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AN INVESTIGATION OF CONVERGENCE AND ERROR ESTIMATE OF APPROXIMATE SOLUTION FOR A QUASILINEAR PARABOLIC INTEGRODIFFERENTIAL EQUATION

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Summary: One parabolic integrodifferential problem in the abstract real Hilbert spaces is studied in this paper. The semidiscrete and full discrete approximate solution is defined and the error estimate of Rothe's function in some function spaces is established.

Keywords: Rothe's method, Galerkin's method, error estimates.

AMS classification: 65M15, 65M20, 35K22.

1. INTRODUCTION

The method of lines (the Rothe method) is a convenient tool for the theoretical analysis of the evolution equations (see [4], [5], [9], [10], [12], [13], etc.), solution of which is reduced to the solution of the corresponding elliptic problems. This fact can be exploited in the numerical analysis, too.

A problem similar to the PC-1 is studied in [4], including the proofs of existence, uniqueness and regularity of solution. The rate of convergence $O(\Delta t^{1/2})$ of Rothe's function in the space $C(J, H)$ and $L_2(J, Y)$ is established. Under certain assumptions on the approximation of Y by the finite dimensional subspaces Y_h, the convergence of the full discrete approximate solution in $C(J, H)$ and $C(J, Y)$ is proved.

The convergence of the semidiscrete PC-1 solution in $C(J, H)$, $C(J, Y)$ is obtained analogously as in [4]. The rate of convergence in $C(J, H)$, $L_2(J, Y)$ and $C(J, Y)$ is respectively $O(\Delta t)$, $O(\Delta t)$ and $O(\Delta t^{1/2})$, while for $\partial_t u$ in $L_2(J, H)$ it is $O(\Delta^{1/2})$. Similar results hold for the full discretization (in time and in space), but the error estimates depend on the approximation of Y by Y_h.

For another approach to numerical analysis of parabolic equations we refer the reader for example to [2], [3], [6], [10], [11], [14]–[17] and to the references given therein.

2. NOTATION AND PRELIMINARIES

Let H, H_1, Y, Y_1 be real Hilbert spaces with norms $|\cdot|$, $|\cdot|_1$, $\|\cdot\|$, $\|\cdot\|_1$, where $H \cap Y$ is dense in H and Y. Let $\langle z, w\rangle_{H'}$ $\langle u, v\rangle_Y$ be the continuous pairings for $z \in H_1$, $w \in H$, $u \in Y_1$, $v \in Y$. The interval $\langle 0, T\rangle$ is denoted by J. In the following we work in the function spaces of the types $C(J, X)$, $L_\infty(J, X)$, $L_2(J, X)$, where X is a Banach space, the basic properties of which can be found in [7]. By $\rightarrow$ we denote the strong convergence.

If X, Y are Banach spaces, $\alpha \in (0, 1\rangle$ then:

– By $Lip_\alpha(X, Y)$ we denote the set of all functions $g: X \rightarrow Y$ satisfying

$$\|g(u) - g(v)\|_Y \leqq C\|u - v\|_X^\alpha \quad \forall u, v \in X .$$

For $\alpha = 1$ the notation $Lip(X, Y) \equiv Lip_1(X, Y)$ will be used.

– By $Lip(J \times X, Y)$ we denote the set of all functions $g: J \times X \rightarrow Y$ satisfying

$$\|g(t, u) - g(t', v)\|_Y \leqq C(|t - t'| + |t - t'|\ \|u\|_X + \|u - v\|_X)$$

$\forall t, t' \in J$; $\forall u, v \in X$.

Definition 2.1. *The operator* $E: L_\infty(J, H) \rightarrow L_\infty(J, H)$ *is said to be a Volterra operator iff*

$$(u(s) = v(s) \quad \textit{for a.e.} \quad s \in \langle 0, t\rangle, t \in J) \Rightarrow$$
$$(E(u)(s) = E(v)(s) \quad \textit{for a.e.} \quad s \in \langle 0, t\rangle) .$$

Let us fix $f \in Lip(J \times H \times Y, Y_1)$, $e \in Lip(J \times Y \times H \times Y, H_1)$ and continuous bilinear forms $p(t; u, v)$, $a(t; x, y)$ for u, $v \in H$ and $x, y \in Y$. Let $E: Lip(J, H) \rightarrow \rightarrow Lip(J, H)$ be a Volterra operator and let $G: L_\infty(J, Y) \rightarrow L_\infty(J, Y)$ be in the form

$$G(z)(t) = \textstyle\int_0^t K(t, s)\, z(s)\, \mathrm{d}s \quad \partial_t K, K \in L_\infty(J \times J) . \tag{2.2}$$

