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PERIODIC AUTOREGRESSION WITH EXOGENOUS VARIABLES AND PERIODIC VARIANCES

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Summary. The periodic autoregressive process with non-vanishing mean and with exogenous variables is investigated in the paper. It is assumed that the model has also periodic variances. The statistical analysis is based on the Bayes approach with a vague prior density. Estimators of the parameters and asymptotic tests of hypotheses are derived.

Keywords: Bayes approach, estimating parameters, exogenous variables, periodic autoregressive process, periodic variances, testing hypotheses.

AMS subject classification: 62M10.

1. INTRODUCTION

The classical autoregressive process $\{X_t\}$ is given by the model

$$X_t = b_1 X_{t-1} + \ldots + b_n X_{t-n} + Y_t ,$$

where $b_1, \ldots, b_n$ are autoregressive parameters and $\{Y_t\}$ is an innovation process of uncorrelated random variables with $\mathsf{E} Y_t = 0$, $\operatorname{Var} Y_t = \sigma^2$. For describing a seasonal time series the autoregressive model can be generalized in such a way that periodic functions $\{b_{ki}\}_{i=1}^n$, $k = 1, \ldots, p$, are used instead of single values $\{b_i\}_{i=1}^n$ if p is the known period of the seasonal series. More precisely, we assume that the variables $X_1, \ldots, X_n$ are given and that X_t for $t > n$ are generated by

$$X_{n+(j-1)p+k} = \sum_{i=1}^{n} b_{ki} X_{n+(j-1)p+k-i} + Y_{n+(j-1)p+k} , \tag{1.1}$$

$k = 1, \ldots, p$; $j = 1, 2, \ldots$. If $\operatorname{Var} Y_t = \sigma^2$ does not depend on t, we have the model with equal variances. If $\operatorname{Var} Y_{n+(j-1)p+k} = \sigma_k^2$ depends on k, the model has periodic variances.

The model (1.1) was investigated by Pagano [5]. A statistical analysis of (1.1) is described by Anděl [1]. Since the periodic autoregression (1.1) has a connection with multidimensional autoregressive models, it was recommended by Newton [4] to use it for estimating parameters and other characteristics of multidimensional models.

If $\mathsf{E}X_t$ is a non-vanishing p-periodic function, the model (1.1) can be modified to

$$(1.2)\qquad X_{n+(j-1)p+k} = \mu_k + \sum_{i=1}^{n} b_{ki} X_{n+(j-1)p+k-i} + Y_{n+(j-1)p+k}\,,$$

where $\mu_1, \ldots, \mu_p$ are constants. This model was analyzed in detail by Anděl et al.[3].

If we wish to take into account also the influence of some exogenous variables $\varphi_1, \ldots, \varphi_S$, we come to the model

$$(1.3)\qquad X_{n+(j-1)p+k} = \mu_k + \sum_{i=1}^{n} b_{ki} X_{n+(j-1)p+k-i} + \\ + \sum_{s=1}^{S} \sum_{r=0}^{m_s} a_{ksr} \varphi_{s,n+(j-1)p+k-r} + Y_{n+(j-1)p+k}\,,$$

$k = 1, \ldots, p$, $j = 1, 2, \ldots$. Here μ_k, b_{ki} and a_{ksr} are unknown parameters and φ_{st} are given values of the exogenous variables φ_s. Our statistical analysis of the model (1.3) will be based on a realization $x_1, \ldots, x_N$ of the random variables $X_1, \ldots, X_N$. The case when the variables Y_t have equal variances was considered by Anděl [2]. In the present paper we assume that Y_t are independent normal variables such that

$$(1.4)\qquad Y_{n+(j-1)p+k} \sim N(0, \sigma_k^2)\,.$$

Further we assume that $(X_1, \ldots, X_n)$ and $(Y_{n+1}, Y_{n+2}, \ldots)$ are independent sets of variables. The analysis is based on the Bayes approach in which the parameters are supposed to be random variables with a vague prior distribution. In the case of equal variances it was possible to derive explicit results for finite values of N. In the model with periodic variances most results are asymptotic. Our estimators of the parameters μ_k, b_{ki} and a_{ksr} are identical with the maximum likelihood estimators. Moreover, it is well known that under general assumptions the asymptotic posterior distribution does not depend on the prior distribution. The Bayesian procedure makes in this case the computation of the asymptotic distributions and asymptotic tests easier.

