CONTINUITY OF HYSTERESIS OPERATORS IN SOBOLEV SPACES

PAVEL KREJČÍ, VLADIMÍR LOVICAR

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Summary. We prove that the classical Prandtl, Ishlinskii and Preisach hysteresis operators are continuous in Sobolev spaces $W^{1,p}(0, T)$ for $1 \leqq p < +\infty$, (locally) Lipschitz continuous in $W^{1,1}(0, T)$ and discontinuous in $W^{1,\infty}(0, T)$ for arbitrary $T > 0$. Examples show that this result is optimal.

Keywords: Hysteresis operators, Preisach operator, Ishlinskii operator.

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A classical result by Krasnoselskii and Pokrovskii (see [1]) gives sufficient conditions for the continuity of a wide class of hysteresis operators (including the Preisach operators) in the space of continuous functions $C([0, T])$. Moreover, in typical cases hysteresis operators map Sobolev spaces $W^{1,p}(0, T)$ into $W^{1,p}(0, T)$. When studying e.g. the continuous dependence of the solution of (partial) differential equations with hysteresis on given data, we often have to make use of the continuity of hysteresis operators in these spaces. We present here a collection of elementary proofs and counterexamples connected with this problem.

Definition 1. *Let $u \in C([0, T])$, $T > 0$ be piecewise monotone and let $h > 0$ be a given number. Then the elementary hysteresis operators l_h, f_h are defined as follows:*

$$l_h(u)(t) = \begin{cases} \max\{l_h(u)(t_k), u(t) - h\}, & t \in (t_k, t_{k+1}] \\ \quad \text{if } u \text{ is nondecreasing in } [t_k, t_{k+1}], \\ \min\{l_h(u)(t_k), u(t_k) + h\}, & t \in (t_k, t_{k+1}] \\ \quad \text{if } u \text{ is nonincreasing in } [t_k, t_{k+1}] \end{cases}$$

$$l_h(u)(0) = \begin{cases} \max\{0, u(0) - h\} & \text{if } u(0) \geqq 0, \\ \min\{0, u(0) + h\} & \text{if } u(0) < 0, \end{cases}$$

$$f_h(u)(t) = u(t) - l_h(u)(t), \quad t \in [0, T].$$

The following lemma is proved in [1], p. 16.

Lemma 1. *Let $u, v \in C([0, T])$ be piecewise monotone, $h > 0$. Then for every $t \in [0, T]$ we have*

$$|l_h(u)(t) - l_h(v)(t)| \leqq \|u - v\|_{[0,t]},$$

where $\|u - v\|_{[0,t]}$ stands for $\max\{|u(s) - v(s)|, s \in [0, t]\}$.

Using this lemma and the density of the set of piecewise monotone functions in $C([0, T])$ we conclude that the domain of definition of l_h, f_h can be extended to the whole $C([0, T])$ and these operators are Lipschitz continuous.

Notice that $l_h(u), f_h(u)$ are absolutely continuous if u is absolutely continuous and, by Definition 1, if $u'(t)$, $(l_h(u))'(t)$ exist, then either $(l_h(u))'(t) = 0$, $(f_h(u))'(t) = u'$ or vice versa.

Definition 2. *Let $\eta \in L^1(0, \infty)$ be a given nonnegative function. Then for every $u \in C([0, T])$ the value of the Ishlinskii operator F is defined by the formula*

$$F(u)(t) = \int_0^\infty f_h(u)(t)\, \eta(h)\, dh.$$

Definition 3. *Let μ: $(0, \infty) \times R^1 \to R^1$, $\mu_0 \in C^1(R^1)$ be given functions, $\mu(h, \varrho) = -\mu(h, -\varrho)$, $\mu_0(\varrho) = -\mu_0(-\varrho)$, $\partial\mu/\partial\varrho \in L^1(0, \infty; C_0(R^1))$, where $C_0(R^1)$ denotes the space of continuous functions w: $R^1 \to R^1$ such that $w(t) \to 0$ as $|t| \to \infty$. Then for every $u \in C([0, T])$ the value of the Preisach operator W is defined by the formula*

$$W(u)(t) = \mu_0(u(t)) + \int_0^\infty \mu(h, l_h(u)(t))\, dh.$$

This definition is different from the "classical" one (cf. [1], [4]), but it is shown in [3] that it is equivalent for a large class of Preisach operators. It makes sense because for $h \geqq \|u\|_{[0,T]}$ we have $l_h(u)t = 0$.