We consider the following problem:

PC-1. *To find u such that*

(i) $u \in Lip_{1/2}(J, Y) \cap Lip(J, H)$

$\partial_t u \in L_2(J, Y) \cap L_\infty(J, H)$

(ii) $u(0) = \alpha \in Y \cap H$

(iii) (2.3) *is satisfied*

(2.3) $$p(t; \partial_t u, v) + a(t; u, v) = \langle f(t, E(u), G(u)), v\rangle_Y + \langle e(t, u, E(u), G(u)), v\rangle_H$$

$\forall v \in H \cap Y$, *for a.e.* $t \in J$.

The following assumptions on p, a, E ($\forall t \in J; \forall x, y \in Y; \forall u, v \in H$; C and C_1 are positive constants) are sufficient for our approach:

(2.4) $$p(t; u, v) = p(t; v, u)$$

(2.5) $$p(t; u, u) \geqq C_1|u|^2$$

(2.6) $$|p^{(1)}(t; u, v)| \leqq C|u|\,|v|$$

(2.7) $$|a^{(1)}(t; x, y)| \leqq C\|x\|\,\|y\|$$

(2.8) $$a(t; x, x) \geqq C_1\|x\|^2 - C|x|^2$$

(2.9) $$|E(z)(t) - E(z)(t')| \leqq \leqq |t - t'|\,\Theta(\|z\|_{C(\langle 0,t\rangle, H)})\,(1 + \|\partial_t z\|_{L_\infty(\langle 0,t\rangle, H)})$$
$$\forall t, t' \in J,\ t' < t,\ \Theta \in C(R_+, R_+), \quad \forall z \in Lip(J, H)$$

(2.10) $\exists U_1 \in H$ such that
$$p(0; U_1, w) + a(0, \alpha, w) = \langle f(0, E(\alpha)(0), 0), w\rangle_Y + \langle e(0, \alpha, E(\alpha)(0), 0), w\rangle_H$$
for all $w \in Y \cap H$

(where $r^{(1)}(t; \xi, \eta) = \partial_t r(t; \xi, \eta)$).

Remark 2.11. C, ε, C_ε denote generic positive constants which do not depend on n, h and which are note necessarily the same at any two places (ε is a small constant and $C_\varepsilon = C(\varepsilon^{-1})$).

The interval J is divided into n subintervals $\langle t_{i-1}, t_i\rangle$, $i = 1, \dots, n$, where $t_i = i\tau$ and $\tau = T/n$. For a given function $w(t)$ we introduce the notation
$$w_i = w(t_i)\,, \quad \delta w_i = (w_i - w_{i-1})/\tau \quad \text{for} \quad i = 1, \dots, n\,.$$

The following functions will be often used ($i = 1, \dots, n$):

(2.12) $$u_n(t) = \begin{cases} \alpha & t = 0 \\ u_{i-1} + (t - t_{i-1})\,\delta u_i & t \in (t_{i-1}, t_i\rangle\,, \end{cases}$$

(2.13) $$\bar{u}_n(t) = \begin{cases} \alpha & t = 0 \\ u_i & t \in (t_{i-1}, t_i\rangle\,, \end{cases}$$

(2.14) $$\tilde{u}_{i-1} = \tilde{u}_{i-1,n}(t) = \begin{cases} \alpha & t \in \langle 0, \tau\rangle \\ u_{j-1} + (t - t_j)\,\delta u_j & t \in \langle t_j, t_{j+1}\rangle\,, \\ u_{i-1} & t \in \langle t_i, T\rangle\,, \end{cases}$$

where $j = 1, \dots, i - 1$; ($\tilde{u}_n \equiv \tilde{u}_{n-1,n}$)

$$(2.15) \qquad R_n(\tilde{u}_n)(t) = R(\tilde{u}_n)(t_i) \quad t \in (t_{i-1}, t_i\rangle\,; \quad R = E, G\,,$$

$$(2.16) \qquad r_n(t, \xi) = r(t_i, \xi) \quad t \in (t_{i-1}, t_i\rangle\,; \quad r = p, a, e, f\,.$$

The Gronwall lemma will be used in the following form:

Lemma 2.17. *Let $r(t)$, $h(t)$, $y(t)$ be real integrable functions on the interval $I = \langle a, b\rangle$, $r(t) \geqq 0$.*

