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A REMARK ON λ-REGULAR ORTHOMODULAR LATTICES

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Summary. A finite orthomodular lattice in which every maximal Boolean subalgebra (block) has the same cardinality k is called λ-regular, if each atom is a member of just λ blocks. We estimate the minimal number of blocks of λ-regular orthomodular lattices to be lower than or equal to λ^2 regardless of k.

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INTRODUCTION

The most powerful tool for constructions of finite orthomodular lattices (abbr. OMLs) are graphical methods – see [1, 2, 3, 5]. Utilization of these methods has brought forth also purely combinatorial problems. Significant classes are formed by OMLs in which every maximal Boolean subalgebra has the same cardinality. Following Köhler [4], we shall call such on OML λ-regular provided every atom is a member of just λ different blocks. It was proved in [4] that, for any cardinality of blocks and any natural number λ, a λ-regular OML always exists. The question of minimal cardinality of λ-regular OMLs was also formulated there, and, using the technique of Greechie diagrams [2], the minimal number of blocks, n, of a λ-regular OML with k atoms in every block was estimated by

$$n \leqq \lambda k^{k(\lambda-1)} .$$

In this paper we strengthen this estimate – we show that $n \leqq \lambda^2$ for any $k \geqq 4$.

NOTIONS. RESULTS

Let $\mathscr{B}$ be a family of Boolean algebras. We denote $[0, a]_B = \{b \in B \mid b \leqq a\}$ for $B \in \mathscr{B}$ and $a \in B$. The n-cycle in $\mathscr{B}$ is a sequence $((B_0, b_0), (B_1, b_1), \ldots, (B_{n-1}, b_{n-1}))$

of (not necessarily distinct) algebras $B_i \in \mathscr{B}$ and (not necessarily distinct) elements $b_i \in B_i \cap B_{i+1}$, $b_i \neq 0$, such that $b_i \leqq b'_{i+1}$, $[0, b_i]_{B_i} = [0, b_i]_{B_{i+1}}$ (indices mod n).

Definition 1. *$\mathscr{B}$ is pasted if for any $A, B \in \mathscr{B}$, $A \neq B$ the following conditions hold:*
(i) *A is not contained in B,*
(ii) *$A \cap B$ is a subalgebra of A and of B on which the operations of A of B coincide,*
(iii) *for each $a \in A \cap B$, $a \notin \{0, 1\}$, there exists a 4-cycle $((A, a), (C_1, a'), (B, a), (C_3, a'))$ (C_1, C_3 arbitrary).*

The system of all blocks of an OML is pasted [6]. Dichtl [1] derived conditions for a pasted family of Boolean algebras to form an OML.

Theorem 2. *Let $\mathscr{B}$ be pasted. On $L = \bigcup_{B \in \mathscr{B}} B$ we define a partial ordering and an orthocomplementation as follows: $a \leqq b$ ($a = b'$) if there is $B \in \mathscr{B}$ such that $a \leqq_B b$ ($a = b'^B$). Then L is an OML if and only if the following two conditions hold true:*
(i) *for any 3-cycle $((B_i, b_i))_{i=0}^2$ in $\mathscr{B}$ there is a member $B \in \mathscr{B}$ such that $[0, b_i]_{B_i} \subset B$ for $i = 0, 1, 2$,*
(ii) *for any 4-cycle $((B_i, b_i))_{i=0}^3$ in $\mathscr{B}$ there is a 4-cycle $((C_0, a), (C_1, a'). (C_2, a), (C_3, a'))$ in $\mathscr{B}$ such that $b_0, b_2 \leqq a \leqq b'_1, b'_3$.*

Proof. See [1, 3 or 5].

Let $\mathscr{L}^{(k)}$ be the class of OMLs whose blocks (maximal Boolean algebras) are formed by the Boolean algebras 2^k.