The following lemma is announced without proof in [1] and [2].

Lemma 2. *Let $u, v \in W^{1,1}(0, T)$ be piecewise monotone. Then we have*

$$\int_0^T |(f_h(u))'(t) - (f_h(v))'(t)|\, dt \leqq |u(0) - v(0)| + 2\int_0^T |u'(t) - v'(t)|\, dt.$$

Proof. We construct a partition $0 = t_0 < t_1 < \ldots < t_N = T$ such that in every interval $J_i = [t_{i-1}, t_i]$ both functions u, v are monotone and in every interval $J_i \cup J_{i+1}$ at least one of the functions u, v is not monotone. We find the subset $B = \{b_1, \ldots, b_k\} \subset \{1, \ldots, N\}$ such that for $i \in B$ we have $u'(t) \,.\, v'(t) \geqq 0$ a.e. in J_i and for $i \notin B$ we have $u'(t) \,.\, v'(t) \leqq 0$ a.e. in J_i.

We introduce the following notation. For $i \notin B$ we denote $G_i = J_i$ ("good intervals"), for $i \in B$ there exists a partition $t_{i-i} \leqq \tau_{i-1} \leqq \tau_i \leqq t_i$ such that if we denote $G_i = [t_{i-1}, \tau_{i-1}]$, $B_i = [\tau_{i-1}, \tau_i]$ ("bad intervals"), $G_i^0 = [\tau_i, t_i]$, $x(t) = f_h(u)(t)$, $y(t) = f_h(v)(t)$, then

$$x'(t) = u'(t), \quad y'(t) = v'(t) \quad \text{a.e. in} \quad G_i, \quad i \in B,$$

$$x'(t) = 0\,,\quad y'(t) = v'(t) \quad \text{or} \quad x'(t) = u'(t)\,,\quad y'(t) = 0 \quad \text{a.e. in} \quad B_i\,,$$
$$x'(t) = y'(t) = 0 \quad \text{a.e. in} \quad G_i^0\,.$$

Indeed, any one of these intervals may degenerate. On the other hand we have $B_i \cap B_j = \emptyset$ for $i \neq j$.

For an arbitrary interval $J = [a, b] \subset [0, T]$ we denote

$$l(J) = |x(a) - y(a)|\,,$$
$$r(J) = |x(b) - y(b)|\,.$$

The following properties are obvious:

(1) $$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t = \sum_{i=1}^{N} \int_{G_i} |x'(t) - y'(t)|\,\mathrm{d}t + \sum_{i\in B} \int_{B_i} |x'(t) - y'(t)|\,\mathrm{d}t\,,$$

(2) $$|x(0) - y(0)| \leqq |u(0) - v(0)|\,,$$

(3) $$|r(G_i) - l(G_i)| = \left|\int_{G_i} \mathrm{d}/\mathrm{d}t\, |x(t) - y(t)|\,\mathrm{d}t\right| \leqq$$
$$\leqq \int_{G_i} |x'(t) - y'(t)|\,\mathrm{d}t \leqq \int_{G_i} |u'(t) - v'(t)|\,\mathrm{d}t\,,$$

(4) $$\int_{B_i} |x'(t) - y'(t)|\,\mathrm{d}t = l(B_i) - r(B_i) = r(G_i) - l(G_{i+1}) \quad \text{for} \quad i \neq b_k\,,$$

(5) $$\int_{B_i} |x'(t) - y'(t)|\,\mathrm{d}t = l(B_i) - r(B_i) \leqq r(G_i) \quad \text{for} \quad i = b_k\,.$$

From (3), (4), (5) we obtain

$$\sum_{i\in B} \int_{B_i} |x'(t) - y'(t)|\,\mathrm{d}t \leqq |x(0) - y(0)| +$$
$$+ \sum_{i=1}^{N} (r(G_i) - l(G_i)) \leqq |x(0) - y(0)| +$$
$$+ \sum_{i=1}^{N} \int_{G_i} |u'(t) - v'(t)|\,\mathrm{d}t\,.$$

It remains to use (1), (2), (3) and the proof is complete.

