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DISCRETE SMOOTHING SPLINES AND DIGITAL FILTRATION. THEORY AND APPLICATIONS

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Summary. Two universally applicable smoothing operations adjustable to meet the specific properties of the given smoothing problem are widely used: 1. Smoothing splines and 2. Smoothing digital convolution filters. The first operation is related to the data vector $\boldsymbol{r} = (r_0, \ldots, r_{n-1})^T$ with respect to the operations $\mathscr{A}$, $\mathscr{L}$ and to the smoothing parameter α. The resulting function is denoted by $\sigma_\alpha(t)$. The measured sample $\boldsymbol{r}$ is defined on an equally spaced mesh $\Delta = \{t_i = ih\}_{i=0}^{n-1}$ $T = nh$. The smoothed data vector $\boldsymbol{y}$ is then $\boldsymbol{y} = \{\sigma_\alpha(t_i)\}_{i=0}^{n-1}$. The other operation gives $\boldsymbol{y} \in E^n$ computed by $\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{r}$, where $*$ stands for the discrete convolution, the running weighted mean by $\boldsymbol{h}$. The main aims of the present contribution: to prove the existence of close interconnection between the two smoothing approaches (Cor. 2.6 and [11]), to develop the transfer function, which characterizes the smoothing spline as a filter in terms of α and λ_{ik} (the eigenvalues of the discrete analogue of $\mathscr{L}$) (Th. 2.5), to develop the reduction ratio between the original and the smoothed data in the same terms (Th. 3.1).

Key words: Discrete smoothing spline (*DS*-spline), smoothing parameter, digital convolution filter, transfer function

AMS classification: 41A15, 93E11, 93E14, 65D07, 65D10.

1. THEORY

1. Problem statement

In technical and physical sciences most research techniques use observation and experiment as one of the fundamental tools. On their basis one tries to construct one or more models which describe the investigated phenomenon as a whole and/or are focused only on some of its selected characteristics. Methods of mathematical statistics and functional analysis play an ever growing role in the process of model construction and verification.

As a rule, various dependent and independent variable quantities, data and parameters enter the model. If the model is given in the form of an analytical function, one usually applies statistical methods when looking for the estimates of its unknown parameters. In case that no analytical model is known and only experimental data loaded with measurement errors describe the relationship between the dependent and independent quantities, it is recommendable to apply a suitable smoothing operation derived by means of methods of mathematical and functional analysis which would both remove the unwanted error fluctuations as much as possible and keep the destortion of the searched physical dependence to a minimum.

The additive model of the experimental measurement $r_i = f(t_i) + e_i$, $i = 0(1)\,n-1$ is used, where $\boldsymbol{r} = \{r_i\}_{i=0}^{n-1}$ are the experimental data, $f(t)$ an unknown function and e_i the measurement errors which may be considered to be the observed values of uncorrelated random variables ε_i with zero mean and common variance. We want to find $y_i \approx f(t_i)$ on the mesh $\Delta = \{ih\}_{i=0}^{n-1}$ $(h > 0,\, T = nh)$ by using a smoothing operation. The smoothing operation should take into account the knowledge of the physical phenomenon.

Two universally applicable smoothing operations adjustable to meet the specific properties of the given smoothing problem are widely used: 1. Smoothing splines and 2. Smoothing digital convolution filters.

1. *Smoothing spline* of the data vector $\boldsymbol{r} = (r_0, \ldots, r_{n-1})^{\mathrm{T}}$ with respect to $\mathscr{A}$, $\mathscr{L}$ and α is a minimizer $\sigma_\alpha(t)$ of the functional

$$(1.1) \qquad \mathscr{F}_{\alpha,\mathscr{L}}(f) = \|\mathscr{A}f - \boldsymbol{r}\|_{E^n}^2 + \alpha\|\mathscr{L}f\|_{L_2}^2 \quad (f \in W^{2,\nu} \text{ complex}),$$

where $\mathscr{A}$: $W^{2,\nu} \to E^n$ is the sampling operator on the mesh Δ, $\mathscr{A}f = (f(t_0), \ldots, f(t_{n-1}))^{\mathrm{T}}$, $\mathscr{L}$: $W^{2,\nu} \to L_2$ a linear bounded operator (e.g. a differential one) and $\alpha > 0$ a real number, the smoothing parameter.

Then $\boldsymbol{r}^\alpha = \{r_i^\alpha\}_{i=0}^{n-1}$ with $r_i^\alpha = \sigma_\alpha(t_i)$ is the smoothed data vector.

2. *Smoothing digital convolution filter* assigns to the input data $\boldsymbol{r}$ its smoothed output $\boldsymbol{y} \in E^n$ computed by

$$(1.2a) \qquad \boldsymbol{y} = \boldsymbol{h} * \boldsymbol{r}.$$

Here $*$ stands for the discrete convolution, $y_i = \sum_{k=0}^{n-1} h_k r_{i-k}$ (= the running weighted mean by $\boldsymbol{h}$), where for $i < 0$ or $i \geqq n$

$$r_i = \begin{cases} r_{i(\mathrm{mod}\, n)} & \text{for implied } n\text{-periodicity} \\ 0 & \text{otherwise.} \end{cases}$$

$\boldsymbol{h}$ is an impulse response of the filter modifying the smoothing effect. Without loss of generality we can further assume a periodic convolution $*$. Then an equivalent formulation of (1.2a) is as follows

$$(1.2b) \qquad \boldsymbol{y} = H\boldsymbol{r},$$

where H is a circulant matrix with the generating vector

$$\boldsymbol{h} = (h_0, \ldots, h_{n-1})^{\mathrm{T}}.$$

Denote by

$$\mathrm{DFT}^+ : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathrm{DFT}^+(\boldsymbol{x}) = \boldsymbol{X} = W^+\boldsymbol{x}$$

the discrete Fourier transformation (DFT) of length n and by DFT$^-$ its inverse

$$\mathrm{DFT}^-(\boldsymbol{X}) = \boldsymbol{x} = \frac{1}{n} W^- \boldsymbol{X},$$

where

$$W^{\pm} = (\varepsilon^{\pm jk})_{j,k=0}^{n-1}, \quad \varepsilon = \mathrm{e}^{\mathrm{i}2\pi/n}. \tag{1.2c}$$

$(1/\sqrt{n})\, W^+ = U$ is a unitary matrix, i.e. $UU^* = I_n$, and $U^* = (1/\sqrt{n})\, W^-$, see Lemma 2.3; as usual, U^* stands for $\bar{U}^{\mathrm{T}}$. Then (1.2a) can be rewritten equivalently as

$$\begin{aligned} &\boldsymbol{y} = \mathrm{DFT}^- (\mathrm{DFT}^+(\boldsymbol{h}) \circ \mathrm{DFT}^+(\boldsymbol{r})) \\ &\text{or} \\ &\boldsymbol{Y} = \boldsymbol{H} \circ \boldsymbol{R} \text{ (i.e. } Y_k = H_k R_k, \quad k = 0(1)\, n-1). \end{aligned} \tag{1.2d}$$

In view of the well-known convolution theorem [4], p. 98, $\boldsymbol{H}$ is called a "transfer function" (or "frequency response") of the filter saying how each of $n/2$ sinusoidal wave components of $\boldsymbol{r}$ are "damped down" as $\boldsymbol{r}$ passes through the filter.

The main aims of the present contribution:

- to prove the existence of close interconnection between the two smoothing approaches
- to develop the transfer function which characterizes the smoothing spline as a filter in terms of α and λ_k (the eigenvalues of the discrete analogue of $\mathscr{L}$)
- to develop the reduction ratio between the original and smoothed data in the same terms.

Restrictions:

n-periodicity and equally spaced mesh, $\Delta = \{t_i = ih\}_{i=0}^{n-1}$, $T = nh$, $h > 0$.

2. Discrete analogue of a (periodic) smoothing spline

This analogue is obtained from (1.1) by replacing the integral

$$\|\mathscr{L}f\|_{L_2}^2 = \int_0^T |\mathscr{L} f(t)|^2 \,\mathrm{d}t \tag{2.0}$$

by its discrete approximation

$$\int_0^T |\mathscr{L} f(t)|^2 \,\mathrm{d}t = h \sum_{i=0}^{n-1} |(\mathscr{L}f)(t_i)|^2 + O(h^2); \tag{2.1}$$

thus we approximate the operator $\mathscr{L}$ on the mesh Δ by its discrete analogue L_D (a complex $n \times n$ matrix) and define $L_I = \sqrt{h} L_D$. Then

$$h \sum_{i=0}^{n-1} |\mathscr{L}(f)(t_i)|^2 = h \|L_D \mathscr{A}(f)\|_{E^n}^2 + O(h^2) = \|L_I \mathscr{A}(f)\|_{E^n}^2 + O(h^2). \tag{2.2}$$

The foregoing observation enables us to define a discrete smoothing criterion $F_{\alpha,L}$: $\mathbb{C}^n \to \mathbb{R}$ by

$$F_{\alpha,L}(\boldsymbol{y}) = \|\boldsymbol{y} - \boldsymbol{r}\|_{E^n}^2 + \alpha \|L\boldsymbol{y}\|_{E^n}^2, \quad \boldsymbol{y} \in E^n, \quad L\ n \times n \text{ matrix}. \tag{2.3}$$

It should be pointed out that the specification $L = L_I (= \sqrt{h} L_D)$ has the following desirable property

$$|F_{\alpha,L_I}(\mathscr{A}f) - \mathscr{F}_{\alpha,\mathscr{L}}(f)| \to 0\,, \quad h \to 0\,, \quad f \in W^{2,\nu}\,.$$

2.1. Definition. *Let $\mathscr{L}$ be an operator as above, $L = L_I$. Let $y = r^\alpha$ minimize (2.3). Then a (periodic) spline $\tau_\alpha(t)$ interpolating r^α with respect to $\mathscr{L}$ is called a discrete (periodic) smoothing spline of r* (shortly a *DS*-spline). □

Indeed, for the smoothed data vector r^α we have $r^\alpha = \mathscr{A}\,\tau_\alpha(t) = (\tau_\alpha(t_0), \ldots$ $\ldots, \tau_\alpha(t_{n-1}))^{\mathrm{T}}$.

