat. MFF UK, Mlynská dolina, 842 15 Bratislava.