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CAYLEY'S PROBLEM

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Summary. Newton's method for computation of a square root yields a difference equation which can be solved using the hyperbolic cotangent function. For the computation of the third root Newton's sequence presents a harder problem, which already Cayley was trying to solve. In the present paper two mutually inverse functions are defined in order to solve the difference equation, instead of the hyperbolic cotangent and its inverse. Several coefficients in the expansion around the fixed points are obtained, and the expansions are glued together in the region of overlapping.

Keywords: Newton method, difference equation, series expansion, fixed point, discrete dynamical system, Julia set.

AMS classification: 58F08, 65Q05.

I. CAYLEY'S PROBLEM. FUNCTIONAL EQUATIONS

Already in 1879 Cayley (see [1]) considered the Newton method for solving polynomial equations in the complex plane, asking what would be the basin $A(\zeta_k)$ of attraction of a certain zero of the polynomial $p(z)$. Newton's sequence takes the form of a difference equation

$$z_{n+1} = z_n - p(z_n)/p'(z_n)\,. \tag{1}$$

Every quadratic equation can be reduced by a linear transformation to the simplest case $z^2 - 1 = 0$, and the corresponding sequence

$$z_{n+1} = (z_n + z_n^{-1})/2 \tag{2}$$

can be expressed analytically using the hyperbolic cotangent and its inverse

$$z_n = \operatorname{cth}(2^n \operatorname{Ar\,cth} z_0)\,. \tag{3}$$

The complex plane is divided into two basins of attraction along the imaginary axis, while the imaginary axis in this case is the *Julia set*. The dynamics on it is chaotic and also rather interesting [2].

What is now known as *Cayley's problem* arises from Newton's sequence of the specific cubic equation $z^3 - 1 = 0$. The difference equation is now

$$z_{n+1} = (2z_n^3 + 1)/3z_n^2 \,. \tag{4}$$

Cayley tried in vain to find an analytic solution similar to (3) or to detect the basins of attraction to the three third roots of one. Julia and Fatou [3, 4] worked in a more general setting, iterating rational functions in the complex plane, and showed some of the properties of the singular Julia set. Computer graphic pictures [5] of the Julia set for the specific Cayley case give some idea of the complexity of the problem.

Here we construct an analytic function $f(w)$ to play the role of the hyperbolic cotangent, and its inverse $g(z)$. The two functions should therefore obey the functional equations

$$f(\lambda w) = (2f^3(w) + 1)/3f^2(w) \,, \tag{5}$$

$$\lambda\, g(z) = g\left(\frac{2z^3 + 1}{3z^2}\right), \tag{6}$$

where the coefficient λ has been chosen appropriately. Suppose we have solutions of (5), (6) with a given λ. Denote these solutions by g_λ, f_λ, respectively. Then to solve (5), (6) with μ instead of λ we can take

$$f_\mu(w) = f_\lambda(w^{1/\alpha}) \,, \quad g_\mu(z) = (g_\lambda(z))^\alpha \,, \quad \alpha = \ln \mu / \ln \lambda \,. \tag{7}$$

Thus we may get a new branching point for $w = g(z) = 0$ or, if we put it the other way, by choosing λ properly branching can be avoided.

The equation (6) is homogeneous and linear, so its solutions form a vector space. The solutions of course make sense only in some neighbourhoods of the fixed points of the iterated function $R(z) = (2z^3 + 1)/3z^2$. Of the four fixed points, the zeroes of the cubic equation $1, -\frac{1}{2} \pm i/2\sqrt{3}$ are *superattractors*, while infinity is a *repellor*. From the series representation around a superattractor it is obvious that g must have a logarithmic sigularity there and that it is defined uniquely up to a constant factor. As for the infinity, taking the transformation $z \to z^{-1}$ and the inverse R^{-1}, the origin $z = 0$ becomes an attractor of R^{-1} with the derivative $\lambda = \frac{2}{3}$ and $g(z)$ can again be defined uniquely up to a constant factor, but here as a regular analytic function with a simple zero. The solution space of (6) is thus one dimensional, which in turn defines the one parameter family of solutions of (5): given one solution $f(w), f(Cw)$ is the family.