Remark. The inequality in Lemma 1 holds if we replace f_h by l_h. It suffices to modify the proof slightly, namely for $i \in B$ we put $[t_{i-1}, \tau_{i-1}] = G_i^0$, $[\tau_i, t_i] = G_i$. For $x(t) = l_h(u)(t)$, $y(t) = l_h(v)(t)$ we obtain (1), (2), (3) without any change. Using the inequality

(6) $$\int_{B_i} |(l_h(u))'(t) - (l_h(v))'(t)|\,\mathrm{d}t \leqq \int_{B_i} |u'(t) - v'(t)|\,\mathrm{d}t +$$
$$+ \int_{B_i} |(f_h(u))'(t) - (f_h(v))'(t)|\,\mathrm{d}t$$

together with (4), (5) we proceed as above.

Lemma 3. *Let $\{u_n\}, \{x_n\} \subset L^p(I)$ be given sequences, where $I \subset R^1$ is a bounded interval and $1 < p < \infty$. Let $x_n \to x_\infty$ in $L^1(I)$, $u_n \to u_\infty$ in $L^p(I)$ and let the inequalities $|x_n(t)| \leqq |u_n(t)|$, $n \in N \cup \{\infty\}$ hold almost everywhere in I. Then $x_n \to x_\infty$ in $L^p(I)$.*

Proof. The proof is extremely simple. Let us assume that x_n does not converge in $L^p(I)$. As the space $L^p(I)$ is uniformly convex, we can find subsequences $\{x_k\}$, $\{u_k\}$ such that

$$\int_I |x_k(t)|^p \, dt \to K > \int_I |x_\infty(t)|^p \, dt ,$$

$$x_k \to x_\infty \quad \text{a.e.}, \quad u_k \to u_\infty \quad \text{a.e.}$$

Put $f_k(t) = |u_k(t)|^p + |u_\infty(t)|^p - \left||x_k(t)|^p - |x_\infty(t)|^p\right| \geqq 0$. By Fatou's lemma we have $\int_I \liminf f_k(t) \, dt \leqq \liminf \int_I f_k(t) \, dt$, hence

$$\limsup \int_I \left||x_k(t)|^p - |x_\infty(t)|^p\right| dt \leqq 0 ,$$

which is a contradiction.

Theorem. *The operators l_h, f_h, F, W are continuous in $W^{1,p}(0, T)$ for $1 \leqq p < \infty$. Moreover, l_h, f_h, F are Lipschitz in $W^{1,1}(0, T)$ and W is locally Lipschitz in $W^{1,1}(0, T)$ provided μ_0' is Lipschitz in R^1 and there exists $\xi \in L^1_{\text{loc}}(0, \infty)$ such that*

$$\left|\frac{\partial \mu}{\partial \varrho}(h, \varrho_1) - \frac{\partial \mu}{\partial \varrho}(h, \varrho_2)\right| \leqq \xi(h) \, |\varrho_1 - \varrho_2| \quad \textit{for every} \quad \varrho_1, \varrho_2 \in R^1 .$$

