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Dalibor Volný

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ON NON ERGODIC VERSIONS OF LIMIT THEOREMS

DALIBOR VOLNÝ

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Summary. The author investigates non ergodic versions of several well known limit theorems for strictly stationary processes. In some cases, the assumptions which are given with respect to general invariant measure, guarantee the validity of the theorem with respect to ergodic components of the measure. In other cases, the limit theorem can fail for all ergodic components, while for the original invariant measure it holds.

Key-words: central limit theorem for dependent r. v's, invariance principle and law of iterated logarithm for dependent r. v's, strictly stationary sequence of r. v's, invariant measure, ergodic decomposition.

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1. FACTORS AND EXISTENCE OF AN ERGODIC DECOMPOSITION

Let $(\Omega, \mathscr{A}, \mu)$ be a probability space, where $\mathscr{A}$ is a σ-algebra of subsets of Ω and μ is a probability measure on $(\Omega, \mathscr{A})$. T is a 1-1 bimeasurable mapping of Ω onto itself, $\mu T^{-1} = \mu$. The quadruple $(\Omega, \mathscr{A}, T, \mu)$ is called a dynamical system. The collection $\mathscr{I} = \{A \in \mathscr{A}: TA = A\}$ is a σ-algebra. If $\mu(A) = 0$ or $\mu(A) = 1$ for each $A \in \mathscr{I}$, we say that μ is ergodic. If there exists a family $(m_\omega;\ \omega \in \Omega)$ of regular conditional probabilities (r.c.p.) induced by $\mathscr{I}$ with respect to μ, and m_ω are ergodic probability measures, we say that $(m_\omega;\ \omega \in \Omega)$ is an ergodic decomposition of μ.

Let f be a measurable function; the process $(f \circ T^i)$ is (strictly) stationary. We shall deal with limit theorems for such processes. Assumptions of our theorems can be expressed in terms of a dynamical system $(\Omega, \mathscr{C}, T, \mu)$ where $\mathscr{C} \subset \mathscr{A}$ is a separable σ-algebra, $T^{-1}\mathscr{C} = \mathscr{C} = T\mathscr{C}$ (see Section 3); we say that $\mathscr{C}$ is separable if these exists a countable collection of sets generating $\mathscr{C}$. (Correctly, we should use the restriction $\mu/\mathscr{C}$ of μ onto $\mathscr{C}$ intsead of μ.) For a separable σ-algebra $\mathscr{C}$, there exists a function g which generates $\mathscr{C}$. For example, we can put $g = \sum_{k=1}^{\infty} (1/3^k)\, \chi_{A_k}$ where $\{A_1, A_2, \ldots\}$ generates $\mathscr{C}$. By ψ we denote the mapping of Ω into $\boldsymbol{R}^Z$ defined by

$$(\psi\omega)_i = g(T^i\omega), \quad i \in \boldsymbol{Z}.$$

Let $\mathscr{B}^Z$ be the Borel σ-algebra on $\boldsymbol{R}^Z$ and S: $\boldsymbol{R}^Z \to \boldsymbol{R}^Z$ be defined by $(S(x))_i = x_{i+1}$, $i \in \boldsymbol{Z}$. The sets $A_{I,i} = \{x \in \boldsymbol{R}^Z: x_i \in I\}$, where I is an interval in $\boldsymbol{R}$ and $i \in \boldsymbol{Z}$, generate the σ-algebra $\mathscr{B}^Z$; the sets $\psi^{-1}(A_{I,0})$, where I is an interval, generate $\mathscr{C}$. We have $S^k A_{I,i} = A_{I,i-k}$ for $k \in \boldsymbol{Z}$. Hence, ψ is a measurable mapping, $\psi^{-1}(\mathscr{B}^Z) = \mathscr{C}$, and S is a 1-1 bimeasurable mapping of $\boldsymbol{R}^Z$ onto itself. Moreover,

$$S \circ \psi = \psi \circ T .$$

From this identity we can easily derive that $\nu = \mu\psi^{-1}$ is an S-invariant measure, i.e. $\nu S^{-1} = \nu$. Thus, $(\boldsymbol{R}^Z, \mathscr{B}^Z, S, \nu)$ is a dynamical system. In the language of ergodic theory (see [1]) we say that it is a factor of $(\Omega, \mathscr{A}, T, \mu)$. For a $\mathscr{C}$-measurable function f there exists a $\mathscr{B}^Z$-measurable function f' such that

$$f = f' \circ \psi . \tag{1}$$

This can be easily seen if f is a simple function. For $f \geqq 0$, (1) follows from the fact that there exists a monotone sequence of $\mathscr{C}$-measurable simple functions converging to f; the general function f can be expressed as a difference of two nonnegative functions. For a more detailed proof see [19].