(i) *If*

$$y(t) \leqq h(t) + \int_a^t r(s)\, y(s)\, \mathrm{d}s \quad \forall t \in I$$

then

$$y(t) \leqq h(t) + \int_a^t h(s)\, r(s) \exp\left(\int_s^t r(x)\, \mathrm{d}x\right) \mathrm{d}s \quad \forall t \in I\,.$$

In particular if $r(s) \equiv C$ and $h(s)$ is nondecreasing, then

$$y(t) \leqq h(t)\, \mathrm{e}^{C(t-a)} \quad \forall t \in I\,.$$

(ii) *If $y(t) \geqq 0$ and*

$$y(t) \leqq h(t) + \int_a^t r(s)\, y(s)\, \mathrm{d}s \quad \text{for a.e.} \quad t \in I$$

then

$$y(t) \leqq h(t) + \int_a^t h(s)\, r(s) \exp\left(\int_s^t r(x)\, \mathrm{d}x\right) \mathrm{d}s \quad \text{for a.e.} \quad t \in I\,.$$

In particular, if $r(s) \equiv C$, $h(s)$ is nondecreasing, then

$$y(t) \leqq h(t)\, \mathrm{e}^{C(t-a)} \quad \text{for a.e.} \quad t \in I\,.$$

Proof. (i) The proof is the same as in [5, Lemma 1. 3. 19].

(ii) Let

$$y(t) \leqq h(t) + \int_a^t r(s)\, y(s)\, \mathrm{d}s \quad \forall t \in I - A\,, \quad \operatorname{meas} A = 0\,.$$

Then

$$y(t)\, \chi_{I-A}(t) \leqq h(t)\, \chi_{I-A}(t) + \chi_{I-A}(t) \int_a^t r(s)\, y(s)\, \mathrm{d}s \leqq$$
$$\leqq h(t)\, \chi_{I-A}(t) + \int_a^t r(s)\, y(s)\, \chi_{I-A}(s)\, \mathrm{d}s \quad \forall t \in I\,,$$

where $\chi_E(t)$ is the characteristic function of E. The assertion (i) implies (ii).

3. SEMIDISCRETIZATION IN TIME

Let us consider this semidiscrete problem:

PD-1. *To find $u_i \in Y \cap H$ $(i = 1, \ldots, n)$ such that*

(ii) $u_0 = \alpha \in Y \cap H$

(ii) (3.1) *holds*

$$(3.1) \qquad p(t_i; \delta u_i, v) + a(t_i; u_i, v) = \langle f(t_i, E(\tilde{u}_{i-1})(t_i), G(\tilde{u}_{i-1})(t_i)), v\rangle_Y +$$
$$+ \langle e(t_i, u_{i-1}, E(\tilde{u}_{i-1})(t_i), G(\tilde{u}_{i-1})(t_i)), v\rangle_H \quad \text{for all} \quad v \in Y \cap H\,.$$

The Rothe function u_n (see (2.12)) is constructed by using the solution of PD-1. The following a priori estimates hold for all $n \in N$ and $\tau \leqq \tau_0$:

$$|\partial_t u_n(t)| \leqq C \quad \text{for a.e.} \quad t \in J \tag{3.2}$$

$$\|\partial_t u_n\|_{L_2(J,Y)} + \tau^{1/2}\|\partial_t u_n\|_{L_\infty(J,Y)} \leqq C \tag{3.3}$$

$$\|u_n - \bar{u}_n\|_{L_2(J,Y)} + \|u_n - \tilde{u}_n\|_{L_2(J,Y)} \leqq C\tau \tag{3.4}$$

$$|u_n(t) - \bar{u}_n(t)| + |u_n(t) - \tilde{u}_n(t)| \leqq C\tau \quad \forall t \in J \tag{3.5}$$

$$\|u_n(t) - \bar{u}_n(t)\| \leqq C\tau^{1/2} \qquad \forall t \in J \tag{3.6}$$

$$|u_n(t) - u_n(t')| \leqq C|t - t'|, \quad \|u_n(t) - u_n(t')\| \leqq C|t - t'|^{1/2} \quad \forall t, t' \in J. \tag{3.7}$$

Using (3.2)–(3.7) we can prove that

$$u_n \to u \quad \text{in} \quad C(J, H) \cap C(J, Y)$$

where u is a solution of PC-1.