Proof. The set of piecewise monotone absolutely continuous functions is dense in $W^{1,1}(0, T)$. Consequently, the Lipschitz continuity of l_h, f_h, F in $W^{1,1}(0, T)$ with the norm $\|u\|_{1,1} = |u(0)| + \int_0^T |u'(t)| \, dt$ follows directly from Lemma 2. In order to prove the continuity of W in $W^{1,1}(0, T)$ we make use of the inequality

$$\begin{aligned}
&\int_0^T |(W(u))'(t) - (W(v))'(t)| \, dt \leqq \\
&\leqq \|\mu_0'(u) - \mu_0'(v)\|_{[0,T]} \int_0^T |u'(t)| \, dt + \\
&+ \|\mu_0'(v)\|_{[0,T]} \int_0^T |u'(t) - v'(t)| \, dt + \\
&+ \int_0^\infty \left\|\frac{\partial \mu}{\partial \varrho}(h, l_h(u)) - \frac{\partial \mu}{\partial \varrho}(h, l_h(v))\right\|_{[0,T]} \int_0^T |(l_h(u))'(t)| \, dt \, dh + \\
&+ \int_0^\infty \left\|\frac{\partial \mu}{\partial \varrho}(h, l_h(v))\right\|_{[0,T]} \int_0^T |(l_h(u))'(t) - (l_h(v))'(t)| \, dt \, dh
\end{aligned}$$

and of Lemmas 1, 2.

Let us now consider a sequence $v_n \to v_\infty$ in $W^{1,p}(0, T)$ for a fixed value $p \in (1, \infty)$. For the operator W we have

$$|(W(v_n))'(t)| \leqq |v_n'(t)| \left(\int_0^K \left|\frac{\partial \mu}{\partial \varrho}(h, l_h(v_n)(t))\right| dh + |\mu_0'(v_n(t))|\right)$$

for some $K > \max_k \|v_k\|_{[0,T]}$ and for every $n \in N \cup \{\infty\}$.

Denoting by x_n and u_n the left-hand side and the right-hand side of the last inequality, respectively, and using the continuous embedding $W^{1,p}(0, T) \subset W^{1,1}(0, T) \subset C([0, T])$ we see that the hypotheses of Lemma 3 are satisfied. The cases of the operators l_h, f_h, F are analogous. The theorem is proved. □

A natural question is whether these results are optimal. We try to give the answer in the following examples.

Example 1. The constants in Lemma 2 cannot be improved. Let us choose numbers $0 < \delta < \varepsilon < h$ and put

$$u(t) = \begin{cases} h - \delta + t, & t \in [0, \delta] \\ h + \varepsilon(t - \delta), & t \in [\delta, 1], \end{cases} \quad v(t) = h - \varepsilon + \varepsilon t, \quad t \in [0, 1].$$

We have

$$x(t) = f_h(u)(t) = \begin{cases} u(t), & t \in [0, \delta], \\ h, & t \in [\delta, 1] \end{cases}$$

$$y(t) = f_h(v)(t) = v(t), \quad t \in [0, 1],$$

and hence

$$\int_0^1 |x'(t) - y'(t)| \, \mathrm{d}t = (\varepsilon - \delta) + 2\delta(1 - \varepsilon) =$$
$$= |u(0) - v(0)| + 2 \int_0^1 |u'(t) - v'(t)| \, \mathrm{d}t.$$

Example 2. The operators l_h, f_h, F, W are discontinuous in $W^{1,\infty}(0, T)$, except in the trivial cases $\eta \equiv 0$, $\mu \equiv 0$. We illustrate this fact by choosing the operator F (the other cases are similar). We choose $h > 0$ such that $\int_{h/2}^h \eta(\sigma) \, \mathrm{d}\sigma > 0$ (cf. Definition 2) and put

$$u(t) = \begin{cases} h, & t \in [0, 1] \\ ht, & t \in [1, 2] \end{cases}, \quad u_n(t) = \begin{cases} h\left(1 - \dfrac{t}{n}\right), & t \in [0, 1] \\ h\left(t - \dfrac{1}{n}\right), & t \in [1, 2] \end{cases}.$$