In virtue of (1), we can investiagte $(f' \circ S^i)$ instead of $(f \circ T^i)$. As we shall see in Section 3, the assumptions of the limit theorems which we use here, are preserved in $(\boldsymbol{R}^Z, \mathscr{B}^Z, S, \nu)$. The advantage of the dynamical system $(\boldsymbol{R}^Z, \mathscr{B}^Z, S, \nu)$ is in the existence of an ergodic decomposition of the measure ν. Let $\mathscr{I}'$ denote the σ-algebra of all $A \in \mathscr{B}^Z$ such that $A = SA$.

Proposition 1. *There exists a family $(m_x; x \in \boldsymbol{R}^Z)$ of r.c.p. induced by $\mathscr{I}'$ with respect to ν where each m_x is an S-invariant and ergodic probability measure.*

The existence of a family of r.c.p. is a well known fact, see [12]. The invariance and the ergodicity of the measures m_x is proved in [16].

Let $\mathscr{G}$ be a countable algebra generating $\mathscr{B}^Z$. Put

$$D_x = \{y \in \boldsymbol{R}^Z: (1/n) \sum_{j=1}^{n} \chi_A(S^j y) \to m_x(A) \quad \text{for each} \quad A \in \mathscr{G}\} .$$

Each D_x is $\mathscr{I}'$ – (hence $\mathscr{B}^Z$ –) measurable and following the Birkhoff ergodic theorem (see [1]), $m_x(D_x) = 1$. For a.e. (μ) $\omega \in \Omega$ we thus have $\psi(\omega) \in D_x$ for some x. The realization of the process $(f \circ T^i)$ at ω can be described in terms of a dynamical system $(D_x, \mathscr{B}^Z \cap D_x, S/D_x, m_x/D_x)$ where $\mathscr{B}^Z \cap D_x = \{A \cap D_x: A \in \mathscr{B}^Z\}$, S/D_x is the restriction of S onto D_x, and m_x/D_x is the restriction of m_x onto $\mathscr{B}^Z \cap D_x$; the measure m_x/D_x is ergodic.

However, there need not exist any ergodic decomposition of the measure μ, even if $\mathscr{A}$ is a separable σ-algebra. Each $\psi^{-1}(D_x)$ is $\mathscr{I}$-measurable and $T/\psi^{-1}(D_x)$ is a 1-1 mapping of $\psi^{-1}(D_x)$ onto itself, bimeasurable with respect to $\mathscr{A} \cap \psi^{-1}(D_x)$. Nevertheless, there need not exist any invariant probability measure on $(\psi^{-1}D_x, \mathscr{A} \cap \psi^{-1}D_x))$ (see [20]).

The existence of the measures m_x can be derived using the Krylov-Bogoliubov theory without any reference to the measure μ, see [13].

2. MAIN THEOREMS

We shall present several limit theorems now. Theorems 1A–5A are given with proofs in [7] and all of them use the assumption that μ is an ergodic probability measure. Theorems 1B–5B are generalizations of the previous ones to an arbitrary T-invariant probability measure μ (i.e. such that $\mu T^{-1} = \mu$). In Theorems 1C–5C we suppose that an ergodic decomposition $(m_\omega; \omega \in \Omega)$ of measure μ exists. The measures m_ω are refered to as the ergodic components of μ. We shall deal with the problem whether – with assumptions of Theorems 1B–5B fulfilled – the limit theorems which hold with respect to the measure μ (due to Theorems 1B–5B) hold with respect to its ergodic components as well.

First, let us introduce several definitions and the necessary notation. Let λ be a probability measure on $(\Omega, \mathscr{A})$ and let $\mathscr{D} \subset \mathscr{A}$ be a σ-algebra. By $L^1(\mathscr{D}, \lambda)$, $L^2(\mathscr{D}, \lambda)$ we denote respectively the Banach and the Hilbert space of all measurable functions f on Ω for which $\int |f| \, d\lambda < \infty$, $\int f^2 \, d\lambda < \infty$ and there exists a $\mathscr{D}$-measurable function g such that $f = g$ a.s. (λ); functions which are equal a.s. (λ) are considered equal. We shall sometimes write $L^1(\mathscr{D})$, $L^2(\mathscr{D})$ respectively, instead of $L^1(\mathscr{D}, \mu)$, $L^2(\mathscr{D}, \mu)$. A σ-algebra $\mathscr{M} \subset \mathscr{A}$ is said to be (T)-invariant if $\mathscr{M} \subset T^{-1}\mathscr{M}$. The orthogonal projection onto $L^2(T^{-i-1}\mathscr{M}) \ominus L^2(T^{-i}\mathscr{M})$ is denoted by P_i and called the difference projection operator (d.p.o.) generated by $T^{-i}\mathscr{M}$ in $L^2(\mathscr{A}, \mu)$, $i \in \mathbf{Z}$; notice that for $f \in L^2(\mathscr{A})$,

$$P_i f = \mathsf{E}(f \mid T^{-i-1}\mathscr{M}) - \mathsf{E}(f \mid T^{-i}\mathscr{M}) \quad \text{a.s.} \quad (\mu) .$$

In $L^2((\mathscr{A}), m_\omega)$ where m_ω is an ergodic component of μ, the invariant σ-algebra $T^{-i}\mathscr{M}$ generates the d.p.o. P_i^ω.