2. PRELIMINARIES

We devote this section to some auxiliary theorems which will be needed in the main part of this paper. Their proofs can be found in Anděl [1].

We will use the symbol c for constants. If not necessary, we will not distinguish between different constants.

Theorem 2.1. *Let $\mathbf{Q}_1, \ldots, \mathbf{Q}_p$ be $n \times n$ symmetric positive definite matrices. Let $\mathbf{Q} = \mathbf{Q}_1 + \ldots + \mathbf{Q}_p$. If $p \geqq 2$, then the matrix*

$$\mathbf{H} = \begin{Vmatrix} \mathbf{Q}_1 & \mathbf{O} & \dots & \mathbf{O} \\ \dots & \dots & \dots & \dots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{Q}_{p-1} \end{Vmatrix} - \begin{Vmatrix} \mathbf{Q}_1\mathbf{Q}^{-1}\mathbf{Q}_1 & \dots & \mathbf{Q}_1\mathbf{Q}^{-1}\mathbf{Q}_{p-1} \\ \dots & \dots & \dots \\ \mathbf{Q}_{p-1}\mathbf{Q}^{-1}\mathbf{Q}_1 & \dots & \mathbf{Q}_{p-1}\mathbf{Q}^{-1}\mathbf{Q}_{p-1} \end{Vmatrix}$$

is positive definite.

Theorem 2.2. *Let* $\mathbf{V}$ *be an* $n \times n$ *symmetric positive definite matrix and let a random vector* $\mathbf{X} = (X_1, \dots, X_n)'$ *have the density*

$$q(\mathbf{x}) = c(1 + \mathbf{x}'\mathbf{V}\mathbf{x})^{-m/2}, \tag{2.1}$$

where $m \geqq n + 1$. *Introduce a random vector*

$$\mathbf{Z} = (Z_1, \dots, Z_s)' = (X_{i_1}, \dots, X_{i_s})',$$

where $1 \leqq s < n$ *and* $1 \leqq i_1 < \dots < i_s \leqq n$. *Let* $\mathbf{W}$ *be the matrix arising from the rows* $i_1, \dots, i_s$ *and from the columns* $i_1, \dots, i_s$ *of the matrix* $\mathbf{V}^{-1}$. *Then the marginal density of the vector* $\mathbf{Z}$ *is*

$$q_1(\mathbf{z}) = c(1 + \mathbf{z}'\mathbf{W}^{-1}\mathbf{z})^{-(m-n+s)/2}.$$

Theorem 2.3. *Let a vector* $\mathbf{X} = (X_1, \dots, X_n)'$ *have the density* (2.1). *Then the random variable*

$$F = \frac{m-n}{n}\mathbf{X}'\mathbf{V}\mathbf{X}$$

has the $F_{n,m-n}$ *distribution.*

3. STATISTICAL ANALYSIS OF THE MODEL

We consider the model (1.3) with the assumption (1.4). At the beginning of this section we introduce the necessary notation. Put

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)', \quad \mathbf{b}_k = (b_{k1}, \dots, b_{kn})', \quad \mathbf{b} = (\mathbf{b}'_1, \dots, \mathbf{b}'_p)',$$

$$\mathbf{a}_{ks} = (a_{ks0}, \dots, a_{ksm_s})', \quad \mathbf{a}_k = (\mathbf{a}'_{k1}, \dots, \mathbf{a}'_{kS})', \quad \mathbf{a} = (\mathbf{a}'_1, \dots, \mathbf{a}'_p)',$$

$$\mathbf{x} = (x_1, \dots, x_N)', \quad \mathbf{x}^0_t = (x_{t-1}, \dots, x_{t-n})',$$

$$\boldsymbol{\varphi}^0_{st} = (\varphi_{st}, \dots, \varphi_{s,t-m_s})', \quad m = m_1 + \dots + m_S,$$

$$\boldsymbol{\Theta}_k = (\mu_k, \mathbf{b}'_k, \mathbf{a}'_k)', \quad \boldsymbol{\Theta} = (\boldsymbol{\Theta}'_1, \dots, \boldsymbol{\Theta}'_p)', \quad \boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_p)',$$