Definition 3. *Let a natural number λ be given. We say that an* OML *$L \in \mathscr{L}^{(k)}$ is λ-regular if every atom $a \in L$ belongs to exactly λ blocks.*

For any $k \geqq 3$ and any $\lambda \geqq 1$ there exists a λ-regular OML $L \in \mathscr{L}^{(k)}$ [4]. A natural question arises to optimize its cardinality. We denote by $N_\lambda^{(k)}$ the minimal number of atoms of a λ-regular OML $L \in \mathscr{L}^{(k)}$ and by $n_\lambda^{(k)}$ the minimal number of blocks of such an OML. It is obvious that $\lambda N_\lambda^{(k)} = k n_\lambda^{(k)}$. Köhler [4] proved that

$$n_\lambda^{(k)} \leqq \lambda k^{k(\lambda-1)}$$

for any $k \geqq 3$ and any $\lambda \geqq 1$. We improve this result as follows.

Proposition 4. *Let $\lambda \geqq 1$ be given. Then $n_\lambda^{(k)} \leqq \lambda^2$ for any $k \geqq 4$.*

Proof. Let $\lambda \geqq 1$ and $k \geqq 4$ be given. Let $B_1, B_2, \ldots, B_\lambda$ be copies of the Boolean algebra 2^k. We denote the atoms of B_i by $b_{i1}, b_{i2}, \ldots, b_{ik}$ and put $x_i = b_{i1} \vee b_{i2}$ (and $x'_i = b_{i3} \vee b_{i4} \vee \ldots \vee b_{ik}$) for any $i = 1, 2, \ldots, \lambda$. Let us unify all x_i's $i = 1, 2, \ldots, \lambda$, and denote this element by x. Put $\mathscr{B} = \{[0, x]_{B_i} \times [0, x']_{B_j} \mid i = 1, 2, \ldots, \lambda, j = 1, 2, \ldots, \lambda\}$. Then $\mathscr{B}$ is pasted. Indeed, if $C, D \in \mathscr{B}$, $C \neq D$, then C is not contained in D and $C \cap D$ equals either $\{0, 1, x, x'\}$ or $\{0, 1\} \cup [0, x]_C \cup$

$\cup [x', 1]_C$ or $\{0, 1\} \cup [0, x']_C \cup [x, 1]_C$. Hence $C \cap D$ is a subalgebra of C and of D and the operations of C and of D coincide on $C \cap D$. As for (iii) of Definition 1, denote $[0, x]_C \times [0, x']_D = A_1$, $[0, x]_D \times [0, x']_C = A_2$. Trivially, $A_1, A_2 \in \mathscr{B}$, and if $a \in C \cap D$, then $a \in A_1 \cap A_2$. If $a = x$ then $((C, x), (A_1, x'), (D, x), (A_2, x'))$ forms a 4-cycle. If $a \leqq x$, $a \notin \{0, x\}$, then $[0, x]_C = [0, x]_D$ and $((C, a), (A_1, a'), (D, a), (A_2, a'))$ is a 4-cycle. The same result can be obtained if $x \leqq a$. If $a \leqq x'$ or $x' \leqq a$, then, analogously, there is a 4-cycle $((C, a'), (A_1, a), (D, a'), (A_2, a))$. We have proved that $\mathscr{B}$ is pasted.