II. THE FUNCTION $g(z)$

Equation (6) requires $\lambda\, g(z) = g(R(z))$. Putting $t = z^{-1}$ we get

$$\lambda\, g(t^{-1}) = g\left(\frac{2 + t^3}{3t}\right)$$

or, denoting $g(t^{-1}) = \bar{g}(t)$, we have

$$\lambda\, \bar{g}(t) = \bar{g}(\bar{R}(t)), \quad \bar{R}(t) = \tfrac{3}{2}t/(1 + t^3/2)$$

and $\bar{R}'(0) = \frac{3}{2}$, so the right λ for $t = 0$ ($z = \infty$) is $\lambda = \frac{3}{2}$. The expansion of $g(z)$ at infinity gives that only every third coefficient is nonzero:

$$g(z) = z^{-1}(A_0 + A_1 z^{-3} + A_2 z^{-6} + \ldots)\,. \tag{8}$$

Applying (6) we get

$$\tfrac{3}{2}\, z^{-1}(A_0 + A_1 z^{-3} + A_2 z^{-6} + \ldots) =$$
$$\tfrac{3}{2}\, z^{-1}/(1 + z^{-3}/2)\,(A_0 + A_1 z^{-3}/(1 + z^{-3}/2)^3 + \ldots)\,,$$

which in turn produces an infinite system for the coefficients A_k:

$$A_k(1 - (\tfrac{3}{2})^{3k}) = \sum_{j=1}^{k} (\tfrac{3}{2})^{3k-3j}\,(-\tfrac{1}{2})^j \binom{3k-2j}{j} A_{k-j}\,. \tag{9}$$

The coefficients A_k have been computed, setting $A_0 = 1$:

1	·210526316	6	·0374698606	11	·0198737245
2	·112702810	7	·0319217372	12	·0181318560
3	·0758967224	8	·0277665223	13	·0166629179
4	·0568261024	9	·0245425293	14	·0154080218
5	·0452330629	10	·0219709593	15	·0143240028

Does the series defined through the coefficients A_k converge? A look at the computed values of coefficients suggests that it should converge for $|z| > 1$. This is actually the case.

For $|z| > 1$ there exists exactly one analytic solution of the functional equation $\frac{3}{2}\, g(z) = g(\frac{2}{3}z + \frac{1}{3}z^{-2})$ with $\lim_{z\to\infty} z\, g(z) = 1$.

The coefficients A_k and the resulting series give uniqueness and a construction of the function, provided the series converges outside the unit circle. Denote by S this outside region. Then from the three branches of R^{-1} we can choose one so as to be analytic on S,

$$R^{-1}\colon S \to S\,.$$

Indeed, suppose $|z| > 1$ and let us try to solve the equation $\frac{2}{3}y + \frac{1}{3}y^{-2} = z$ with $|y| > 1$. This can be done by iteration setting $y_0 = z$ and $y_{n+1} = \frac{3}{2}z - y_n^{-2}$. This sequence gives $|y_n| > 1$ and it converges since the derivative of the iteration is absolutely below 1.

Now, infinity is an attractor for R^{-1}, attracting all of S. So it suffices to have the solution defined in a neighbourhood of infinity and then extended over S by the functional equation. However, as the derivative of R^{-1} at infinity equals $\frac{2}{3}$, an old result of Schröder gives the local existence [7].

Can we extend $g(z)$ defined by (8) beyond the unit circle? We shall determine the function g in a neighbourhood of the other three fixed points, the attractors 1, $-\frac{1}{2} \pm \mathrm{i}\sqrt{3}$. For reasons of symmetry it is sufficient to take $z = 1$ and develop g there.

If we set $z = 1 + u$, then $R(z) = 1 + u^2(1 + \frac{2}{3}u)/(1 + u)^2$. Thus it is clear that if we take the expansion around 1, g cannot be regular with any λ, but due to a result of Böttcher [6], it must have a logarithmic singularity with $\lambda = 2$:

$$g(1 + u) = \ln\left(\frac{1}{u}\right) + a_0 + a_1 u + a_2 u^2 + \dots. \tag{10}$$

Here we have chosen the sign so as to make g positive for $u > 0$. The coefficients a_i are again obtained from the expansion (10) and the equation (6). First we take the expansions

$$(1 + \tfrac{2}{3}u)^i (1 + u)^{-2i} = \sum_{j=0}^{\infty} b_{ij} u^j \tag{11}$$

and comparing the coefficients of the series we get

$$2a_i = (-1)^i ((\tfrac{2}{3})^i - 2)/i + a_1 b_{1,i-2} + a_2 b_{2,i-4} + \dots. \tag{12}$$