$$\mathbf{w}_{kj} = (1, \mathbf{x}^{0\prime}_{n+(j-1)p+k}, \boldsymbol{\varphi}^{0\prime}_{1,n+(j-1)p+k}, \dots, \boldsymbol{\varphi}^{0\prime}_{S,n+(j-1)p+k})',$$

$$\alpha_k = \left[\frac{N-n-k}{p}\right] + 1,$$

where [ ] denotes the integer part,

$$\varkappa_k = \sum_{j=1}^{\alpha_k} x^2_{n+(j-1)p+k}, \quad \mathbf{q}_k = \sum_{j=1}^{\alpha_k} x_{n+(j-1)p+k}\mathbf{w}_{kj}, \quad \mathbf{Q}_k = \sum_{j=1}^{\alpha_k} \mathbf{w}_{kj}\mathbf{w}'_{kj}.$$

We shall assume that the matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_p$ are positive definite. This is no substantial restriction, because for $\alpha_k > 1 + n + m$ this condition is satisfied with probability one.

The conditional density of $X_{n+1}, \ldots, X_N$, given $X_1 = x_1, \ldots, X_n = x_n, \boldsymbol{\Theta}, \boldsymbol{\sigma}$ is

$$f(x_{n+1}, \ldots, x_N \mid x_1, \ldots, x_n, \boldsymbol{\Theta}, \boldsymbol{\sigma}) = (2\pi)^{-(N-n)/2} \times$$
$$\times \prod_{k=1}^{p} \sigma_k^{-\alpha_k} \exp\left\{-\frac{1}{2\sigma_k^2} \sum_{j=1}^{\alpha_k} z_{kj}^2\right\},$$

where

$$z_{kj} = x_{n+(j-1)p+k} - \mu_k - \mathbf{b}_k' \mathbf{x}^0_{n+(j-1)p+k} - \sum_{s=1}^{S} \mathbf{a}_{ks}' \boldsymbol{\varphi}^0_{s,n+(j-1)p+k} =$$
$$= x_{n+(j-1)p+k} - \boldsymbol{\Theta}_k' \mathbf{w}_{kj}.$$

Theorem 3.1. *Let $\boldsymbol{\Theta}$ and $\boldsymbol{\sigma}$ have the prior density*

$$\pi(\boldsymbol{\Theta}, \boldsymbol{\sigma}) = \begin{cases} \sigma_1^{-1} \ldots \sigma_p^{-1} & \text{for} \quad \sigma_1 > 0, \ldots, \sigma_p > 0 \quad \text{and} \quad \boldsymbol{\Theta} \in \boldsymbol{R}_{p(1+n+m)}, \\ 0 & \text{otherwise}, \end{cases}$$

and let $(\boldsymbol{\Theta}, \boldsymbol{\sigma})$ be independent of $(X_1, \ldots, X_n)$. Then the posterior density of $\boldsymbol{\Theta}$ and $\boldsymbol{\sigma}$ is

$$g(\boldsymbol{\Theta}, \boldsymbol{\sigma} \mid \mathbf{x}) = c\sigma_1^{-\alpha_1 - 1} \ldots \sigma_p^{-\alpha_p - 1} \times \tag{3.1}$$
$$\times \exp\left\{-\sum_{k=1}^{p} \frac{1}{2\sigma_k^2} [(\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \mathbf{Q}_k (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*) + v_k]\right\}$$

for $\sigma_1 > 0, \ldots, \sigma_p > 0$ and zero otherwise, where

$$\boldsymbol{\Theta}_k^* = \mathbf{Q}_k^{-1} \mathbf{q}_k, \quad v_k = \varkappa_k - \boldsymbol{\Theta}_k^{*\prime} \mathbf{q}_k.$$

Proof. From the Bayes theorem we have

$$g(\boldsymbol{\Theta}, \boldsymbol{\sigma} \mid \mathbf{x}) = cf(x_{n+1}, \ldots, x_N \mid x_1, \ldots, x_n, \boldsymbol{\Theta}, \boldsymbol{\sigma})\, \pi(\boldsymbol{\Theta}, \boldsymbol{\sigma}).$$