Let us now put $L = \bigcup_{B \in \mathscr{B}} B$. We use Theorem 2 to prove that L is an OML. As for (i), observe that if $((A_1, a_1), (A_2, a_2), (A_3, a_3))$ is a 3-cycle in $\mathscr{B}$, then $a_1, a_2, a_3 \in C$ for some $C \in \{A_1, A_2, A_3\}$ – otherwise there would be a block $[0, x]_{B_k} \times [0, x]_{B_l}$ or $[0, x']_{B_k} \times [0, x']_{B_l}$ in $\mathscr{B}$ (for some $k, l \in \{1, 2, \ldots, \lambda\}$). To prove (ii), suppose that $((A_1, a_1), (A_2, a_2), (A_3, a_3), (A_4, a_4))$ is a 4-cycle in $\mathscr{B}$. Suppose first that $A_1 \neq A_2 \neq A_3 \neq A_4 \neq A_1$. Then $A_1 = [0, x]_{A_1} \times [0, x']_{A_1}$, $A_2 = [0, x]_{A_1} \times [0, x']_{A_3}$, $A_3 = [0, x]_{A_3} \times [0, x']_{A_3}$, $A_4 = [0, x]_{A_3} \times [0, x']_{A_1}$ (if necessary, the role of A_2 and A_4 is interchanged). Now $a_1 \in A_1 \cap A_2$ implies that $a_1 \leqq x$ or $a_1 = x' \vee c_1$ for some $c_1 \leqq x$. Similarly, $a_2 \leqq x'$ or $a_2 = x \vee c_2$, $c_2 \leqq x'$, $a_3 \leqq x$ or $a_3 = x' \vee c_3$, $c_3 \leqq x$, and $a_4 \leqq x'$ or $a_4 = x \vee c_4$, $c_4 \leqq x'$. If $a_1, a_3 \leqq x \leqq a_2', a_4'$, then $((A_1, x), (A_2, x'), (A_3, x), (A_4, x'))$ is the desired 4-cycle. If $a_1 \nleqq x$ then $a_1 = x' \vee c_1$, $c_1 \leqq x$. Since $a_1 \leqq a_2'$, we have $a_2 \leqq x \wedge c_1'$ which is possible only if $a_2 = x$. Therefore $c_1 = 0$, $a_1 = x'$ and $a_2 = x$. Moreover, $a_3 \leqq a_2' = x'$ and $a_4 \leqq a_1' = x$. Hence $a_1, a_3 \leqq x' \leqq a_2', a_4'$. The 4-cycle is constructed similarly as above. The same result can be obtained also if $a_2 \nleqq x'$, if $a_3 \nleqq x$ or if $a_4 \nleqq x'$. Finally, we shall analyze the case $A_1 = A_2$ (due to the symmetry, it solves also the other cases with an equality). Now $a_1, a_2, a_4 \in A_1$. Therefore $a_2, a_4 \leqq a_2 \vee a_4 \leqq a_1', a_3'$ and there is a 4-cycle in $\mathscr{B}$, namely $((A_1, (a_2 \vee a_4)'), (A_1, a_2 \vee a_4), (A_1, (a_2 \vee a_4)'), (A_1, a_2 \vee a_4))$.

Since each block of the logic $L = \bigcup_{B \in \mathscr{B}} B$ can be identified with some B, $B \in \mathscr{B}$, the logic L has λ^2 blocks (see eg. [3], Lemma 14, p. 51). It is easily seen that each atom from L belongs to exactly λ blocks. The proof is complete.

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Souhrn

POZNÁMKA O λ-REGULÁRNÍCH ORTOMODULÁRNÍCH SVAZECH

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Konečný ortomodulární svaz, v němž každá maximální Booleova podalgebra (blok) má stejnou kardinalitu k, se nazývá λ-regulární, jestliže každý atom leží právě v λ blocích. Dokážeme, že nejmenší počet bloků λ-regulárního ortomodulárního svazu je menší nebo roven λ^2 bez ohledu na k.

Резюме

ЗАМЕЧАНИЕ О λ-РЕГУЛЯРНЫХ ОРТОМОДУЛЯРНЫХ РЕШЕТКАХ

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Конечная ортомодулярная решетка, в которой каждая максимальная булевская подалгебра (блок) имеет одинаковую кардинальность k, называется λ-регулярной, если каждый атом лежит точно в λ блоках. В статье доказано, что наименьшее число блоков λ-регулярной ортомодулярной решетки меньше или равно λ^2 независимо от k.

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