Remember that the spline interpolating r with respect to $\mathscr{L}$ is a function of a given Hilbert space $W^{2,\nu}$, which minimizes the functional (2.0) on the subclass of functions of $W^{2,\nu}$ that interpolate the vector r. In case of $\mathscr{L} = \mathscr{D}^{(\nu)}$ the spline is a polynomial of order $2\nu - 1$.

2.1a. Convention. *In all what follows let $\Delta = \{t_i = ih\}_{i=0}^{n-1}$ $(h > 0)$ be an equally spaced mesh on the interval $[0, T = nh]$, $r = (r_0, \ldots, r_{n-1})^{\mathrm{T}}$ the data vector to be smoothed, $r^\alpha = (r_0^\alpha, \ldots, r_{n-1}^\alpha)^{\mathrm{T}}$ the smoothed vector. Further, let us assume that the matrix L has an orthonormal system $u_0, \ldots, u_{n-1}$ of eigenvectors along with the respective eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$.* □

Then the matrix $U^* = (u_0, \ldots, u_{n-1})$ formed by the $n \times 1$ columns u_i is unitary and (see also [16] 4.10.1 and 4.10.3)

$$ULU^* = \operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1})\,. \tag{2.4}$$

E.g. the circulant matrix L has the mentioned property; the corresponding matrix U is equal to $(1/\sqrt{n})\, W^+$ (see the following Lemma 2.4 or [16] 4.9 and 4.8.3; cf. also (1.2c)).

2.2. Theorem. *Let L have the above property. Then*

$$r^\alpha = U^* R^\alpha\,,$$

where

$$R^\alpha = (R_0^\alpha, \ldots, R_{n-1}^\alpha)^{\mathrm{T}}\,, \quad R_s^\alpha = \frac{R_s}{1 + \alpha|\lambda_s|^2}\,, \quad s = 0(1)\,n-1\,, \tag{2.5}$$

$$R = Ur\,. \tag{2.6}$$

(R and R^α are base U^*-coordinates of r and r^α, respectively). Moreover, $r \in \mathbb{R}^n$ implies $r^\alpha \in \mathbb{R}^n$.

Proof. By Parseval's theorem for unitary transforms ([16] 4.7.14 (ii))

$$F_{\alpha,L}(y) = \|Uy - Ur\|_{E^n}^2 + \alpha\|ULy\|_{E^n}^2 = \|Y - R\|_{E^n}^2 + \alpha\|ULU^*Uy\|_{E^n}^2 =$$
$$= \|Y - R\|_{E^n}^2 + \alpha\|\operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1})\, Y\|_{E^n}^2\,.$$

The vector $\boldsymbol{Y} = \boldsymbol{R}^\alpha$ minimizing $F_{\alpha,L}(y)$ is a solution of the SLE:

$$\frac{\partial F_{\alpha,L}(Y)}{\partial Y} = 0\,, \quad k = 0(1)\,n-1\,.$$

It is easy to see that this SLE in $Y_0, \ldots, Y_{n-1}$ has the diagonal matrix $S = \operatorname{diag}(1 + \alpha|\lambda_0|^2, \ldots, 1 + \alpha|\lambda_{n-1}|^2)$ and the right-hand side $\boldsymbol{R}$, i.e.

$$\boldsymbol{r}^\alpha = U^*\boldsymbol{R}^\alpha = U^*S^{-1}\boldsymbol{R} = U^*S^{-1}U\boldsymbol{r}\,.$$

Moreover, $U^*S^{-1}U = (U^*S^{-1}U)^*$ implies that $U^*S^{-1}U$ is real, hence $\boldsymbol{r}^\alpha$ is real if $\boldsymbol{r}$ is real. □

2.3. Lemma

$$\sum_{j=0}^{n-1} \varepsilon^{\pm kj} = \begin{cases} n & \text{if} \quad k \equiv 0(\text{mod}\ n)\,, \\ 0 & \textit{otherwise}\,, \end{cases} \qquad \varepsilon = e^{2\pi i/n}\,.$$

Proof. If $k \equiv 0(\text{mod}\ n)$, then $\varepsilon^{\pm kj} = 1$ for all j; if $k \not\equiv 0(\text{mod}\ n)$, then

$$\varepsilon^{\pm k} \neq 1\,, \quad \text{thus} \quad \sum_{j=0}^{n-1} \varepsilon^{\pm kj} = (\varepsilon^{\pm nk} - 1)/(\varepsilon^{\pm k} - 1) = 0\,. \quad \square$$

2.4. Lemma. *Let L be a circulant matrix with a generating vector $l = (l_0, \ldots, l_{n-1})^{\mathrm{T}}$. Then the Fourier base $U^* = (1/\sqrt{n})\,W^-$ (i.e. the system of columns of U^*) is the system of eigenvectors of L with the respective eigenvalues*

(2.7) $\qquad (\lambda_0, \ldots, \lambda_{n-1})^{\mathrm{T}} = W^+\boldsymbol{l}\,.$

If L is real, then $\lambda_k = \bar{\lambda}_{n-k}$, $k = 1(1)\,n-1$.

Proof. By Lemma 2.3 the matrix U^* is unitary. The matrix W^+ transforms the j-th column $\{l_{k-j(\text{mod}\ n)}\}_{k=0}^{n-1}$ of the matrix L to the column $\{\sum_{k=0}^{n-1} \varepsilon^{ik} l_{k-j(\text{mod}\ n)}\}_{i=0}^{n-1} = \{\sum_{k=0}^{n-1} \varepsilon^{i(k+j)} l_k\}_{i=0}^{n-1} = \{\varepsilon^{ij}\lambda_i\}_{i=0}^{n-1} =$ the j-th column of the matrix $\operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1})\,.\,W^+$. Thus

(2.7′) $\qquad W^+L = \operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1})\,W^+$

completing the proof. □

Note. In fact, we have proved (1.2d), because from (2.7′) we have

$$W^+L\boldsymbol{r} = \operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1})\,W^+\boldsymbol{r} = \operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1})\,\mathrm{DFT}^+(\boldsymbol{r}) =$$
$$= \mathrm{DFT}^+(\boldsymbol{l}) \circ \mathrm{DFT}^+(\boldsymbol{r}) = \mathrm{DFT}^+(\boldsymbol{l} * \boldsymbol{r})\,.$$

If L is real, the relation $\lambda_k = \bar{\lambda}_{n-k}$ follows evidently from (2.7). □

By virtue of the above result we have $U = (1/\sqrt{n})\,W^+$ provided L is circulant.

2.5. Theorem (Main Theorem). *Let L be a circulant matrix. Then $U = (1/\sqrt{n})\,W^+$ and*

$$(2.8) \qquad r^\alpha = h * r = \frac{1}{n} W^- H W^+ r = \tilde{H} r ,$$

where

$$(2.9) \qquad H = W^+ h = \left(\frac{1}{1 + \alpha|\lambda_0|^2}, \ldots, \frac{1}{1 + \alpha|\lambda_{n-1}|^2} \right)^T$$

is the associated transfer function (*see* (1.2d)),

$$(2.10) \qquad h = \frac{1}{n} W^- H, \quad H = \operatorname{diag} \left(\frac{1}{1 + \alpha|\lambda_0|^2}, \ldots, \frac{1}{1 + \alpha|\lambda_{n-1}|^2} \right)$$

and the matrix

$$(2.11) \qquad \tilde{H} = \frac{1}{n} W^- H W^+$$

is circulant with the generating vector h.