We insert for f and make use of the fact that

$$\sum_{j=1}^{\alpha_k} z_{kj}^2 = (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \mathbf{Q}_k (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*) + v_k. \quad \square$$

Theorem 3.2. (i) *Modus of the posterior density* (3.1) *is*

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}^*, \quad \sigma_k^{*2} = v_k/(\alpha_k + 1) \quad \text{for} \quad k = 1, \ldots, p.$$

(ii) *The marginal posterior density of $\boldsymbol{\Theta}$ is*

$$g_1(\boldsymbol{\Theta} \mid \mathbf{x}) = c \prod_{k=1}^{p} [1 + v_k^{-1} (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \mathbf{Q}_k (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)]^{-\alpha_k/2}.$$

(iii) *The marginal posterior density of* $\boldsymbol{\sigma}$ *is*

$$g_2(\boldsymbol{\sigma} \mid \mathbf{x}) = c \prod_{k=1}^{p} \sigma_k^{-\alpha_k+n+m} \exp\{-v_k/(2\sigma_k^2)\}$$

for $\sigma_1 > 0, \ldots, \sigma_p > 0$.

Proof. (i) It is clear that $g(\boldsymbol{\Theta}, \boldsymbol{\sigma} \mid \mathbf{x}) \leqq g(\boldsymbol{\Theta}^*, \boldsymbol{\sigma} \mid \mathbf{x})$. Using calculus we can derive that $g(\boldsymbol{\Theta}^*, \boldsymbol{\sigma} \mid \mathbf{x})$ reaches its maximum for $\sigma_1 = \sigma_1^*, \ldots, \sigma_p = \sigma_p^*$.

(ii) The density $g_1(\boldsymbol{\Theta} \mid \mathbf{x})$ can be calculated from

$$g_1(\boldsymbol{\Theta} \mid \mathbf{x}) = \int_0^\infty \ldots \int_0^\infty g(\boldsymbol{\Theta}, \boldsymbol{\sigma} \mid \mathbf{x})\, \mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_p \,.$$

(iii) We have

$$g_2(\boldsymbol{\sigma} \mid \mathbf{x}) = \int_{\boldsymbol{R}_{p(1+n+m)}} g(\boldsymbol{\Theta}, \boldsymbol{\sigma} \mid \mathbf{x})\, \mathrm{d}\boldsymbol{\Theta} =$$

$$= c \prod_{k=1}^{p} \sigma_k^{-\alpha_k-1} \exp\left\{-\frac{v_k}{2\sigma_k^2}\right\} \int_{\boldsymbol{R}_{1+n+m}} \exp\left\{-\frac{1}{2\sigma_k^2}(\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \mathbf{Q}_k(\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)\right\} \mathrm{d}\boldsymbol{\Theta}_k \,.$$

To evaluate the last integral we use the substitution

$$\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^* = \sigma_k \mathbf{y}_k \,.$$

The corresponding Jacobian is σ_k^{1+n+m} and from here we get the assertion (iii). □

An important conclusion from Theorem 3.2 is that $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_p$ are, given $\mathbf{x}$, conditionally independent, and also $\sigma_1, \ldots, \sigma_p$ are, given $\mathbf{x}$, conditionally independent.

The first problem which we are going to investigate is a construction of tests of fit.

Theorem 3.3. *The posterior distribution of the variable*

$$F_k = \frac{\alpha_k - 1 - n - m}{(1 + n + m)\, v_k} (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \mathbf{Q}_k(\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*) \qquad (3.2)$$

is $F_{1+n+m,\alpha_k-1-n-m}$ *for* $k = 1, \ldots, p$.

Proof. The assertion follows from Theorem 3.2 (ii) and from Theorem 2.3. □

Theorem 3.3 can be used for testing that $\boldsymbol{\Theta}_k$ for a given fixed k is equal to a fixed vector $\boldsymbol{\Theta}_k^0$. We insert $\boldsymbol{\Theta}_k = \boldsymbol{\Theta}_k^0$ into (3.2) and if the result exceeds the critical value of the corresponding F distribution, the hypothesis $\boldsymbol{\Theta}_k = \boldsymbol{\Theta}_k^0$ can be rejected.