In the sequel, we shall use the notation

$$s_n(f) = \sum_{j=1}^{n} \frac{f \circ T^j}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$

Theorem 1.A. (P. Billingsley, I. A. Ibragimov) *Let μ be an ergodic measure and let $(f \circ T^i)$ be a sequence of square integrable martingale differences. Then the measures $\mu s_n^{-1}(f)$ weakly converge to a distribution with the characteristic function* $\exp(-\frac{1}{2}\sigma^2 t^2)$, *where* $\sigma^2 = \mathsf{E}f^2$.

Theorem 1.A is one of the first limit theorems for strictly stationary (not independent) processes. It was proved independently by P. Billingsley ([2]) in 1961 and by I. A. Ibragimov ([11]) in 1963.

Theorem 1.B. (C. C. Heyde) *Let $(f \circ T^i)$ be a sequence of square integrable martingale differences. Then the measures $\mu s_n^{-1}(f)$ weakly converge to a distribution with the characteristic function* $\mathsf{E} \exp(-\frac{1}{2}\eta^2 t^2)$, *where* $\eta^2 = \mathsf{E}(f^2 \mid \mathscr{I})$ *a.s.* (μ).

Theorem 1.B is a corollary to Theorem 3.2 from [7], as noticed in the fifth chapter of [7].

Theorem 1.C. *Let an ergodic decomposition* $(m_\omega;\ \omega \in \Omega)$ *of the measure* μ *exist, and let* $(f \circ T^i)$ *be a sequence of square integrable martingale differences. Then for a.e.* (μ) *measure* m_ω, *the measures* $m_\omega s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\exp\left(-\frac{1}{2}\eta^2(\omega)\, t^2\right)$.

Theorem 2.A. (M. I. Gordin) *Let* μ *be an ergodic measure and let* Q *be the set of all* $g \in L^2(\mathscr{A})$ *for which an invariant* σ*-algebra* $\mathscr{M} \subset \mathscr{A}$ *and a positive integer* n *such that* $g \in L^2(T^{-n}\mathscr{M}) \ominus L^2(T^n\mathscr{M})$ *exist.*

If $f \in L^2(\mathscr{A})$ *and*

(2)
$$\inf_{g \in Q} \limsup_{n \to \infty} \mathsf{E} s_n^2(f - g) = 0\,,$$

then there exists a limit $\sigma^2 = \lim_{n \to \infty} \mathsf{E} s_n^2(f)$, *and the measures* $\mu s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\exp\left(-\frac{1}{2}\sigma^2\, t^2\right)$.

Theorem 2.A was given by M. I. Gordin in 1969 (see [5]).

Theorem 2.B. (G. K. Eagleson, D. Volný) *Let* $f \in L^2(\mathscr{A})$ *satisfy Condition* (2). *Then there exists a limit (in the sense of the norm of* $L^1(\mathscr{A})$) $\eta^2 = \lim_{n \to \infty} \mathsf{E}(s_n^2(f) \mid \mathscr{I})$, *and the measures* $\mu s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\mathsf{E} \exp\left(-\frac{1}{2}\eta^2\, t^2\right)$.

In [3] G. K. Eagleson gave a theorem claiming that if $f \in L^2(\mathscr{A})$ satisfies Condition (2), then there exists an $\mathscr{I}$-measurable function η, $\eta > 0$ a.s. (μ), and the measures $\mu s_n^{-1}(f/\eta)$ weakly converge to the standard normal distribution $\mathsf{N}(0, 1)$. It is a matter of fact, however, that without additional assumptions, the inequality $\eta > 0$ a.s. (μ) is not guaranteed. Theorem 2.B which is an improvement of Eagleson's contribution (also in calculating η^2), was given in [14], [15]. The proof was done by a technique different from that used by G. K. Eagleson.