In this way the proof of existence, uniqueness and continuous dependence on the right-hand side and on the initial function of the solution of PC-1 proceeds in [12, Th. 4.18, Th. 4.32]). Therefore we can write without proof:

Theorem 3.8. *Let $E \in Lip(C(J, H), C(J, H))$ and let (2.2), (2.4)–(2.10) be satisfied. Then there exists a unique solution of PC-1 which depends continuously on the right-hand side and on the initial function.*

Now, we can start with the derivation of the error estimates for u and $\partial_t u$ in some function spaces. The main tools will be the a priori estimates, a suitable choice of the test function, Young's and Hölder's inequalities.

Theorem 3.9. *Let the assumptions of Theorem 3.8 hold. Then for all $t \in J$ we have*

(i) $$\max_{\langle 0,t\rangle} |u_n - u|^2 + \int_0^t \|u_n - u\|^2 \, ds \leqq C\tau^2$$

(ii) $$\max_{\langle 0,t\rangle} \|u_n - u\|^2 \leqq C\tau$$

(iii) $$\int_0^t |\partial_s u_n - \partial_s u|^2 \, ds \leqq C\tau.$$

Proof: The relation (3.1) implies

$$p_n(t; \partial t\, u_n(t), v) + a_n(t; \bar{u}_n, v) = \langle f_n(t, E_n(\tilde{u}_n)(t), G_n(\tilde{u}_n)(t)), v\rangle_Y + \\ + \langle e_n(t, \bar{u}_n(t - \tau), E_n(\tilde{u}_n)(t), G_n(\tilde{u}_r)(t)), v\rangle_H \quad \text{for all} \quad v \in Y \cap H. \tag{3.10}$$

(i) Let us subtract (2.3) from (3.10) for $v = u_n - u$ and then integrate it over $(0, t)$. Using (3.2)–(3.7) we estimate

$$(3.11)\qquad \int_0^t [p(s; \partial_s(u_n - u), u_n - u) + a(s; u_n - u, u_n - u)]\, ds \leqq$$
$$\leqq \int_0^t [\langle f(s, E(u_n), G(u_n)) - f(s, E(u), G(u)), u_n - u\rangle_Y +$$
$$+ \langle e(s, u_n, E(u_n), G(u_n)) - e(s, u, E(u), G(u)), u_n - u\rangle_H]\, ds +$$
$$+ C_\varepsilon \tau^2 + \varepsilon \int_0^t [\|u_n - u\|^2 + |u_n - u|^2]\, ds\,.$$

From this (for sufficiently small ε) we deduce

$$(3.12)\qquad |u_n(t) - u(t)|^2 + \int_0^t \|u_n - u\|^2\, ds \leqq C\tau^2 +$$
$$+ C \int_0^t [\max_{\langle 0,s\rangle} |u_n - u|^2 + \int_0^s \|u_n - u\|^2\, d\xi]\, ds$$

and Gronwall's lemma implies (i).

(ii) Let us subtract (2.3) from (3.10) for $v = u_n - u$. Owing to (3.2)–(3.7) it is easy to see that

$$(3.13)\qquad a(t; u_n - u, u_n - u) \leqq C_\varepsilon \tau + \varepsilon[\|u_n - u\|^2 + |u_n - u|^2 -$$
$$- p(t; \partial_t(u_n - u), u_n - u) +$$
$$+ \langle f(t, E(u_n), G(u_n)) - f(t, E(u), G(u)), u_n - u\rangle_Y +$$
$$+ \langle e(t, u_n, E(u_n), G(u_n)) - e(t, u, E(u), G(u)), u_n - u\rangle_H \quad \text{for a.e.} \quad t \in J$$

and for sufficiently small ε we have

$$\|u_n(t) - u(t)\|^2 \leqq C[\tau + |u_n(t) - u(t)|^2 + \max_{\langle 0,t\rangle} |u_n - u|^2 + \int_0^t \|u_n - u\|^2\, ds]$$
$$\text{for a.e.} \quad t \in J\,.$$

Using (i) we obtain

$$(3.14)\qquad \|u_n(t) - u(t)\|^2 \leqq C\tau + C \int_0^t \|u_n - u\|^2\, ds \quad \text{for a.e.} \quad t \in J\,.$$

Both sides of (3.14) are continuous in t, thus (3.14) is satisfied for all $t \in J$ and Gronwall's lemma implies (ii).