If we wish to test the hypothesis that the whole vector $\boldsymbol{\Theta}$ is equal to a given vector $\boldsymbol{\Theta}^0$, we must combine the statistics F_k. Because of the conditional independence, we can use the so called "combination of independent tests of fit". Let H_k be the distribution function of $F_{1+n+m,\alpha_k-1-n-m}$. Put $\pi_k = 1 - H_k(F_k)$. It is well known that then the posterior distribution of

$$\varrho = -2 \sum_{k=1}^{p} \ln \pi_k$$

is χ^2_{2p}. Therefore, a test of $H_0\colon \boldsymbol{\Theta} = \boldsymbol{\Theta}^0$ can be based on ϱ. If ϱ exceeds the critical value $\chi^2_{2p}(\alpha)$, we reject H_0 on the level α. However, the calculation of $H_k(F_k)$ can be a little difficult. Below we derive an asymptotic method, which is easier from the numerical point of view.

Theorem 3.4. *Denote* $\hat{\sigma}_k^2 = v_k/(\alpha_k - n - m - 1)$. *Then, given* $\mathbf{x}$, *the variables* $\hat{\sigma}_1^2, \ldots, \hat{\sigma}_p^2$ *are independent and*

$$(\alpha_k - n - m - 1)\,\hat{\sigma}_k^2/\sigma_k^2 \sim \chi^2_{\alpha_k - n - m - 1}\,.$$

Proof. The marginal posterior density of σ_k is

$$c\sigma_k^{-\alpha_k+n+m} \exp\{-v_k/(2\sigma_k^2)\}\,, \tag{3.3}$$

which we get from Theorem 3.2 (iii) as well as the assertion about the independence. From (3.3) we obtain that $v_k/\sigma_k^2 \sim \chi^2_{\alpha_k - n - m - 1}$. Now, the assertion is an easy consequence of this fact. □

Let us introduce explicitly two important special cases of the previous theorem. If $p = 2$, then

$$\hat{\sigma}_1^2/\hat{\sigma}_2^2 \sim F_{\alpha_1 - n - m - 1, \alpha_2 - n - m - 1}\,.$$

The hypothesis $\sigma_1^2 = \sigma_2^2$ can be tested in the same way as the classical comparison of two variances in two independent samples from two normal distributions. If $\alpha_k = \alpha_0$ do not depend on k, we can use Cochran's test for testing the hypothesis $\sigma_1^2 = \ldots = \sigma_p^2$. This test is based on the statistic

$$g' = (\max_{1 \le k \le p} \hat{\sigma}_k^2)/\sum_{k=1}^{p} \hat{\sigma}_k^2$$

and its critical values can be found in statistical tables.

Write $\mathbf{Q}_k^{-1}$ in the form

$$\mathbf{Q}_k^{-1} = \begin{Vmatrix} Q_k^{11} & \mathbf{Q}_k^{12} & \mathbf{Q}_k^{13} \\ \mathbf{Q}_k^{21} & \mathbf{Q}_k^{22} & \mathbf{Q}_k^{23} \\ \mathbf{Q}_k^{31} & \mathbf{Q}_k^{32} & \mathbf{Q}_k^{33} \end{Vmatrix},$$

where Q_k^{11}, $\mathbf{Q}_k^{22}$ and $\mathbf{Q}_k^{33}$ are 1×1, $n \times n$ and $m \times m$ blocks, respectively. We have defined $\boldsymbol{\Theta}_k = (\mu_k, \mathbf{b}_k', \mathbf{a}_k')'$. Similarly we shall write $\boldsymbol{\Theta}_k^*$ in the form $\boldsymbol{\Theta}_k^* = (\mu_k^*, \mathbf{b}_k^{*\prime}, \mathbf{a}_k^{*\prime})'$.

Theorem 3.5. (i) *The marginal posterior density of* $\boldsymbol{\mu}$ *is*

$$g_3(\boldsymbol{\mu} \mid \mathbf{x}) = c \prod_{k=1}^{p} [1 + (\mu_k - \mu_k^*)^2/(v_k Q_k^{11})]^{-(\alpha_k - n - m)/2}\,.$$

(ii) *The marginal posterior density of* $\mathbf{b}$ *is*

$$g_4(\mathbf{b} \mid \mathbf{x}) = c \prod_{k=1}^{p} [1 + v_k^{-1}(\mathbf{b}_k - \mathbf{b}_k^*)'\,(\mathbf{Q}_k^{22})^{-1}\,(\mathbf{b}_k - \mathbf{b}_k^*)]^{-(\alpha_k - 1 - m)/2}\,.$$