Proof. Denoting $H = (1/(1 + \alpha|\lambda_0|^2), \ldots, 1/(1 + \alpha|\lambda_{n-1}|^2))^T$ we have by Theorem 2.2

$$U r^\alpha = U U^* R^\alpha = R^\alpha = \left(\frac{R_0}{1 + \alpha|\lambda_0|^2}, \ldots, \frac{R_{n-1}}{1 + \alpha|\lambda_{n-1}|^2} \right)^T =$$

$$= H \circ U r = H \circ (1/\sqrt{n}) \, W^+ r .$$

Thus

$$r^\alpha = \frac{1}{n} W^-(H \circ W^+ r) .$$

If we define $h = (1/n) \, W^- H$ where H is the $n \times n$ diagonal matrix with the diagonal H, we obtain from the preceding

$$r^\alpha = h * r = \frac{1}{n} W^- H W^+ r .$$

Write $\hat{H}$ for the circulant matrix generated by h. Then using Lemma 2.4 we obtain

$$W^+ \hat{H} = W^+ \hat{H} \frac{1}{n} W^- W^+ = H W^+ .$$

Thus

$$\hat{H} = \frac{1}{n} W^- H W^+ = \tilde{H} .$$

Clearly, the matrix $\tilde{H}$ is circulant and its first column is h. □

2.6. Corollary. *The periodic DS-spline smoothing data vector* r *is a periodic interpolation spline of the data* r^α *which were obtained by filtration of* r *according to the transfer function* (2.9) *provided that* L *is a circulant matrix.* □

2.7 Corollary. *Let L be a circulant matrix such that* $L = L_I$ *where* L_I *is associated with the differential operator* $\mathscr{L} = \mathscr{D}^{(\nu)}$ *of order* ν (cf. §2). *Then the operator* $\boldsymbol{r} \to \boldsymbol{r}^\alpha$ *is linear.The operator* $\boldsymbol{r} \to$ (*the periodic DS-spline smoothing* $\boldsymbol{r}$) *is linear, too* (*supposing* $n \geqq \nu$).

Proof. The first assertion follows immediately from (2.8). The second assertion: Let $\boldsymbol{r} \to \boldsymbol{r}^\alpha$ and $\boldsymbol{s} \to \boldsymbol{s}^\alpha$ be two discrete smoothing operations and $\tau_\alpha(t)$ and $\varphi_\alpha(t)$ two periodic splines of $(2\nu - 1)$-th degree which interpolate $\boldsymbol{r}^\alpha$ and $\boldsymbol{s}^\alpha$, respectively. Then $a\,\tau_\alpha(t) + b\,\varphi_\alpha(t)$ (a and b arbitrary constants) is a polynomial spline of $(2\nu - 1)$-th degree which interpolates $a\boldsymbol{r}^\alpha + b\boldsymbol{s}^\alpha$. The first assertion implies that the discrete smoothing of $a\boldsymbol{r} + b\boldsymbol{s}$ results in $a\boldsymbol{r}^\alpha + b\boldsymbol{s}^\alpha$. Now, let $\psi_\alpha(t)$ be the periodic polynomial spline of degree $2\nu - 1$ which interpolates the data $a\boldsymbol{r}^\alpha + b\boldsymbol{s}^\alpha$. The supposition $n \geqq \nu$ implies the uniqueness of the spline of degree $2\nu - 1$ which interpolates the vector $a\boldsymbol{r} + b\boldsymbol{s}$ ([18], Th. 1.5) and thus $\psi_\alpha(t) = a\,\tau_\alpha(t) + b\,\varphi_\alpha(t)$. □

3. Reduction of data $\boldsymbol{r}$ by the DS-spline smoothing

We are going to express the smoothed data r_s^α and/or the reduction $w_s = r_s^\alpha / r_s$ $s = 0(1)\,n - 1$ (if $r_s \neq 0$) explicitly in terms of r_s and λ_s (Fourier base is still assumed) By Theorem 2.5 and in view of (1.2a–d) we have

3.1 Theorem. *Let L be a circulant matrix and* λ_k *its eigenvalues. Then*

$$r_s^\alpha = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\hat{B}_{s,k}}{1 + \alpha|\lambda_k|^2}, \quad s = 0(1)\,n - 1, \tag{3.1a}$$

where

$$\hat{B} = \{\hat{B}_{s,k}\}_{s,k=0}^{n-1} = W^- \operatorname{diag} \boldsymbol{R}.$$

Moreover, if L and r are real, then

$$r_s^\alpha = \frac{1}{n} \sum_{k=0}^{[n/2]} \frac{B_{s,k}}{1 + \alpha|\lambda_k|^2}, \quad s = 0(1)\,n - 1, \tag{3.1b}$$

where

$$\left\{\begin{aligned} B_{s,0} &= \sum_{t=0}^{n-1} r_t, \\ B_{s,k} &= 2 \sum_{t=0}^{n-1} r_t \cos \frac{2\pi k}{n}(t - s), \quad k = 1(1)\left[\frac{n-1}{2}\right], \\ B_{s,n/2} &= (-1)^s \sum_{t=0}^{n-1} (-1)^t\, r_t \quad \textit{if } n \textit{ is even}, \end{aligned}\right. \tag{3.2}$$

$$w_s = \frac{r_s^\alpha}{r_s} = \frac{1}{n r_s} \sum_{k=0}^{[n/2]} \frac{B_{s,k}}{1 + \alpha|\lambda_k|^2} \quad (\text{if } r_s \neq 0). \tag{3.3}$$

Proof. By Theorem 2.5

$$r^{\alpha} = \frac{1}{n} W^{-} \operatorname{diag} \boldsymbol{HR} = \frac{1}{n} W^{-} \operatorname{diag} \boldsymbol{RH} = \frac{1}{n} \hat{B}\boldsymbol{H}, \tag{3.4}$$

where

$$\hat{B} = \{\hat{B}_{s,k}\}_{s,k=0}^{n-1} = W^{-} \operatorname{diag} \boldsymbol{R} \tag{3.5}$$

is a matrix depending on $\boldsymbol{r}$ (we write $\hat{B}_{s,k} = \hat{B}_{s,k}(\boldsymbol{r})$ if necessary). Thus, we have proved (3.1a).

If $\boldsymbol{r}$ is real, then

$$\hat{B}_{s,k} = \hat{B}_{s,n-k}^{-}, \quad k = 1(1)\,[n/2], \tag{3.6}$$

because

$$\hat{B}_{s,k} = \varepsilon^{-sk} R_k = \varepsilon^{-sk} \sum_{t=0}^{n-1} \varepsilon^{kt} r_t = \sum_{t=0}^{n-1} \varepsilon^{k(t-s)} r_t,$$

$$\hat{B}_{s,n-k} = \varepsilon^{-s(n-k)} R_{n-k} = \varepsilon^{sk} \overline{R}_k = (\varepsilon^{-ks} R_k)^{-} = \hat{B}_{s,k}^{-}.$$

Provided that both L and $\boldsymbol{r}$ are real, we obtain (3.1b) and (3.2) if we put

$$\left\{\begin{array}{l} B_{s,0} = \hat{B}_{s,0}, \quad B_{s,k} = \hat{B}_{s,k} + \hat{B}_{s,n-k} = 2\Re\hat{B}_{s,k} = 2\sum_{t=0}^{n-1} r_t \cos\frac{2\pi k}{n}(t-s), \\ \qquad k = 1(1)\left[\frac{n-1}{2}\right], \\ B_{s,n/2} = \hat{B}_{s,n/2} = \sum_{t=0}^{n-1} r_t\, \varepsilon^{n(t-s)/2} = \varepsilon^{-ns/2} \sum_{t=0}^{n-1} r_t \varepsilon^{nt/2} = (-1)^s \sum_{t=0}^{n-1} (-1)^t r_t \\ \qquad \text{if} \quad n \text{ is even}, \end{array}\right. \tag{3.7}$$

following Lemma 2.4 ($\lambda_k = \bar{\lambda}_{n-k}$ and consequently $H_k = H_{n-k}$). □

Sometimes we are interested in the parameter α expressed as a function of the other quantities. As the function $1/(1 + \alpha|\lambda_k|^2)$ has the expansion

$$1 - \alpha|\lambda_k|^2 + \alpha^2|\lambda_k|^4 - \alpha^3|\lambda_k|^6 + \ldots$$

(which converges if and only if $\alpha|\lambda_k|^2 < 1$), we have

$$n r_s w_s (= n r_s^{\alpha}) = \sum_{k=0}^{[n/2]} B_{s,k} - \alpha \sum_{k=0}^{[n/2]} |\lambda_k|^2 B_{s,k} + \alpha^2 \sum_{k=0}^{[n/2]} |\lambda_k|^4 B_{s,k} - \ldots$$

or

$$n r_s w_s (= n r_s^{\alpha}) = \sum_{j=0}^{\infty} (-1)^j \alpha^j \sum_{k=0}^{[n/2]} |\lambda_k|^{2j} B_{s,k}$$

with $\alpha|\lambda_k|^2 < 1$, $k = 0(1)\,[n/2]$, which enables us to determine an approximate value of α in terms of λ_k and w_s.