(iii) Let us subtract (2.3) from (3.10) for $v = \partial_t(u_n - u)$ and then integrate it over $(0, t)$. Owing to (3.2)–(3.7) the following estimate can be obtained

$$(3.15)\qquad \int_0^t p(s; \partial_s(u_n - u), \partial_s(u_n - u))\, ds \leqq$$
$$\leqq \int_0^t [a(s; u - u_n, \partial_s(u_n - u)) + \varepsilon|\partial_s(u_n - u)|^2 +$$
$$+ \langle f(s, E(u_n), G(u_n)) - f(s, E(u), G(u)), \partial_s(u_n - u)\rangle_Y +$$
$$+ \langle e(s, u_n, E(u_n), G(u_n)) - e(s, u, E(u), G(u)), \partial_s(u_n - u)\rangle_H]\, ds + C_\varepsilon \tau$$

and further for sufficiently small ε

$$(3.16)\quad \int_0^t |\partial_s(u_n - u)|^2\, ds \leqq C[\tau + \|u_n - u\|_{C(\langle 0,t\rangle, H)} + \|u_n - u\|_{L_2(\langle 0,t\rangle, Y)}]\,.$$

The assertion (i) implies (iii).

Solving PC-1 we first discretize in time and then in space. Let Y_h be a subspace of Y for $h > 0$.

Suppose that for $\alpha_h \in Y_h \cap H (h > 0)$ we have

$$(4.1) \qquad |a(0; \alpha_h, v)| \leqq C|v| \quad \forall h > 0, \quad \forall v \in Y \cap H$$

$$(4.2) \qquad f(0, E(\alpha_h)(0), 0) = 0 \quad \forall h > 0.$$

It is easy to see that (4.1)–(4.2) imply (2.10) for $v \in Y_h \cap H$.

Let us consider the following discrete problem:

PD-2. *To find* $u_i^h \in Y_h \cap H$ $(i = 1, \ldots, n)$ *such that*

(i) $u_0^h = \alpha_h \in Y_h \cap H$

$\|\alpha_h\|_{Y \cap H} \leqq C$

(ii) (4.3) *is satisfied*

$$(4.3) \qquad p(t_i; \delta u_i^h, v) + a(t_i; u_i^h, v) = \langle f(t_i, E(\tilde{u}_{i-1}^h)(t_i), G(\tilde{u}_{i-1}^h)(t_i)), v \rangle_Y + \\ + \langle e(t_i, u_{i-1}^h, E(\tilde{u}_{i-1}^h)(t_i), G(\tilde{u}_{i-1}^h)(t_i)), v \rangle_H \quad \textit{for all} \quad v \in Y_h \cap H,$$

where δu_i^h, $\tilde{u}_i^h$ *are defined analogously to* δu_i, $\tilde{u}_i$.

$u_\sigma(t)$ $(\sigma = [\tau, h])$ denotes the full discrete Rothe function defined in terms of u_i^h and α_h (see (2.12)). The functions $\bar{u}_\sigma(t)$, $\tilde{u}_\sigma(t)$ are constructed similarly as $\bar{u}_n(t)$, $\tilde{u}_n(t)$.

In the same way as in Part 3 we deduce

$$(4.4) \qquad p_\tau(t; \partial_t u_\sigma(t), v) + a_\tau(t; \bar{u}_\sigma(t), v) = \langle f_\tau(t, E_\tau(\tilde{u}_\sigma)(t), G_\tau(\tilde{u}_\sigma)(t)), v \rangle_Y + \\ + \langle e_\tau(t, \bar{u}_\sigma(t - \tau), E_\tau(\tilde{u}_\sigma)(t), G_\tau(\tilde{u}_\sigma)(t)), v \rangle_H \quad \text{for all} \quad v \in Y_h \cap H,$$

where $r_\tau = r_n$ for $r = p, a, e, f, E, G$;

$$(4.5) \qquad |\partial_t u_\sigma(t)| \leqq C \quad \text{for a.e.} \quad t \in J$$

$$(4.6) \qquad \|\partial_t u_\sigma\|_{L_2(J,Y)} + \tau^{1/2} \|\partial_t u_\sigma\|_{L_\infty(J,Y)} \leqq C$$

$$(4.7) \qquad \|u_\sigma - \bar{u}_\sigma\|_{L_2(J,Y)} + \|u_\sigma - \tilde{u}_\sigma\|_{L_2(J,Y)} \leqq C\tau$$

$$(4.8) \qquad |u_\sigma(t) - \bar{u}_\sigma(t)| + |u_\sigma(t) - \tilde{u}_\sigma(t)| \leqq C\tau \quad \forall t \in J$$

$$(4.9) \qquad \|u_\sigma(t) - \bar{u}_\sigma(t)\| \leqq C\tau^{1/2} \quad \forall t \in J$$

$$(4.10) \qquad |u_\sigma(t) - u_\sigma(t')| \leqq C|t - t'|, \quad \|u_\sigma(t) - u_\sigma(t')\| \leqq C|t - t'|^{1/2} \quad \forall t, t' \in J$$

(the constant C is independent of σ).