(iii) *The marginal posterior density of* $\boldsymbol{a}$ *is*

$$g_5(\boldsymbol{a} \mid \mathbf{x}) = c \prod_{k=1}^{p} [1 + v_k^{-1}(\boldsymbol{a}_k - \boldsymbol{a}_k^*)' (\mathbf{Q}_k^{33})^{-1} (\boldsymbol{a}_k - \boldsymbol{a}_k^*)]^{-(\alpha_k - 1 - n)/2} .$$

Proof. All the three formulas follow from Theorem 3.2 (ii) and from Theorem 2.2. □

It is clear that $\mu_1, \ldots, \mu_p$ are conditionally independent and the same is true also for $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p$ and for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p$.

4. SOME APPROXIMATIONS

If a vector $\boldsymbol{X} = (X_1, \ldots, X_n)'$ has the density (2.1), then the vector $\boldsymbol{Y} = m^{1/2}\boldsymbol{X}$ has the density

$$c(1 + \mathbf{y}'\mathbf{V}\mathbf{y}/m)^{-m/2} .$$

As $m \to \infty$, the distribution of $\boldsymbol{Y}$ converges to $N(0, \mathbf{V}^{-1})$. Thus for large m we can approximate the distribution of $\boldsymbol{X}$ by $N(0, m^{-1}\mathbf{V}^{-1})$. Hence it follows that

$$m\boldsymbol{X}'\mathbf{V}\boldsymbol{X} \sim \chi_n^2 \tag{4.1}$$

approximately holds. Denote

$$\boldsymbol{U}_k = v_k^{-1}\alpha_k\mathbf{Q}_k , \quad \boldsymbol{U} = \boldsymbol{U}_1 + \ldots + \boldsymbol{U}_p ,$$

$$\boldsymbol{L} = \left\| \begin{matrix} \boldsymbol{U}_1 & \mathbf{O} & \ldots & \mathbf{O} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{O} & \mathbf{O} & \ldots & \boldsymbol{U}_{p-1} \end{matrix} \right\| - \left\| \begin{matrix} \boldsymbol{U}_1\boldsymbol{U}^{-1}\boldsymbol{U}_1 & \ldots & \boldsymbol{U}_1\boldsymbol{U}^{-1}\boldsymbol{U}_{p-1} \\ \ldots & \ldots & \ldots \\ \boldsymbol{U}_{p-1}\boldsymbol{U}^{-1}\boldsymbol{U}_1 & \ldots & \boldsymbol{U}_{p-1}\boldsymbol{U}^{-1}\boldsymbol{U}_{p-1} \end{matrix} \right\| .$$

Further we introduce

$$U_k^{(1)} = (\alpha_k - n - m)(v_k Q_k^{11})^{-1} , \quad \boldsymbol{U}_k^{(2)} = (\alpha_k - 1 - m)(v_k\mathbf{Q}_k^{22})^{-1} ,$$

$$\boldsymbol{U}_k^{(3)} = (\alpha_k - 1 - n)(v_k\mathbf{Q}_k^{33})^{-1} ,$$

$$\boldsymbol{U}^{(i)} = \boldsymbol{U}_1^{(i)} + \ldots + \boldsymbol{U}_p^{(i)} \quad \text{for} \quad i = 1, 2, 3 .$$

Define matrices $\boldsymbol{L}_i$ $(i = 1, 2, 3)$ with help of $\boldsymbol{U}_k^{(i)}$ and $\boldsymbol{U}^{(i)}$ in the same way as the matrix $\boldsymbol{L}$ with help of $\boldsymbol{U}_k$ and $\boldsymbol{U}$. Then the densities $g_1(\boldsymbol{\Theta} \mid \mathbf{x})$, $g_3(\boldsymbol{\mu} \mid \mathbf{x})$, $g_4(\boldsymbol{b} \mid \mathbf{x})$ and $g_5(\boldsymbol{a} \mid \mathbf{x})$ can be approximated by the densities

$$\tilde{g}_1(\boldsymbol{\Theta} \mid \mathbf{x}) = c \exp\{-\tfrac{1}{2}\sum_{k=1}^{p} (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \boldsymbol{U}_k(\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)\} ,$$