We have

$$u(t) - u_n(t) = \begin{cases} \dfrac{ht}{n}, & t \in [0, 1] \\ \dfrac{h}{n}, & t \in [1, 2] \end{cases},$$

hence $u_n \to u$ in $W^{1,\infty}(0, 2)$. On the other hand,

$$(F(u))'(t) - (F(u_n))'(t) = \begin{cases} \dfrac{h}{n} \displaystyle\int_{ht/2n}^{\infty} \eta(\sigma) \, \mathrm{d}\sigma, & t \in (0, 1), \\ h \displaystyle\int_{h(t-1)/2}^{ht} \eta(\sigma) \, \mathrm{d}\sigma, & t \in \left(1, 1 + \dfrac{1}{n}\right), \\ h \displaystyle\int_{h(t-1/n)}^{ht} \eta(\sigma) \, \mathrm{d}\sigma, & t \in \left(1 + \dfrac{1}{n}, 2\right). \end{cases}$$

For $t \in (1, 1 + (1/n))$ we have $(F(u))'(t) - (F(u_n))'(t) \geqq h \int_{h/2}^h \eta(\sigma) \, \mathrm{d}\sigma \geqq \text{const.}$, hence $F(u_n)$ does not converge to $F(u)$ in $W^{1,\infty}(0, 2)$.

Example 3. The operators l_h, f_h, F, W are not locally Lipschitz in $W^{1,p}(0, T)$ for $p > 1$, except in the trivial cases. Again we choose the operator F. Let us introduce the functions $z_n: [0, 1] \to R^1$ for $n \in N$:

$$z_n(t) = \begin{cases} \varrho_n t, & t \in [0, \tau_n] \\ \varrho_n \tau_n, & t \in [\tau_n, 1] \end{cases}, \quad \text{where} \quad \varrho_n = n^{1/p-1}, \quad \tau_n = n^{-p/p-1}.$$

Put

$$u_n(t) = \begin{cases} h, & t \in [0, 1] \\ h + z_n(t-1), & t \in [1, 2] \end{cases}, \quad v_n(t) = \begin{cases} h - \frac{t}{n}, & t \in [0, 1] \\ h - \frac{1}{n} + z_n(t-1), & t \in [1, 2] \end{cases},$$

where h is the same as in Example 2.

We have

$$u_n(t) - v_n(t) = \begin{cases} \frac{t}{n}, & t \in [0, 1] \\ \frac{1}{n}, & t \in [1, 2] \end{cases}, \quad \text{hence}$$

$$\|u_n - v_n\|_{1,p} \leqq c/n, \quad \text{and} \quad \|u_n\|_{1,p}, \quad \|v_n\|_{1,p} \leqq \text{const.}$$

Analogous computation as in Example 2 yields

$$\|F(u_n) - F(v_n)\|_{1,p} \geqq \int_1^2 |(F(u_n))'(t) - (F(v_n))'(t)|^p \, dt \geqq \text{const.}$$

for $n \geqq 1/h$.

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Souhrn

SPOJITOST HYSTEREZNÍCH OPERÁTORŮ V SOBOLEVOVÝCH PROSTORECH

PAVEL KREJČÍ, VLADIMÍR LOVICAR

Je dokázáno, že klasické hysterezní operátory (Prandtlův, Išlinského a Preisachův) jsou spojité v Sobolevově prostoru $W^{1,p}(0, T)$ pro $1 \leqq p < +\infty$, (lokálně) lipschitzovské ve $W^{1,1}(0, T)$ a nespojité ve $W^{1,\infty}(0, T)$ pro libovolné $T > 0$. Příklady ukazují, že tento výsledek je optimální.

Резюме

НЕПРЕРЫВНОСТЬ ГИСТЕРЕЗИСНЫХ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ СОБОЛЕВА

PAVEL KREJČÍ, VLADIMÍR LOVICAR

Доказывается, что классические гистерезисные операторы Прандтля, Ишлинского и Прейсаха непрерывны в пространстве Соболева $W^{1,p}(0, T)$ для $1 \leqq p < +\infty$, (локально) непрерывны по Липшицу в $W^{1,1}(0, T)$ и разрывны в $W^{1,\infty}(0, T)$ для произвольного $T > 0$. Примеры показывают, что этот результат оптимален.

Authors' addresses: RNDr. *Pavel Krejčí*, CSc., RNDr. *Vladimír Lovicar*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.