4. Digital filtration

By Lemma 2.4, the columns of the matrix $U^* = (1/\sqrt{n}) W^-$ form an orthonormal system of eigenvectors for each circulant matrix, and the respective eigenvalues are obtained as the transform $W^+ \boldsymbol{l}$ of its generating vector $\boldsymbol{l}$. The matrices W^+ and $(1/n) W^-$ define mutually inverse linear transforms

(4.1) $\quad \mathrm{DFT}^+(\boldsymbol{x}) := W^+\boldsymbol{x} = \boldsymbol{X} \quad \text{and} \quad \mathrm{DFT}^-(\boldsymbol{X}) := (1/n) W^-\boldsymbol{X} = \boldsymbol{x}$

called the discrete Fourier transform and the inverse discrete Fourier transform, respectively (see [4], p. 98). Hereafter, instead of $\boldsymbol{x}' = L\boldsymbol{x}$ we shall use the common notation

(4.2) $\quad \boldsymbol{x}' = \boldsymbol{l} * \boldsymbol{x}, \quad x'_k = \sum_{j=0}^{n-1} l_{k-j(\mathrm{mod}\,n)}\, x_j = \sum_{j=0}^{n-1} l_j x_{k-j(\mathrm{mod}\,n)}\,.$

This bilinear and commutative operation $*$ is known as the discrete cyclic (periodic) convolution (see [4], p. 110). Denoting by the symbol $\circ$ the componentwise multiplication of vectors, we obtain from (2.7′)

$$W^+(\boldsymbol{l} * \boldsymbol{x}) = W^+L\boldsymbol{x} = \mathrm{diag}\,(\lambda_0, \ldots, \lambda_{n-1})\, W^+\boldsymbol{x} = W^+\boldsymbol{l} \circ W^+\boldsymbol{x}$$

or equivalently

(4.3) $\quad \mathrm{DFT}^+(\boldsymbol{l} * \boldsymbol{x}) = \mathrm{DFT}^+(\boldsymbol{l}) \circ \mathrm{DFT}^+(\boldsymbol{x}),$

which is the so called discrete convolution theorem (see [4], p. 118). Hence we arrive at

(4.4) $\quad \boldsymbol{l} * \boldsymbol{x} = \mathrm{DFT}^-(\mathrm{DFT}^+(\boldsymbol{l}) \circ \mathrm{DFT}^+(\boldsymbol{x}))\,.$

The operation $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{l} * \boldsymbol{x}$ is known as the digital convolution filter, the vector $\boldsymbol{l}$ is the impulse response of the filter (in view of $\boldsymbol{l} * (1, 0, \ldots, 0)^{\mathrm{T}} = \boldsymbol{l}$) and $\boldsymbol{l} = \mathrm{DFT}^+(\boldsymbol{l})$ is the transfer function (or frequency response) of the filter. Inspecting (4.2) we see that in the process of convolution filtration each value x_k of the input vector $\boldsymbol{x}$ is replaced on the output by a weighted mean of the neighbouring values with the components of $\boldsymbol{l}$ standing for the weight coefficients.

4.1 Theorem. *If L is a circulant matrix, then the smoothing process described in Theorem* 2.5 *is exactly the convolution filter defined by* (2.8). □

The algorithm of smoothing by discrete smoothing splines may be then sketched as follows:

Step 1: Computing $\mathrm{DFT}^+(\boldsymbol{r}) = \boldsymbol{R}$

Step 2: Multiplying $\boldsymbol{R}$ by the transfer function $\boldsymbol{H}$: $\boldsymbol{R}^\alpha = \boldsymbol{H} \circ \boldsymbol{R}$ (Theorem 2.2)

Step 3: Computing $\mathrm{DFT}^-(\boldsymbol{R}^\alpha) = \boldsymbol{r}^\alpha$ (Theorem 2.2)

Step 4: Interpolating the data $\{t_s, r_s^\alpha\}$, $s = 0(1)\; n-1$ by a periodic interpolation spline.

Steps 1 and 3 may be accomplished very effectively by using a "fast Fourier transform" (FFT) algorithm, e.g. see [4, 5, 7, 17].

5. Discrete analogue of the differential operator $\mathscr{L} = \mathscr{D}^{(\nu)}, 1 \leqq \nu \leqq n - 1$ and its eigenvalues

The differential operator $\mathscr{D}^{(\nu)}$, defined by $\mathscr{D}^{(\nu)}(f) = f^{(\nu)}$, may be discretely approximated by various methods. We shall apply the method of divided differences [3] 2.4, p. 56 ff, [14] III, 3.6, p. 106, which will give rise to an $n \times n$ matrix $D^{(\nu)}$ approximating $\mathscr{D}^{(\nu)}$ and consequently to $L = \sqrt{h}\, D^{(\nu)}$. Both $D^{(\nu)}$ and L are circulant in view of the implied periodicity of f. Supposing the mesh $\Delta \equiv t_0 < t_1 < \ldots < t_{n-1}$ to be equidistant, the ν-th divided difference of a function $f(t)$ assumes the form

$$f[t_i, \ldots, t_{i+\nu}] = \frac{1}{h^\nu} \sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} f(t_{i+j}) .$$

If $\nu = 2z - 1$ is odd, the value of the ν-th derivative of f at the point t_k is approximated as follows

$$f^{(\nu)}(t_k) \approx f[t_{k-z+1}, \ldots, t_{k+z}] = \frac{1}{h^\nu} \sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{\nu - j} f(t_{k-z+1+j}) \tag{5.1a}$$

and if $\nu = 2z$ is even,

$$f^{(\nu)}(t_k) \approx f[t_{k-z}, \ldots, t_{k+z}] = \frac{1}{h^\nu} \sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{\nu - j} f(t_{k-z+j}) . \tag{5.1b}$$

The same result may be obtained from Bessel's interpolation formula [13] III, 1, p. 187–8.

After some formal rearrangements we can rewrite (5.1) into the matrix form (independently of the parity of ν) as follows.

5.1. Theorem.

$$(f^{(\nu)}(t_0), \ldots, f^{(\nu)}(t_{n-1}))^\mathsf{T} \approx D^{(\nu)}(f(t_0), \ldots, f(t_{n-1}))^\mathsf{T} , \tag{5.2}$$

where $D^{(\nu)} = (1/\sqrt{h})\, L$, L *is circulant with the generating (first column) vector* $l = (l_0, \ldots, l_{n-1})^\mathsf{T}$,

$$\begin{cases} l_j = h^{(1/2)-\nu}(-1)^{z+j} \binom{\nu}{z + j} & \text{for } j = 0(1)\left[\frac{\nu}{2}\right], \\ l_j = 0 & \text{for } j = \left[\frac{\nu}{2}\right] + 1(1)\, n - z - 1 , \\ l_j = h^{(1/2)-\nu}(-1)^{j-n+z} \binom{\nu}{j - n + z} & \text{for } j = n - z(1)\, n - 1 , \end{cases} \tag{5.3}$$

where

$$z = \left[\frac{\nu + 1}{2}\right].$$

Proof. Denoting $c_j = h^{(1/2)-\nu}(-1)^{\nu-j}\binom{\nu}{\nu - j}$ for $0 \leqq j \leqq \nu$ and $c_j = 0$ for $\nu + 1 \leqq j \leqq n - 1$ we have by (5.1a) and (5.1b)

$$f^{(\nu)}(t_k) = \frac{1}{\sqrt{h}} \sum_{j=0}^{n-1} c_j f(t_{k-(z-\alpha-j)})$$

where $\nu = 2z - \alpha$, $\alpha \in \{0, 1\}$ and $[\nu/2] = z - \alpha \geqq 0$. Putting $k - (z - \alpha - j) = j'(\text{mod } n)$, i.e. $j = (z - \alpha - (k - j')(\text{mod } n))(\text{mod } n)$, we obtain

$$f^{(\nu)}(t_k) = \frac{1}{\sqrt{h}} \sum_{j'=0}^{n-1} l_{(k-j')\text{mod} n} f(t_{j'}) \quad \text{where} \quad l_j = c_{(z-\alpha-j)\text{mod} n} .$$

We have to distinguish two cases:

(1) $0 \leqq j \leqq z - \alpha$. Then $(z - \alpha - j) \bmod n = z - \alpha - j$ and so $0 \leqq z - \alpha - j \leqq z - \alpha \leqq 2z - \alpha = \nu$. Hence in view of $\nu - (z - \alpha - j) = 2z - \alpha - (z - \alpha - j) = z + j$ we get $l_j = c_{z-\alpha-j} = h^{(1/2)-\nu}(-1)^{z+j}\binom{\nu}{z+j}$.

(2) $z - \alpha + 1 \leqq j \leqq n - 1$. Then $(z - \alpha - j)(\text{mod } n) = n + z - \alpha - j$. In this case the inequality $0 \leqq n + z - \alpha - j \leqq 2z - \alpha = \nu$ holds if and only if $n - z \leqq j \leqq n - 1$. Indeed, $\{n + z - \alpha - j \leqq 2z - \alpha \Leftrightarrow n - z \leqq j\}$ and $j \leqq n - 1$ implies $j \leqq n - \alpha$ whence $0 \leqq z - \alpha \leqq z = j + z - j \leqq n + z - \alpha - j$. Hence in view of $\nu - (n + z - \alpha - j) = 2z - \alpha - (n + z - \alpha - j) = j - n + z$ we get $l_j = c_{n+z-\alpha-j} = h^{(1/2)-\nu}(-1)^{j-n+z}.\binom{\nu}{j-n+z}$ if and only if $n - z \leqq j \leqq n - 1$, in particular $l_j = 0$ for $z - \alpha + 1 \leqq j \leqq n - z - 1$. □

Let us note that the matrices L_D and L_I from Sec. 2 are now specified by $D^{(\nu)}$ and L, respectively.