Again we give 15 computed values of the coefficients a_i:

1	·666666667	6	·396204846	11	−·161102456
2	−·055555557	7	−·194656737	12	·933372487
3	−·160493827	8	−·155483159	13	−1·22566744
4	·302469136	9	·439798587	14	·0228503829
5	−·405761317	10	−·377459402	15	3·40248459

Böttcher's result guarantees the convergence of the series in a neighbourhood of $z = 1$ $(u = 0)$. Can we make any conclusion regarding the radius of convergence? Certainly any point of the Julia set will be a singular point for the series, since in any neighbourhood of such points there are preimages of $z = 1$, where we have the logarithmic singularity. Thus the point nearest to 1 shall determine the radius of convergence.

Consulting an image of the Julia set (see e.g. [5]) we find $r \doteq ·6$, which we can confirm by computation. First we find a cycle of period 2 and solve the equation $R(R(z)) = z$. This is an equation of degree 9, dividing out the three fixed points leaves us with degree six, giving three cycles located symmetrically. The cycle nearest to $z = 1$ is

$$\zeta_{1,2} = 10^{-1/6} \,.\, \exp\left(\mathrm{i}\left(\frac{\pi}{3} - \frac{1}{3}\operatorname{arc\,tg}\sqrt{\frac{27}{5}}\right)\right), \tag{13}$$

$$\zeta_{1,2} = ·538608673 \pm ·417204483\,\mathrm{i}\,,$$

and the distance is

$$r = |1 - \zeta| = \cdot 622046251 .$$

To see that this is also the radius of convergence of the series (10) we shall follow the circles $u = \varrho \, . \, e^{i\varphi}$, $\varrho \leqq r$ and consider their images under the iteration

$$u \to u^2(1 + \tfrac{2}{3}u)(1 + u)^{-2} .$$

The curve obtained by iteration winds around $z = 1$ twice, and it may go out of the circle $|u| = r$. If we iterate once again, the curve winds 4 times, and so on. Depending on how close to r is the initial radius ϱ, we have to take the number of iterations to push the curve inside the circle $|u| = r$ and eventually to any neighbourhood of $z = 1$. Since g can be defined in the neighbourhood, its definition can be extended to the whole inside of the circle by the functional equation.

Denote by C_0, C_1 and C_2 the insides of the circles with radius r and with centers at the three roots of unity respectively. We have just defined the function $g(z)$ on C_0.

However, the functions $g(z)$ defined on S and C_0 by (8) and (10) do not match as they belong to different λ. Nonetheless, by virtue of (7) we can make them coincide on the intersection. Since S intersects also C_1 and C_2, we shall take the "common denominator" $\lambda = \frac{3}{2}$. In the intersection we have

(14) $$g_{3/2}(z) = M(g_2(z))^\alpha , \quad \alpha = \ln(3/2)/\ln 2 .$$

To determine M, take a small positive u, set $y_0 = 1 + u$, take the sequence $y_n = R^{-1}(y_{n-1})$, which converges to infinity. For u small enough g_2 is just the logarithm, for y_n big enough, $g_{3/2}$ is just y_n^{-1}. Therefore

(15) $$M = \lim_{\substack{n \to \infty \\ u \to 0}} \left(\ln \left(\frac{1}{u} \right) \right)^{-\alpha} . \, (\tfrac{3}{2})^n / y_n = \cdot 71515 .$$

Of course we have got new branching at $z = 1$ due to the irrational power α.

On C_1 and C_2 we can define $g_{3/2}$ by symmetry. Denote $\omega = -\frac{1}{2} + (i/2)\sqrt{3}$, then from (8) we conclude

$$g(\omega z) = \omega^{-1} g(z)$$

and this serves for the definition in C_1 and C_2. Thus g is now defined on the union $D = S \cup C_0 \cup C_1 \cup C_2$.

Mapping the unit circle with the Cayley map R, we get the hypocycloid

$$z = \tfrac{2}{3}e^{i\varphi} + \tfrac{1}{3}e^{-2i\varphi} .$$

For z between this cycloid and the unit circle there is exactly one $R^{-1}(z) \in S$ and so $g(z)$ can be defined by (6). We cut the inside of the cycloid symmetrically along the radii $\varphi = 0$, $2\pi/3$, $4\pi/3$. For z between these cuts and the cycloid there is again a unique $R^{-1}(z)$, lying in the neighbouring sector of the cycloid, and again the functional equation (6) applies.