$$\tilde{g}_3(\boldsymbol{\mu} \mid \mathbf{x}) = c \exp\{-\tfrac{1}{2}\sum_{k=1}^{p} (\mu_k - \mu_k^*)^2 U_k^{(1)}\} ,$$

$$\tilde{g}_4(\boldsymbol{b} \mid \mathbf{x}) = c \exp\{-\tfrac{1}{2}\sum_{k=1}^{p} (\boldsymbol{b}_k - \boldsymbol{b}_k^*)' \boldsymbol{U}_k^{(2)}(\boldsymbol{b}_k - \boldsymbol{b}_k^*)\} ,$$

and

$$\tilde{g}_5(\boldsymbol{a} \mid \boldsymbol{x}) = c \exp \{-\tfrac{1}{2} \sum_{k=1}^{p} (\boldsymbol{a}_k - \boldsymbol{a}_k^*)' \boldsymbol{U}_k^{(3)} (\boldsymbol{a}_k - \boldsymbol{a}_k^*)\},$$

respectively. Therefore, if we denote

$$\gamma_{\boldsymbol{\Theta}} = \sum_{k=1}^{p} (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*)' \boldsymbol{U}_k (\boldsymbol{\Theta}_k - \boldsymbol{\Theta}_k^*), \quad \gamma_{\boldsymbol{\mu}} = \sum_{k=1}^{p} (\mu_k - \mu_k^*)^2 U_k^{(1)},$$

$$\gamma_{\boldsymbol{b}} = \sum_{k=1}^{p} (\boldsymbol{b}_k - \boldsymbol{b}_k^*)' \boldsymbol{U}_k^{(2)} (\boldsymbol{b}_k - \boldsymbol{b}_k^*), \quad \gamma_{\boldsymbol{a}} = \sum_{k=1}^{p} (\boldsymbol{a}_k - \boldsymbol{a}_k^*)' \boldsymbol{U}_k^{(3)} (\boldsymbol{a}_k - \boldsymbol{a}_k^*),$$

then

$$\gamma_{\boldsymbol{\Theta}} \sim \chi^2_{p(1+n+m)}, \quad \gamma_{\boldsymbol{\mu}} \sim \chi^2_p, \quad \gamma_{\boldsymbol{b}} \sim \chi^2_{np}, \quad \gamma_{\boldsymbol{a}} \sim \chi^2_{mp}$$

approximately hold. These results can be used in the following way. If we have a hypothesis $H_{\boldsymbol{\mu}}$: $\boldsymbol{\mu} = \boldsymbol{\mu}^0$, then we calculate

$$\gamma_{\boldsymbol{\mu}} = \sum_{k=1}^{p} (\mu_k^0 - \mu_k^*)^2 U_k^{(1)}.$$

If $\gamma_{\boldsymbol{\mu}} \geqq \chi_p^2(\alpha)$, we reject $H_{\boldsymbol{\mu}}$ on a level which is approximately equal to α. Similarly we can test hypotheses about $\boldsymbol{b}$, $\boldsymbol{a}$ and about the whole $\boldsymbol{\Theta}$.

Now, we will discuss how to test the hypothesis H_0: $\boldsymbol{\Theta}_1 = \ldots = \boldsymbol{\Theta}_p$. Put

$$\Delta_k = \boldsymbol{\Theta}_k - \boldsymbol{\Theta}_p - (\boldsymbol{\Theta}_k^* - \boldsymbol{\Theta}_p^*) \quad \text{for} \quad k = 1, \ldots, p-1, \quad \Delta_p = \boldsymbol{\Theta}_p - \boldsymbol{\Theta}_p^*,$$

$$\Delta = (\Delta_1', \ldots, \Delta_{p-1}')', \quad \boldsymbol{h} = \sum_{k=1}^{p-1} \boldsymbol{U}_k \Delta_k, \quad \boldsymbol{G} = \sum_{k=1}^{p-1} \Delta_k' \boldsymbol{U}_k \Delta_k - \boldsymbol{h}' \boldsymbol{U}^{-1} \boldsymbol{h}.$$