5.2 Theorem. *The eigenvalues λ_k of the matrix $L = \sqrt{(h)}\, D^{(\nu)}$ fulfil*

$$|\lambda_k|^2 = h^{1-2\nu}\left(2 \sin \frac{\pi k}{n}\right)^{2\nu}, \quad k = 0(1)\, n - 1 . \tag{5.4}$$

Proof. By Lemma 2.4

$$\lambda_k = \sum_{j=0}^{n-1} l_j \varepsilon^{jk}, \quad k = 0(1)\, n - 1.$$

Substituting (5.3) for l_j we obtain

$$\begin{aligned} h^{\nu-(1/2)}\lambda_k &= \sum_{j=z}^{\nu}(-1)^j \binom{\nu}{j} \varepsilon^{(j-z)k} + \sum_{j=0}^{z-1}(-1)^j \binom{\nu}{j} \varepsilon^{(j-z)k} = \\ &= \sum_{j=0}^{\nu}(-1)^j \binom{\nu}{j} \varepsilon^{-(-j+z)k} = \varepsilon^{(\nu-z)k} \sum_{j=0}^{\nu}(-1)^j \binom{\nu}{j} \varepsilon^{-(\nu-j)k} = \\ &= \varepsilon^{(\nu-z)k}(\varepsilon^{-k} - 1)^\nu = \varepsilon^{(\nu-z)k}\left(\varepsilon^{-k/2}(\varepsilon^{-k/2} - \varepsilon^{k/2})\right)^\nu = \\ &= \varepsilon^{(\nu-z)k}\varepsilon^{-\nu k/2}\left(-2\mathrm{i} \sin \frac{\pi k}{n}\right)^\nu, \end{aligned}$$

$$\lambda_k = \varepsilon^{(\nu-2z)k/2}\, h^{(1/2)-\nu}(-\mathrm{i})^\nu \left(2 \sin \frac{\pi k}{n}\right)^\nu . \tag{5.5}$$

Now, if $v = 2z - 1$, then

(5.6) $$\varepsilon^{(v-2z)k/2}(-\mathrm{i})^v = (-1)^z \left(\sin\frac{\pi k}{n} + \mathrm{i}\cos\frac{\pi k}{n}\right)$$

and if $v = 2z$, then

(5.7) $$\varepsilon^{(v-2z)k/2}(-\mathrm{i})^v = (-1)^z\,.$$

Now (5.4) follows from (5.5) to (5.7) □

Conclusions

We believe that all results stated in the present paper are approximately valid also for the classical theory of smoothing splines because the smoothing criteria (1.1) and (2.3) do not differ much. The following conclusions can be drawn from our results.

1. On an equally spaced mesh the periodic smoothing spline is not a new quality among the existing smoothing operations. Approximately the same effect may be achieved by applying a digital convolution filter with the transfer function

$$\boldsymbol{H} = \left(\frac{1}{1 + \alpha|\lambda_0|^2}, \ldots, \frac{1}{1 + \alpha|\lambda_{n-1}|^2}\right)^{\mathrm{T}}.$$

The convolution filtering is preferable also from the point of view of the computational effectivity because there exist fast algorithms with time consumption proportional to $n \lg n$, see [4].

2. The transfer function $\boldsymbol{H}$ may serve as a guideline allowing to choose α and L which are best-suited to satisfy our smoothing requirement. It is preferable to choose directly the eigenvalues λ_k when looking for a suitable L. In case of the differential operator, we choose a suitable order v considering formula (5.4) for $|\lambda_n|^2$.

3. The smoothing effect depends strongly on the sampling rate. If $\boldsymbol{r}$ are n samples and $\boldsymbol{r}'$ are n' samples of the same function on a fixed interval $T = nh = n'h'$, then $\boldsymbol{r}^\alpha$ and $(\boldsymbol{r}')^\alpha$ may show a qualitatively different behaviour caused by an irregular dependence of λ_k (and consequently of the transfer function $\boldsymbol{H}$) on n and h, as can be seen from (5.4) – n enters the sin function.

II. APPLICATIONS

6. p-Periodic data $\boldsymbol{r}$

In all what follows $\boldsymbol{r}$ and L are real.

If the data $\boldsymbol{r}$ have some specified properties such as

- p-periodicity: $r_s = r_{s(\mathrm{mod}\,p)}$, where p is an integer $\geqq 2$ and n is divisible by p,
- partial symmetry: $r_s = r_{n-s}$,
- full symmetry: p even, $r_s = r_{n-s}$ and $r_s = -r_{(p/2)-s}$,

then simpler formulas can be derived from (3.1)–(3.3) in the sense that the summation terms are shortened to p instead of n in case of p-periodicity, those for $B_{s,k}$ even to $[(p-1)/2]$ if additional partial symmetry is present. For p-periodic and fully symmetrical data both summation terms have only $[(p/2-1)/2]$ members. For some given p-periodic wave functions the sums for $B_{s,k}$ may be evaluated explicitly.

First, we shall confine our discussion to p-periodic dara $\boldsymbol{r}$, where p is an integer $\geqq 2$ and n is divisible by p. A representant of this kind is the sequence

$$\boldsymbol{\delta}_{p,q} = \{\delta_{p,q}(t)\}_{t=0}^{n-1},$$

where

$$\delta_{p,q}(t) = \begin{cases} 1 & \text{for} \quad t = t'p + q, \quad q = 0(1)\,p-1, \quad t' = 0(1)\,n/p-1. \\ 0 & \text{otherwise}. \end{cases}$$

Denote $\varepsilon_k = e^{2\pi i/k}$ for every positive integer k, and $\varepsilon := \varepsilon_n$.

The definitions of the matrix $\hat{B}$ (3.5), of the vector $\boldsymbol{R}$ (2.6) and of the matrix W^- (1.2c) imply by Theorem 3.1

$$\hat{B}_{s,k}(\boldsymbol{\delta}_{p,q}) = \varepsilon^{-sk} R_k = \varepsilon^{-sk} \sum_{t=0}^{n-1} \varepsilon^{kt}\, \delta_{p,q}(t) = \varepsilon^{-sk} \sum_{t'=0}^{(n/p)-1} \varepsilon^{k(t'p+q)} =$$
$$= \varepsilon^{(q-s)k} \sum_{t'=0}^{(n/p)-1} \varepsilon_{n/p}^{kt'}.$$

Thus by Lemma 2.3

$$(6.1) \qquad \begin{cases} \hat{B}_{s,k'n/p}(\boldsymbol{\delta}_{p,q}) = \dfrac{n}{p}\, \varepsilon_p^{(q-s)k'} \quad \text{for} \quad k' = 0(1)\,p-1, \\[2ex] \hat{B}_{s,k}(\boldsymbol{\delta}_{p,q}) = 0 \quad \text{for} \quad k \not\equiv 0 \left(\bmod \dfrac{n}{p}\right). \end{cases}$$

Now, let $\boldsymbol{r}$ be an arbitrary p-periodic sequence of length n,

$$\boldsymbol{r} = \sum_{q=0}^{p-1} r_q \boldsymbol{\delta}_{p,q},$$

then by (3.5) and (6.1)

$$(6.2) \qquad \begin{cases} \hat{B}_{s,k'n/p} = \hat{B}_{s,k'n/p}(\boldsymbol{r}) = \hat{B}_{s,k'n/p}\left(\sum\limits_{q=0}^{p-1} r_q \boldsymbol{\delta}_{p,q}\right) = \dfrac{n}{p}\, \varepsilon_p^{-sk'} \sum\limits_{q=0}^{p-1} r_q \varepsilon_p^{k'q}, \\[2ex] \qquad k' = 0(1)\,p-1, \\[2ex] \hat{B}_{s,k} = \hat{B}_{s,k}(\boldsymbol{r}) = 0, \quad k \not\equiv 0 \left(\bmod \dfrac{n}{p}\right). \end{cases}$$

It is easy to see that $\boldsymbol{r}^\alpha$ is p-periodic if $\boldsymbol{r}$ is p-periodic. Indeed, $s \equiv s' (\bmod\ p)$ implies $\varepsilon_p^{-sk'} = \varepsilon_p^{-s'k'}$, thus by (6.2) $\hat{B}_{s,k} = \hat{B}_{s',k}$ and by (3.1) $r_s^\alpha = r_{s'}^\alpha$. Therefore, we may evaluate r_s^α and w_s for $s = 0(1)\,p-1$ only.

Denoting

$$(6.3)\qquad \begin{cases} \tilde{B}_{s,k'} = \dfrac{p}{n}\,\hat{B}_{s,k'n/p}\,, & s,\,k' = 0(1)\,p-1\,, \\ r^{(p)} = (r_0, \ldots, r_{p-1})^{\mathrm{T}} & \text{(the basic period of the data vector } r)\,, \\ W_p^{\pm} = (\varepsilon_p^{\pm jk})_{j,k=0}^{p-1}\,, & \\ R^{(p)+} = W_p^{+} r^{(p)} = \{\sum\limits_{q=0}^{p-1} \varepsilon_p^{k'q} r_q\}_{k'=0}^{p-1} & \end{cases}$$

we obtain by (6.2)

$$(6.4_1)\qquad \tilde{B}_{s,k'} = \varepsilon_p^{-sk'} R_{k'}^{(p)+} = \varepsilon_p^{-sk'} \sum_{q=0}^{p-1} \varepsilon_p^{k'q} r_q\,, \quad s,\,k' = 0(1)\,p-1\,.$$

By virtue of

$$(6.4_2)\qquad \tilde{B}_{s,p-k'} = \tilde{B}_{s,k'}^{-}\,, \quad H_{k'n/p} = H_{n-(k'n/p)}\,, \quad k' = 0(1)\,[p/2]\,,$$

the following assertion is true. (The proof proceeds like in Theorem 3.1.)