Theorem 4.11. *Let the assumptions of Theorem* 3.8 *be fulfilled (except* (2.10)). *Moreover, let* (4.1), (4.2) *be satisfied. Then the following estimates hold for* $\sigma \leqq \sigma_0$

and $u_h \in Y_h \cap H$:

(i)
$$\max_{\langle 0,t\rangle} |u_\sigma - u|^2 + \int_0^t \|u_\sigma - u\|^2 \, ds \leqq C[\tau^2 + |\alpha - \alpha_h|^2 + \int_0^t (|u - u_h| + |u - u_h|^2 + \|u - u_h\|^2) \, ds] , \quad \forall t \in J$$

(ii)
$$\int_0^t |\partial_s u_\sigma - \partial_s u|^2 \, ds \leqq C[\tau + \max_{\langle 0,t\rangle} |u_\sigma - u| + \max_{\langle 0,t\rangle} |u_\sigma - u|^2 + \|u_\sigma - u\|_{L_2(\langle 0,t\rangle, Y)} + \int_0^t \|u_\sigma - u\|^2 \, ds + \int_0^t |\partial_s u - u_h|^2 \, ds + \|\partial_t u - u_h\|_{L_2(\langle 0,t\rangle, Y)} (\tau + \max_{\langle 0,t\rangle} |u_\sigma - u| + \|u_\sigma - u\|_{L_2(\langle 0,t\rangle, Y)})] := C\,\Theta(t) ,$$
$$\forall t \in J$$

(iii)
$$\|u(t) - u_\sigma(t)\|^2 \leqq C[\tau + |u(t) - u_h(t)| + |u(t) - u_h(t)|^2 + \|u(t) - u_h(t)\|^2 + \max_{\langle 0,t\rangle} |u_\sigma - u| + \max_{\langle 0,t\rangle} |u_\sigma - u|^2 + \int_0^t \|u_\sigma - u\|^2 \, ds]$$
$$\textit{for a.e.} \quad t \in J .$$

(iv) *If we suppose*

(4.12)
$$a(t; x, y) = a(t; y, x) \quad \forall t \in J ; \quad \forall x, y \in Y$$

then

$$\max_{\langle 0,t\rangle} \|u - u_\sigma\|^2 \leqq C(\Theta(t) + \|\alpha - \alpha_h\|^2) \quad \forall t \in J .$$

Proof. The assumptions of the theorem guarantee the existence and uniqueness of the solution of PC-1. Let u_h be a fixed element in $Y_h \cap H$.

(i) Let us subtract (2.3) from (4.4) for $v = u_\sigma - u_h$ and then integrate it over $(0, t)$. Owing to (4.5)–(4.10) we have

(4.13)
$$\int_0^t [p(s; \partial_s(u_\sigma - u), u_\sigma - u) + a(s; u_\sigma - u, u_\sigma - u)] \, ds \leqq$$
$$\leqq \int_0^t [\langle f(s, E(u_\sigma), G(u_\sigma)) - f(s, E(u), G(u)), u_\sigma - u_h\rangle_Y + \langle e(s, u_\sigma, E(u_\sigma), G(u_\sigma)) - e(s, u, E(u), G(u)), u_\sigma - u_h\rangle_H] \, ds +$$
$$+ \int_0^t [\varepsilon \|u - u_\sigma\|^2 + C_\varepsilon(|u - u_\sigma|^2 + |u - u_h|^2 + |u - u_h| + \|u - u_h\|^2)] \, ds + C_\varepsilon \tau^2$$

and for sufficiently small ε

(4.14)
$$|u(t) - u_\sigma(t)|^2 + \int_0^t \|u - u_\sigma\|^2 \, ds \leqq C[\tau^2 + |\alpha - \alpha_h|^2 + \int_0^t [|u - u_h| + |u - u_h|^2 + \|u - u_h\|^2] \, ds + \int_0^t [\max_{\langle 0,s\rangle} |u - u_\sigma|^2 + \int_0^s \|u - u_\sigma\|^2 \, d\xi] \, ds] .$$

Using Gronwall's lemma we obtain (i).