If $\boldsymbol{\Theta}$ has the density $\tilde{g}_1(\boldsymbol{\Theta} \mid \boldsymbol{x})$, then the density of $(\Delta_1', \ldots, \Delta_p')'$ is

$$c \exp \{-\tfrac{1}{2}[\boldsymbol{G} + (\Delta_p + \boldsymbol{U}^{-1}\boldsymbol{h})' \boldsymbol{U} (\Delta_p + \boldsymbol{U}^{-1}\boldsymbol{h})]\}.$$

Since $\boldsymbol{G} = \Delta' \boldsymbol{L} \Delta$, the marginal density of Δ is $c \exp\{-\tfrac{1}{2}\Delta' \boldsymbol{L} \Delta\}$. Thus the posterior density of $r_{\boldsymbol{\Theta}} = \Delta' \boldsymbol{L} \Delta$ is approximately $\chi^2_{(p-1)(1+n+m)}$. If $r_{\boldsymbol{\Theta}} \geqq \chi^2_{(p-1)(1+n+m)}(\alpha)$, we reject H_0. Similarly we can describe also the remaining three cases. Let

$$\Delta_k^{(\boldsymbol{\mu})} = \mu_k - \mu_p - (\mu_k^* - \mu_p^*), \quad \Delta_{\boldsymbol{\mu}} = (\Delta_1^{(\boldsymbol{\mu})}, \ldots, \Delta_{p-1}^{(\boldsymbol{\mu})})',$$

$$\Delta_k^{(\boldsymbol{b})} = \boldsymbol{b}_k - \boldsymbol{b}_p - (\boldsymbol{b}_k^* - \boldsymbol{b}_p^*), \quad \Delta_{\boldsymbol{b}} = (\Delta_1^{(\boldsymbol{b})'}, \ldots, \Delta_{p-1}^{(\boldsymbol{b})'})',$$

$$\Delta_k^{(\boldsymbol{a})} = \boldsymbol{a}_k - \boldsymbol{a}_p - (\boldsymbol{a}_k^* - \boldsymbol{a}_p^*), \quad \Delta_{\boldsymbol{a}} = (\Delta_1^{(\boldsymbol{a})'}, \ldots, \Delta_{p-1}^{(\boldsymbol{a})'})',$$

$$r_{\boldsymbol{\mu}} = \Delta_{\boldsymbol{\mu}}' \boldsymbol{L}_1 \Delta_{\boldsymbol{\mu}}, \quad r_{\boldsymbol{b}} = \Delta_{\boldsymbol{b}}' \boldsymbol{L}_2 \Delta_{\boldsymbol{b}}, \quad r_{\boldsymbol{a}} = \Delta_{\boldsymbol{a}}' \boldsymbol{L}_3 \Delta_{\boldsymbol{a}}.$$

Then we have approximately

$$r_{\boldsymbol{\mu}} \sim \chi^2_{p-1}, \quad r_{\boldsymbol{b}} \sim \chi^2_{(p-1)n}, \quad r_{\boldsymbol{a}} \sim \chi^2_{(p-1)m}.$$

These results can be used for testing hypotheses that μ_k, $\boldsymbol{b}_k$ and $\boldsymbol{a}_k$ separately do not depend on k.

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Souhrn

PERIODICKÁ AUTOREGRESE S EXOGENNÍMI VELIČINAMI A S PERIODICKÝMI ROZPTYLY

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V práci je vyšetřován periodický autoregresní proces s nenulovou střední hodnotou a s exogenními veličinami. Předpokládá se, že model má i periodické rozptyly. Statistická analýza je založena na bayesovském přístupu s nevlastní apriorní hustotou. Jsou odvozeny odhady parametrů a asymptotické testy hypotéz.

Резюме

ПЕРИОДИЧЕСКАЯ АВТОРЕГРЕССИЯ С ЭКЗОГЕННЫМИ ПЕРЕМЕННЫМИ И ПЕРИОДИЧЕСКИМИ ВАРИАНЦИЯМИ

JIŘÍ ANDĚL

В работе исследуется периодический процесс авторегрессии с ненулевым средним значением и с экзогенными переменными. Предполагается, что варианции также периодические. Статистический анализ основан на принципе Бейеса с несобственной априорной плотностью. В статье выведены оценки параметров и асимптотические проверки гипотез.

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