6.1 Theorem. *The smoothed vector r^α of a p-periodic data vector r is p-periodic, too, and*

$$(6.5)\qquad r_s^\alpha = \frac{1}{p} \sum_{k'=0}^{[p/2]} \frac{B'_{s,k'}}{1 + \alpha|\lambda_{k'n/p}|^2}\,, \quad s = 0(1)\,p-1\,,$$

where

$$(6.6)\qquad \begin{cases} B'_{s,0} = \tilde{B}_{s,0} = R_0^{(p)+} = \sum\limits_{q=0}^{p-1} r_q\,, \\ B'_{s,k'} = \tilde{B}_{s,k'} + \tilde{B}_{s,p-k'} = 2\sum\limits_{q=0}^{p-1} r_q \cos 2\pi k'(q-s)/p\,, \quad k' = 1(1)\left[\dfrac{p-1}{2}\right], \\ B'_{s,p/2} = \tilde{B}_{s,p/2} = (-1)^s R_{p/2}^{(p)+} = (-1)^s \sum\limits_{q=0}^{p-1} (-1)^q\, r_q \quad \text{for } p \text{ even}\,. \end{cases}$$

If $r_s \neq 0$, then

$$(6.7)\qquad w_s = r_s^\alpha / r_s = \frac{1}{p r_s} \sum_{k'=0}^{[p/2]} \frac{B'_{s,k'}}{1 + \alpha|\lambda_{k'n/p}|^2}\,, \quad s = 0(1)\,p-1\,. \;\square$$

6a. Partially symmetrical p-periodic data r

Consider a vector r of length n and denote by $r^k = (r_{j-k(\mathrm{mod}\,n)})_{j=0}^{n-1}$ the vector cyclically shifted k positions to the right, $k = 0(1)\,n-1$. Then $(r^k)^\alpha$ is the vector $r^\alpha = \tilde{H} r$ cyclically shifted k positions to the right, i.e. $(r^k)^\alpha = (r^\alpha)^k$. Indeed, the i-th component of the smoothed vector $(r^k)^\alpha$ satisfies $(r^k)_i^\alpha = \sum_{j=0}^{n-1} h_{i-j(\mathrm{mod}\,n)} r_{j-k(\mathrm{mod}\,n)} = \sum_{j'=0}^{n-1} h_{i-k-j'(\mathrm{mod}\,n)} r_{j'} = r_{i-k}^\alpha = (r^{\alpha k})_i =$ the i-th component of the shifted vector r^α. Thus without loss of generality we may restrict our considerations to such r or $r^{(p)}$, respectively, for which

$$(6.8)\qquad r_0 = \min(r_0, \ldots, r_{n-1}) = \min(r_0, \ldots, r_{p-1}).$$

6.2 Definition. *A data vector* $\boldsymbol{r}$ *of length* n *is said to be partially symmetric if*

$$r_k = r_{n-k(\mathrm{mod}\, n)} \quad \textit{for} \quad k = 0(1)\,[n/2]\,. \quad \square$$

Clearly a p-periodic vector $\boldsymbol{r}$ of length n is partially symmetric if and only if

$$r_k = r_{p-k(\mathrm{mod}\, p)} \quad \text{for} \quad k = 0(1)\,[p/2]\,.$$

6.3 Theorem. *Let* $\boldsymbol{r}$ *be a* p*-periodic and partially symmetrical data vector. Then*

$$(6.5\mathrm{a})\qquad r_s^\alpha = \frac{1}{p}\sum_{k'=0}^{[p/2]} \frac{B''_{s,k'}}{1 + |\lambda_{k'n/p}|^2}, \quad s = 0(1)\,p-1\,,$$

where

$$(6.6\mathrm{a})\quad \begin{cases} B''_{s,0} = R_0^{(p)+} = r_0 + 2\sum\limits_{q=1}^{[(p-1)/2]} r_q + r_{p/2}\,, \\ B''_{s,k'} = 2\cos\dfrac{2\pi sk'}{p}\,R_{k'}^{(p)+} = 2\cos\dfrac{2\pi sk'}{p}\left(r_0 + 2\sum\limits_{q=1}^{[(p-1)/2]} r_q \cos\dfrac{2\pi k'q}{p} + (-1)^{k'} r_{p/2}\right), \\ \qquad k' = 1(1)\,[(p-1)/2]\,, \\ B''_{s,p/2} = (-1)^s R_{p/2}^{(p)+} = (-1)^s\left(r_0 + 2\sum\limits_{q=1}^{[(p-1)/2]} (-1)^q r_q + (-1)^{p/2}\, r_{p/2}\right) \\ \qquad\qquad \textit{for} \quad p \textit{ even}\,. \\ \textit{For } p \textit{ odd we define } r_{p/2} = 0\,. \end{cases}$$

$$(6.7\mathrm{a})\qquad w_s = r_s^\alpha / r_s \quad (\text{if } r_s \neq 0)\,.$$

Proof. The vector $\boldsymbol{r}$ is p-periodic and partially symmetric, so we only need to evaluate formulas for $s = 0(1)\,[p/2]$. In view of the partial symmetry, by (6.3) and (6.4) we have

$$(6.9)\qquad R_{k'}^{(p)+} = \sum_{q=0}^{p-1} \varepsilon_p^{k'q} r_q = r_0 + 2\sum_{q=1}^{[(p-1)/2]} r_q \cos\frac{2\pi k'q}{p} + (-1)^{k'}\, r_{p/2}\,,$$

$$k' = 0(1)\,[p/2].$$

If $p = 2$, then the sum $\sum\limits_{q=1}^{[(p-1)/2]} \left(= \sum\limits_{q=1}^{0}\right)$ will be omitted; this convention will be applied in the sequel, too. Thus $\boldsymbol{R}^{(p)+}$ is a real partially symmetric vector, which implies (6.6a). Indeed, by (6.6)

$$B''_{s,k'} = \tilde{B}_{s,k'} + \tilde{B}_{s,p-k'} = \varepsilon_p^{-sk'} R_{k'}^{(p)+} + \varepsilon_p^{-s(p-k')} R_{p-k'}^{(p)+} =$$

$$= 2\cos\frac{2\pi sk'}{p}\, R_{k'}^{(p)+} \quad \text{for} \quad k' = 1(1)\,[(p-1)/2]$$

and the cases $k' = 0$ and $k' = p/2$ (p even) follow directly from (6.6).

Hence we see that $\boldsymbol{r}^\alpha$ is partially symmetric because $B''_{p-s,k'} = B''_{s,k'}$. $\square$

6b. Fully symmetrical p-periodic data $\boldsymbol{r}$

6.4 Definition. *The full symmetry of a p-periodic data vector* $\boldsymbol{r}$ *means that*

$$(6.10)\qquad \begin{cases} p = 2m\,, \\ r_q = -r_{m-q} \quad \text{for} \quad q = 0(1)\,m\,, \\ r_q = r_{p-q(\text{mod}\,p)}\,. \quad \square \end{cases}$$

Formulas (6.8) and (6.10) yield

$$r_m = \max(r_0, \ldots, r_{p-1}) = \max(r_0, \ldots, r_m) \quad \text{and} \quad r_0 \leqq 0\,.$$

Indeed, $r_0 > 0$ and $r_0 \leqq r_m$ imply $-r_0 = r_m \geqq r_0$, a contradiction. We may restrict ourselves to the case $r_0 < 0$ (omitting the trivial case $r_0 = 0$). Thus $r_m > 0$ will be assumed in all what follows. As the reduction $w_s = r_s^\alpha / r_s$ does not depend on the data scale, without loss of generality we may suppose $r_0 = -1$, $r_m = +1$.

6.5 Theorem. *Let* $\boldsymbol{r}$ *be a fully symmetrical p-periodic data vector. Assume that* $r_0 = -1$ *and* $r_m = 1$; *define* $r_{m/2} = 0$ *for* m *odd and put* $B'''_{s,l} = \frac{1}{2} B''_{s,2l+1}$, $l = 0(1)\,[(m-1)/2]$. *Then*

$$(6.5\text{b})\qquad r_s^\alpha = \frac{1}{m} \sum_{l=0}^{[(m-1)/2]} \frac{B'''_{s,l}}{1 + \alpha |\lambda_{(2l+1)\pi/2m}|^2}\,, \quad s = 0(1)\,p - 1\,,$$

where

$$(6.6\text{b})\qquad \begin{cases} B'''_{s,l} = \cos \dfrac{\pi s(2l+1)}{m} R^{(p)+}_{2l+1} = \\ \qquad = 2 \cos \dfrac{\pi s(2l+1)}{m} \left(-1 + 2 \displaystyle\sum_{q=1}^{[(m-1)/2]} r_q \cos \dfrac{\pi q(2l+1)}{m} \right), \\ l = 0(1)\,[m/2] - 1\,, \\ B'''_{s,(m-1)/2} = \frac{1}{2}(-1)^s R^{(p)+}_m = (-1)^s \left(-1 + 2 \displaystyle\sum_{q=1}^{(m-1)/2} (-1)^q r_q\right) \textit{ for } m \textit{ odd}; \end{cases}$$

$$(6.7\text{b})\qquad w_s = \frac{r_s^\alpha}{r_s} = \frac{1}{m r_s} \sum_{l=0}^{[(m-1)/2]} \frac{B'''_{s,l}}{1 + \alpha |\lambda_{(2l+1)\pi/2m}|^2}\,, \quad s = 0(1)\,p - 1 \; (r_s \neq 0)\,.$$

The vector $\boldsymbol{r}^\alpha$ is fully symmetric and p-periodic. Moreover,

$$w_{m-s} = w_s\,, \quad s = 0(1)\,p - 1 \quad \text{and if } m \text{ is even}\,, \quad r_{m/2} = 0\,.$$

So we only need to evaluate the formula (6.7b) for $s = 0(1)\,[(m-1)/2]$.