(ii) Let us subtract (2.3) from (4.4) for $v = \partial_t u_\sigma - u_h$ and then integrate it over $(0, t)$. Using (4.5)–(4.10) we get

(4.15)
$$\int_0^t [p(s; \partial_s(u_\sigma - u), \partial_s(u_\sigma - u)) + a(s; u_\sigma - u, \partial_s(u_\sigma - u))] \, ds \leqq$$

$$\leqq \int_0^t [\langle f(s, E(u_\sigma), G(u_\sigma)) - f(s, E(u), G(u)), \partial_s u_\sigma - u_h \rangle_Y +$$
$$+ \langle e(s, u_\sigma, E(u_\sigma), G(u_\sigma)) - e(s, u, E(u), G(u)), \partial_s u_\sigma - u_h \rangle_H] \, ds +$$
$$+ \varepsilon \int_0^t |\partial_s(u - u_\sigma)|^2 \, ds + C_\varepsilon[\tau + \int_0^t |\partial_s u - u_h|^2 \, ds +$$
$$+ (\tau + \|u - u_\sigma\|_{L_2(\langle 0,t \rangle, Y)}) \, \|\partial_t u - u_h\|_{L_2(\langle 0,t \rangle, Y)}]$$

and for sufficiently small ε we conclude that (ii) holds.

(iii) Subtracting (2.3) from (4.4) for $v = u_\sigma - u_h$ and using (4.5)–(4.10) we have

(4.16)
$$a(t; u_\sigma - u, u_\sigma - u) \leqq -p(t; \partial_t(u_\sigma - u), u_\sigma - u_h) +$$
$$+ \langle f(t, E(u_\sigma), G(u_\sigma)) - f(t, E(u), G(u)), u_\sigma - u_h \rangle_Y +$$
$$+ \langle e(t, u_\sigma, E(u_\sigma), G(u_\sigma)) - e(t, u, E(u), G(u)), u_\sigma - u_h \rangle_H +$$
$$+ \varepsilon \|u_\sigma - u\|^2 + C_\varepsilon[\tau + |u - u_h| + |u - u_h|^2 + \|u - u_h\|^2 + |u - u_\sigma|^2]$$
$$\text{for a.e. } t \in J .$$

It is easy to see that (4.16) yields (iii).

(iv) The assertion follows from (4.15) and from the fact that

$$\mathrm{d}/\mathrm{d}t \, (a(t; x, x)) = a^{(1)}(t; x, x) + 2a(t; \partial_t x, x) .$$

If we want to prove the convergence $u_\sigma \to u$ for $\sigma \to 0$ in some function spaces we can use Theorem 4.11 but this is not sufficient to complete the proof. That is why we have to impose some new assumptions on the approximation of Y by its subspaces Y_h:

(4.17) $\forall v \in Y \cap H \quad \exists v_h \in Y_h \cap H$ such that $v_h \to v$ in $Y \cap H$ for $h \to 0$,

(4.18) $\exists C > 0 \quad \forall w \in Y \cap H \quad \forall h > 0 \colon \|w - w_h\|_{Y \cap H} \leqq C\|w\|_{Y \cap H}$

where w_h is the approximation of w in the sense of (4.17).

These assumptions are satisfied for currently used approximations – see Examples 5.1, 5.3.

Collecting all our results, we are ready to establish the following important theorem.

Theorem 4.19. *Let the assumptions of Theorem* 4.11 *be fulfilled. Moreover, let* (4.17), (4.18) *be satisfied. Then* $\forall t \in J$

(i)
$$\max_{\langle 0,t \rangle} |u - u_\sigma|^2 + \max_{\langle 0,t \rangle} \|u - u_\sigma\|^2 \to 0 \quad \textit{for} \quad \sigma \to 0 ,$$

(ii)
$$\int_0^t |\partial_s(u - u_\sigma)|^2 \, ds \to 0 \quad \textit{for} \quad \sigma \to 0 .$$

Proof. In Theorem 4.11 (i), let u_h (α_h) be an approximation of u (α) in the sense of (4.17). Lebesgue dominated convergence theorem, (4.18) and the fact that $u \in L_2(J, Y \cap H)$ enables us to pass to the limit (of the inequality in Theorem 4.11 (i)) for $\sigma \to 0$. This leads to

(4.20)
$$\max_{\langle 0,t \rangle} |u_\sigma - u|^2 + \int_0^t \|u_\sigma - u\|^2 \, ds \to 0 \quad \text{for} \quad \sigma \to 0 .$$

From (4.20), Arzela-Ascoli theorem and

$$\partial_t u,\ \partial_t u_\sigma \in L_2(J,\ Y \cap H)$$

we obtain (i).