III. THE FUNCTION $f(w)$

Since we have chosen $g(z)$, its inverse $f(w)$ is now defined, $\lambda = \frac{3}{2}$. As the initial term of $g(z)$ is z^{-1}, the initial term of $f(w)$ is w^{-1}. From the functional equation (5), $f(w)$ can be presented in the form

$$(16) \qquad f(w) = w^{-1}\,\varphi(w^3)\,, \quad \varphi\left(\frac{27}{8}\,t\right) = \left(\varphi^3(t) + \frac{t}{2}\right)/\varphi^2(t)$$

and $\varphi(0) = 1$. Thus f can be given as a series

$$(17) \qquad f(w) = w^{-1}(B_0 + B_1 w^3 + B_2 w^6 + \ldots)\,.$$

Directly we find $B_0 = 1$, $B_1 = \frac{4}{19}$ and the rest from the recursion

$$(18) \qquad \sum_{i=0}^{n} \sum_{j=0}^{n-i} B_i B_j B_{n-i-j}(\mu^i - 1) = 0\,, \quad \mu = \frac{27}{8}\,.$$

Again we give 15 coefficients B_i:

1	·210526316	6	−3·11548372 E-06	11	3·61407374 E-11
2	−·0202611793	7	3·29098610 E-07	12	−3·62249244 E-12
3	2·31665041 E-03	8	−3·42383403 E-08	13	3·60960550 E-13
4	−2·62335983 E-04	9	3·51887806 E-09	14	−3·57853879 E-14
5	2·89320177 E-05	10	−3·58084230 E-10	15	3·53207176 E-15

From the very coefficients it is easy to conjecture, that the radius of convergence satisfies $|w^3| < 10$. However, we shall show that the actual number is $|w^3| < 10{\cdot}8045851$.

We shall start from the singular point W_0, $f(W_0) = \infty$. Then $W_1 = \frac{2}{3}W_0$ and from the equation (5) we conclude that $f(W_1)$ is zero. Further, $W_2 = \frac{2}{3}W_1 = \frac{4}{9}W_0$ and $f(W_2)$ is one of the solutions of the equation $2z^3 + 1 = 0$. We define a sequence of real negative numbers starting with $y_0 = \infty$, $y_1 = 0$, $y_2 = -2^{-1/3}$, $y_n = R^{-1}(y_{n-1})$ where as before R^{-1} means the solution of the equation

$$\tfrac{2}{3}y_n + \tfrac{1}{3}y_n^{-2} = y_{n-1}$$

which is absolutely above unity.

The corresponding sequence W_n converges to zero and $f(W_n) = y_n$. But for W_n small enough, $f(W_n) \doteq W_n^{-1}$. Thus we can determine W_0 from the limit

$$(19) \qquad W_0 = \lim_{n\to\infty} (\tfrac{3}{2})^n / y_n = -2{\cdot}21073166$$

or compute the radius of convergence

$$W_0^3 = -10{\cdot}8045851\,.$$

Actually we have obtained the radius of convergence because W_0 is the singularity of f nearest to the origin, apart from $w = 0$, of course. If instead of the sequence y_n, defined by R^{-1}, we take another sequence of inverses, we get other singularities. The procedure can be like this: choose an integer k and find all the k-inverses of infinity. There are 3^{k-2} of them. Some of them will be outside the unit circle. Denote then by η_k. We prolong the sequence by $\eta_n = R^{-1}(\eta_{n-1})$. The singularity is then analogously to (19)

$$w = \lim_{n \to \infty} (\tfrac{3}{2})^n / \eta_n .$$

For $|w| < |W_0|$ the function $f(w)$ is defined by the series (17), while for w outside this circle we have to apply the functional equation until we get inside.

Thus having defined the two functions f and g, we can express the solution of the difference equation (4) in terms of them as

$$z_n = f((\tfrac{3}{2})^n g(z_0)) . \tag{20}$$

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Souhrn

CAYLEYŮV PROBLÉM

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Newtonova metoda pro výpočet druhé odmocniny vede na lineární diferenční rovnici, kterou lze řešit v uzavřeném tvaru pomocí hyperbolické kotangenty. V předloženém článku se zavádějí dvě navzájem inverzní funkce, které slouží k témuž účelu v případě aproximace třetí odmocniny a studuji se jejich vlastnosti.

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