Proof. By (6.10) $[(p-1)/2] = m - 1$ and $r_{m/2} = 0$ for m even. Further (6.9) is reduced to the form

$$R^{(p)+}_{k'} = -1 + 2 \sum_{q=1}^{[(m-1)/2]} \left(r_q \cos \frac{\pi k' q}{m} + r_{m-q} \cos \frac{\pi k'(m-q)}{m} + r_{m/2} \right) + (-1)^{k'}\,,$$

$$k' = 0(1)\,m\,,$$

whence with respect to $r_{m-q} = -r_q$ and $r_{m/2} = 0$ it follows for k' odd or even, respectively

$$R^{(p)+}_{2l+1} = 4 \sum_{q=1}^{[(m-1)/2]} r_q \cos \frac{\pi q(2l+1)}{m} - 2 \quad \text{for} \quad l = 0(1)\,[(m-1)/2]$$

$$R^{(p)+}_{2l} = 0 \quad \text{for} \quad l = 0(1)\,[m/2]\,.$$

Then the formulas (6.5a)–(6.7a) reduce to (6.5b)–(6.7b).

Further, for every l there evidently holds $B'''_{m-s,l} = -B'''_{s,l}$, whence $r^{\alpha}_{m-s} = -r^{\alpha}_{s}$, which means that the smoothed data is fully symmetric. Thus the reduction

$$w_{m-s} = \frac{r^{\alpha}_{m-s}}{r_{m-s}} = \frac{-r^{\alpha}_{s}}{-r_s} = w_s \quad (\text{for } r_s \neq 0)$$

is the same for the positive and negative part of the halfwave. □

6.6 Definition. *The number* $|r_0| = r_m$ *is called an amplitude of a a fully symmetric p-periodic data vector* $\boldsymbol{r}$. □

This legitimates us to define the notion of a reduction of the amplitude by $w_0 = w_m$.

According to our convention concerning the data scaling we may suppose (without loss of generality) the unit amplitude of $\boldsymbol{r}$.

6.7 Corollary. *Let* $\boldsymbol{r}$ *be a fully symmetrical p-periodic data vector. Then*

$$w_0 = w_m = \frac{1}{m} \sum_{l=0}^{[(m-1)/2]} \frac{A_l}{1 + \alpha|\lambda_{(2l+1)n/2m}|^2}\,,$$

where

$$A_l = -R^{(p)+}_{2l+1} = 2\left(1 - 2 \sum_{q=1}^{[(m-1)/2]} r_q \cos \frac{\pi q(2l+1)}{m}\right),$$

$$l = 0(1)\,[m/2] - 1\,,$$

$$A_{(m-1)/2} = -\tfrac{1}{2}R^{(p)+}_{m} = 1 - 2 \sum_{q=1}^{(m-1)/2} (-1)^q\, r_q \quad \text{for} \quad m \text{ odd.}$$

For some special periodic data the $R^{(p)}_{k'}$ may be easily evaluated. This will be demonstrated by several examples such as the sinusoidal wave, for which a detailed derivation is given, and others where only the results are presented.

III. EXAMPLES

7a. Sinusoidal wave

A sample of a sinusoidal wave on an equidistant mesh $\Delta = \{th\}_{t=0}^{n-1}$ is the vector

$$(7.1) \qquad \boldsymbol{r} = \left\{-\cos \frac{2\pi th}{P}\right\}_{t=0}^{n-1} = \left\{-\cos \frac{2\pi t}{p}\right\}_{t=0}^{n-1}$$

with $P = ph$ – the period, p an integer $\geqq 2$ and divisor of n.

7.1 Theorem. *For the sinusoidal wave with the sample vector* (7.1) *we have*

$$(7.1a)\qquad \begin{aligned} r_s^\alpha &= \frac{r_s}{1+\alpha|\lambda_{n/p}|^2}\,, \\ w_s &= \frac{1}{1+\alpha|\lambda_{n/p}|^2}\,, \quad (r_s \neq 0)\,, \end{aligned} \qquad s = 0(1)\,[p/2]\,.$$

Thus the reduction w_s does not depend on s.

Proof. Clearly, the vector $\boldsymbol{r}$ is partially symmetric with the basic *period* $\boldsymbol{r}^{(p)} = \{-\cos 2\pi t/p\}_{t=0}^{p-1}$. We have

$$\boldsymbol{r}^{(p)} = -\tfrac{1}{2}\{e^{-2\pi i t/p}\}_{t=0}^{p-1} - \tfrac{1}{2}\{e^{-2\pi i(p-1)t/p}\}_{t=0}^{p-1}\,,$$

which means that $\boldsymbol{r}^{(p)}$ is a linear combination of the second and the last column of the matrix $W_p^- = (\varepsilon_p^{-jk})_{j,k=0}^{p-1}$. By Lemma 2.3

$$\boldsymbol{R}^{(p)+} = W_p^+\boldsymbol{r}^{(p)} = \begin{cases} (0, -p/2, \overbrace{0, \ldots, 0}^{(p-3)\text{times}}, -p/2)^\mathsf{T} & \text{for } p > 2\,, \\ (0, -p)^\mathsf{T} = (0, -2)^\mathsf{T} & \text{for } p = 2\,, \end{cases}$$

$$R_{k'}^{(p)+} = \begin{cases} -p/2 & \text{for } k' = 1 \text{ and } p > 2\,, \\ -p & \text{for } k' = 1 \text{ and } p = 2\,, \end{cases}$$

$$R_{k'}^{(p)+} = 0 \quad \text{for} \quad k' = 0(1)\,[p/2]\,, \quad k' \neq 1\,.$$

Then (7.1a) is obtained from (6.5a)–(6.7a) after subsutituting the above derived $\boldsymbol{R}^{(p)+}$ into (6.6a). □

7.2. Note. As we have seen, the relations (3.1) and (3.2) are equivalent to (2.8) and (2.10). In a similar way, the relations (6.5) and (6.6) are equivalent to

$$(7.2)\qquad \boldsymbol{r}^{(p)\alpha} = \tilde{H}^{(p)}\boldsymbol{r}^{(p)}\,, \quad \tilde{H}^{(p)} = (1/p)\,W_p^- H^{(p)} W_p^+$$

where $\tilde{H}^{(p)}$ is a circulant matrix with the generating vector $\boldsymbol{h}^{(p)} = (1/p)\,W_p^-\boldsymbol{H}^{(p)}$, $W_p^\pm = (\varepsilon_p^{\pm jk})_{j,k=0}^{p-1}$, $\boldsymbol{H}^{(p)}$ is a vector the components of which are $H_{k'}^{(p)} = H_{k'n/p} = 1/(1+\alpha|\lambda_{k'n/p}|^2)$ and $H^{(p)} = \operatorname{diag}(H_0^{(p)}, \ldots, H_{p-1}^{(p)})$; the property $H_{k'}^{(p)} = H_{p-k'(\mathrm{mod}\,p)}^{(p)}$ is an evident consequence of $H_k = H_{n-k(\mathrm{mod}\,n)}$.

Now, the result of Theorem 7.1 may be obtained from (7.2) because $H_1^{(p)} = H_{p-1}^{(p)}$ is the reduction of the base vectors $\{\varepsilon_p^{-t}\}_{p=0}^{p-1}$ and $\{\varepsilon_p^{-(p-1)t} = \varepsilon_p^t\}_{t=0}^{p-1}$ and consequently also of (7.1). In more detail, the equations

$$W_p^- H^{(p)} = W_p^- H^{(p)} W_p^+ (1/p)\,W_p^- \quad \text{and} \quad W^{\mp\alpha} = W_p^- H^{(p)} W_p^+ (1/p)\,W^\mp\,,$$

where $W^- = \{\varepsilon_p^{-t}\}_{t=0}^{p-1}$ and $W^+ = \{\varepsilon_p^t = \varepsilon_p^{-(p-1)t}\}_{t=0}^{p-1}$ are the 2nd and the last column of W_p^-, respectively, say that the respective $W^{\mp\alpha}$ is the 2nd and the last column of $W_p^- H^{(p)}$ which is equal to $H_1^{(p)}W^-$ or $H_{p-1}^{(p)}W^+$, respectively. Then, in view of $H_{p-1}^{(p)} = H_1^{(p)} = 1/(1+\alpha|\lambda_{n/p}|^2)$ and $\boldsymbol{r}^{(p)} = -\tfrac{1}{2}(W^- + W^+)$, we get the desired formula $\boldsymbol{r}^{(p)\alpha} = 1/(1+\alpha|\lambda_{n/p}|^2)\,\boldsymbol{r}^{(p)}$.