(ii) In Theorem 4.11 (ii), let u_h be an approximation of $\partial_t u$ in the sense of (4.17). From (4.18), Lebesgue dominated convergence theorem and $\partial_t u \in L_2(J, Y \cap H)$ we conclude (ii).

5. EXAMPLES

Example 5.1. Let G be a bounded convex domain in R^2 with polygonal boundary ∂G. Let $H = L_2(G)$, $Y = H_0^1(G)$ and let $\| \ \|_2$ denote the norm in $H^2(G)$. ($H^1(G)$, $H_0^1(G)$, $H^2(G)$ are Sobolev spaces.) We consider the standard approximation of Y by linear finite elements with regular triangulation of G.

The relation (4.17) follows from [1, Th. 3.2.3] and (4.18) from [1, Th. 3.1.4] or [8, Th. 6.8].

It is well known that

$$|u - u_h| + h\|u - u_h\| \leqq C\, h^2 \|u\|_2 \quad \forall u \in H^2(G)$$

$$|u - u_h| \leqq C\, h \|u\| \qquad \forall u \in H^1(G)\,.$$

The following result is a consequence of Theorems 4.11 and 4.19.

Theorem 5.2. *Let the assumptions of Theorem* 4.11 *be fulfilled. Then* $\forall t \in J$

(i)
$$\max_{\langle 0,t\rangle} |u - u_\sigma|^2 + \max_{\langle 0,t\rangle} \|u - u_\sigma\|^2 + \int_0^t |\partial_s(u - u_\sigma)|^2 \,\mathrm{d}s \to 0$$

for $\sigma \to 0$;

(ii) *if* $u \in L_2(J, H^2(G))$ *then*

$$\max_{\langle 0,t\rangle} |u_\sigma - u|^2 + \int_0^t \|u_\sigma - u\|^2 \,\mathrm{d}s = O(\tau^2 + h^2)$$

$$\int_0^t |\partial_s(u - u_\sigma)|^2 \,\mathrm{d}s = O(\tau + h)\,;$$

(iii) *if* $u \in C(J, H^2(G))$ *then*

$$\max_{\langle 0,t\rangle} \|u_\sigma - u\|^2 = O(\tau + h)\,;$$

(iv) *if* (4.12), $\alpha \in H^2(G)$ *and* $u \in L_2(J, H^2(G))$ *then*

$$\max_{\langle 0,t\rangle} \|u_\sigma - u\|^2 = O(\tau + h)\,.$$

Example 5.3. Let $X = Y \cap H$ be a real separable Hilbert space with a complete orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Then we have

$$\forall v \in X : v = \sum_{i=1}^{\infty} c_i e_i \quad \|v\|_X = \sum_{i=1}^{\infty} c_i^2\,.$$

For $v \in X$ we put

$$v_n = \sum_{i=1}^{n} c_i e_i \in X_n$$

where $X_n = \text{span}\{e_1, \ldots, e_n\}$.

It is easy to see that

$$v_n \to v \quad \text{in} \quad X \quad \text{for} \quad n \to \infty$$

$$\|v - v_n\|_X \leqq \|v\|_X + \|v_n\|_X \leqq 2\|v\|_X ,$$

i.e. the relations (4.17), (4.18) are satisfied.

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Súhrn

SKÚMANIE KONVERGENCIE I ODHADU CHYBY APROXIMATÍVNEHO RIEŠENIA KVÁZILINEÁRNEJ PARABOLICKEJ INTEGRODIFERENCIÁLNEJ ROVNICE

MARIÁN SLODIČKA

V článku je študovaný jeden parabolický integrodiferenciálny problém v reálnych abstraktných Hilbertových priestoroch. Je definované semidiskretizované i plne diskretizované riešenie a sú určené odhady chýb Rotheho funkcie v niektorých funkcionálnych priestoroch.

Резюме

ИССЛЕДОВАНИЕ СХОДИМОСТИ И ОЦЕНКИ ОШИБКИ ПРИБЛИЖЕННОГО РЕШЕНИЯ КВАЗИЛИНЕЙНОГО ИНТЕРГРОДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

MARIÁN SLODIČKA

В статье рассматривается одна параболическая интегродифференциальная задача в вещественных абстрактных гильбертовых пространствах. Введено определение полудискретного и вполнедискретного приближенных решений и даны оценки ошибок функции Роте в некоторых функциональных пространствах.

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