7b. Saw-like wave

The basic period of the saw-like wave is given by the vector

$$r^{(p)} = \{1 - |q - m|\, 2/m\}_{q=0}^{2m-1}$$

where n is divisible by p, $p = 2m$, $m \geqq 1$. This data vector is p-periodic and fully symmetric.

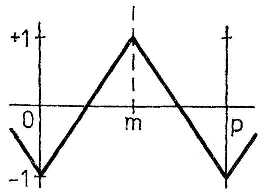

Fig. 1. Saw-like wave.

7.3 Theorem. *For the saw-like wave we have*

$$\text{(7.1b)}\qquad \begin{cases} r_s^\alpha = \dfrac{1}{m} \displaystyle\sum_{l=0}^{[(m-1)/2]} \dfrac{B'''_{s,l}}{1 + \alpha|\lambda_{(2l+1)n/2m}|^2}, & s = 0(1)\,2m - 1\,, \\ B'''_{s,l} = -\dfrac{2}{m} \cos \dfrac{\pi s(2l+1)}{m} \Big/ \sin^2 \dfrac{\pi(2l+1)}{2m} & \text{for } l = 0(1)\,[m/2] - 1\,, \\ B'''_{s,(m-1)/2} = \dfrac{1}{m}\,(-1)^{s+1} & \text{for } m \text{ odd.} \end{cases}$$

For the reduction of the amplitude we obtain

$$w_0 = w_m = \frac{1}{m} \sum_{l=0}^{[(m-1)/2]} \frac{A_l}{1 + \alpha|\lambda_{(2l+1)n/2m}|^2}\,,$$

$$A_l = \frac{2}{m \sin^2 (2l+1)\, \pi/2m}\,, \quad l = 0(1)\,[m/2] - 1\,,$$

$$A_{(m-1)/2} = \frac{1}{m} \quad \textit{for } m \textit{ odd.} \quad \square$$

In particular, for $p = 2\,(m = 1)$ we have

$$w_0 = w_1 = \frac{1}{1 + \alpha|\lambda_{n/2}|^2}\,,$$

which is the same result as in (7.1a) (for $p = 2$ the samples of the sinusoidal and saw-like waves are identical).

7c. Rectangular pulse train

Figure 2 shows the graph of the function representing a sequence of rectangular pulses for m odd. From these functions the following p-periodic fully symmetric discrete function will be deduced:

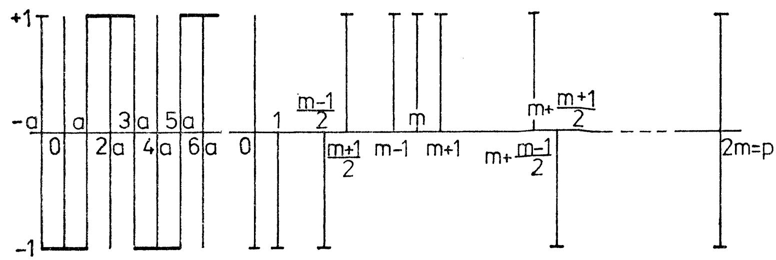

Fig. 2. Rectangular pulse train.

$$
\begin{aligned}
&p = 2m,\ m \geqq 1\,,\\
&r_q = r_{p-q(\bmod p)} = -1, \quad q = 0(1)\,[(m-1)/2]\,,\\
&r_q = r_{p-q(\bmod p)} = +1, \quad q = [m/2] + 1(1)\,m\,,\\
&r_{m/2} = 0 \quad \text{for} \quad m \quad \text{even}\,.
\end{aligned}
$$

7.4 Theorem. *For the rectangular pulse train we have*

$$
(7.1\text{c}) \qquad \begin{cases}
r_s^\alpha = \dfrac{1}{m} \displaystyle\sum_{l=0}^{[(m-1)/2]} \dfrac{B'''_{s,l}}{1 + \alpha\left|\lambda_{(2l+1)n/2m}\right|^2}, \quad s = 0(1)\,p-1\,, & \\
\left.\begin{array}{l} B'''_{s,l} = 2(-1)^{l+1} \dfrac{\cos \pi s(2l+1)/m}{\sin \pi(2l+1)/2m}, \\ B'''_{s,(m-1)/2} = \begin{cases} (-1)^{s+1} & \textit{for } (m-1)/2 \textit{ even} \\ (-1)^s & \textit{for } (m-1)/2 \textit{ odd} \end{cases} \end{array}\right\} & l = 0(1)\,(m-1)/2 \ \textit{for } m \textit{ odd}, \\
B'''_{s,l} = 2(-1)^{l+1} \cos(\pi s(2l+1)/m)\, \operatorname{ctg}(\pi(2l+1)/2m)\,, & \\
l = 0(1)\,[(m-1)/2] \quad \textit{for } m \textit{ even}. \quad \square &
\end{cases}
$$

7d. and e. Other two examples

The p-periodic curve e) results from the curve d) by shifting and multiplying: $r_s(\text{e}) = 2(r_s(d) - 1/2)$. These curves are not symmetrical. The result for the curve d) is

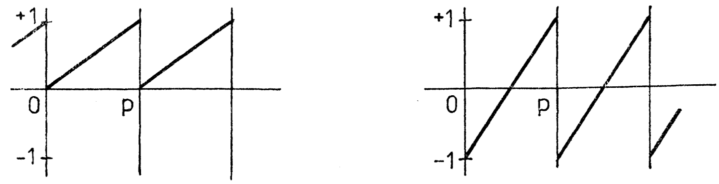

Fig. 3. Cases d) and e).

$$r_s^\alpha(d) = \frac{p-1}{2} + \frac{1}{p}\sum_{k'=1}^{[p/2]} \frac{B'_{s,k'}}{1+\alpha|\lambda_{k'n/p}|^2}, \quad s = 0(1)\,p-1,$$

(7.1d) $$B'_{s,k'} = -\frac{\sin \pi k'(2s+1)/p}{\sin \pi k'/p}, \quad k' = 1(1)\,[(p-1)/2],$$

$$B'_{s,p/2} = \tfrac{1}{2}(-1)^{s+1} \quad \textit{for} \quad p \quad \textit{even};$$

and for e):

(7.1e) $$r_s^\alpha(\mathrm{e}) = p - 1 + \frac{2}{p}\sum_{k'=1}^{[p/2]} \frac{B'_{s,k'}}{1+\alpha|\lambda_{k'n/p}|^2} - \frac{1}{1+\alpha|\lambda_0|^2},$$

$$s = 0(1)\,p-1.$$

Note. Problems similar to those of the present paper are dealt with also by Gautschi [8], Locher [15] and in a more general form by Gutknecht [9]. The first two authors deal with the periodic interpolation problem in one dimension, the third author has generalized their results admitting any linear and translation invariant operator (not only an interpolating one) in several dimensions. In a certain sense our Theorem 2.5 may be viewed as a special case of Gutknecht's result which, when applied to the DS-spline, says that the transfer function coefficients $\boldsymbol{H}$ are to be computed as DFT of the DS-smoothed unit vector $(1, 0. \ldots, 0)^{\mathrm{T}}$. Thus Gutknecht's result implies only that the DS-smoothed data are obtained by discrete convolution. Our Theorem 2.5 is stronger in that it moreover provides explicit formulas for $\boldsymbol{H}$. Using Gutknecht's result the proof of the above mentioned Theorem 2.5 might be modified by considering $\boldsymbol{r} = (1, 0, \ldots, 0)^{\mathrm{T}}$ only.

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Souhrn

DISKRÉTNÍ VYHLAZOVACÍ SPLAJNY A ČÍSLICOVÁ FILTRACE. TEORIE A APLIKACE

JIŘÍ HŘEBÍČEK, FRANTIŠEK ŠIK, VÍTĚZSLAV VESELÝ

Hlavní výsledky práce: důkaz existence úzké souvislosti mezi vyhlazovacími splajny a digitálními konvolučními filtry (Cor. 2.6), vytvoření přenosové funkce, která charakterizuje vyhlazovací splajn jako filtr pomocí vyhlazovacího parametru a vlastních hodnot diskrétního analogu operátoru $\mathscr{L}$ (Th. 2.5) a určení podílu mezi původními a vyhlazenými daty (Th. 3.1).

Резюме

ДИСКРЕТНЫЕ СГЛАЖИВАЮЩИЕ СПЛАЙНЫ И ЦИФРОВАЯ ФИЛЬТРАЦИЯ. ТЕОРИЯ И ПРИМЕНЕНИЯ

Jiří Hřebíček, František Šik, Vítězslav Veselý

Главные результаты работы: Доказательство существования тесной связи между сглаживающими сплайнами и цифровыми конволюционными фильтрами (Cor. 2.6), создание передаточной функции, которая характеризует сглаживающий сплайн как фильтр с помощью сглаживающего параметра и собственных значений дискретного аналога оператора $\mathscr{L}$ (Th. 2.5), и установление отношения между первоначальными и сглаженными данными (Th. 3.1).

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