Theorem 2.C. *There exists a dynamical system* $(\Omega, \mathscr{A}, T, \mu)$ *and* $f \in L^2(\mathscr{A})$ *such that an ergodic decomposition* $(m_\omega;\ \omega \in \Omega)$ *of the measure* μ *exists, the function* f *satisfies Condition* (2), *and for each* m_ω *the sequence* $m_\omega s_n^{-1}(f)$ *has at least two different weak limit points.*

Theorem 3.A. (C. C. Heyde) *Let the measure* μ *be ergodic, let* $\mathscr{M} \subset \mathscr{A}$ *be an invariant* σ*-algebra, and* P_0 *be the difference projection operator generated by* $\mathscr{M}$. *If* $f \in L^2(\mathscr{A})$,

(3)
$$\text{the sum} \quad g = \sum_{i \in \mathbf{Z}} P_0(f \circ T^i) \quad \text{converges in} \quad L^2(\mathscr{A})\,,$$

and

(4) $$\mathsf{E}g^2 = \limsup_{n\to\infty} \mathsf{E}s_n^2(f)\,,$$

then the measures $\mu s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\exp\left(-\frac{1}{2}\sigma^2 t^2\right)$.

This theorem was given by C. C. Heyde in 1974, see [9].

Theorem 3.B. *Let* $f \in L^2(\mathscr{A})$, *and let Conditions* (3), (4) *hold. Then* $\mathsf{E}(s_n^2(f) \mid \mathscr{I}) \xrightarrow[n\to\infty]{} \eta^2$ *in* $L^1(\mathscr{A})$, *and the measures* $\mu s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\mathsf{E}\exp\left(-\frac{1}{2}\eta^2 t^2\right)$.

Theorem 3.C. *There exists a dynamical system* $(\Omega, \mathscr{A}, T, \mu)$ *and* $f \in L^2(\mathscr{A})$ *such that an ergodic decomposition* $(m_\omega;\ \omega \in \Omega)$ *of the measure* μ *exists,* f *satisfies Conditions* (3) *and* (4), *and the sequence* $m_\omega s_n^{-1}(f)$ *has at least two different weak limit points for each* m_ω.

Theorem 4.A. (M. I. Gordin) *Let the measure* μ *be ergodic and let* f *be an integrable function (not necessarily square integrable). If*

(5) $$\sum_{k=0}^{\infty} \{\mathsf{E} \mid \mathsf{E}(f \mid T^k\mathscr{M})| + \mathsf{E}\,|f - \mathsf{E}(f \mid T^{-k}\mathscr{M})|\} < \infty$$

and

(6) $$\limsup_{n\to\infty} (1/\sqrt{n})\, \mathsf{E}|S_n| = \lambda\,, \quad \textit{where} \quad S_n = \sum_{j=1}^{n} f \circ T^j\,,$$

then $\lim_{n\to\infty} (1/\sqrt{n})\, \mathsf{E}|S_n| = \lambda$ *exists for some* λ, $0 \leqq \lambda < \infty$, *and the measures* $\mu s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\exp\left(-\frac{1}{2}\pi\,\frac{1}{2}\lambda^2 t^2\right)$.

Theorem 4.A was presented by M. I. Gordin at the Vilnius Conference on Probability and Statistics in 1973. Proceedings from the conference with the original Gordin's proof seem to be hardly available outside the USSR. The theorem is quoted in [7]; the proof from [7] is corrected in [4].

Theorem 4.B. *Let* f *be an integrable function and let* (5), (6) *hold, or*

(7) $$\sum_{k=0}^{\infty} \{\mathsf{E}(|\mathsf{E}(f|T^k\mathscr{M})| \mid \mathscr{I}) + \mathsf{E}(|f - \mathsf{E}(f|T^{-k}\mathscr{M})| \mid \mathscr{I})\} < \infty \quad \textit{a.s.} \quad (\mu)\,,$$

and

(8) $$\limsup_{n\to\infty} (1/\sqrt{n})\, \mathsf{E}(|S_n| \mid \mathscr{I}) < \infty \quad \textit{a.s.} \quad (\mu)\,.$$

Then there exists a limit $\eta = \lim_{n\to\infty} (1/\sqrt{n})\, \mathsf{E}(|S_n| \mid \mathscr{I})$ *in* $L^1(\mathscr{A})$, *and the measures* $\mu s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\mathsf{E}\exp\left(-\frac{1}{2}\pi\frac{1}{2}\eta^2 t^2\right)$.

Theorem 4.C. *Let an ergodic decomposition* $(m_\omega;\ \omega \in \Omega)$ *of the measure* μ *exist, and let* $f \in L^1(\mathscr{A}, \mu)$ *satisfy the assumption of Theorem 4.B. Then for almost all* (μ) m_ω *we have* $(1/\sqrt{n})\, \mathsf{E}_{m_\omega}(|S_n| \mid \mathscr{I}) \to \eta(\omega)$ *in* $L^1(\mathscr{A}, m_\omega)$, *and the measures* $m_\omega\, s_n^{-1}(f)$ *weakly converge to a distribution with the characteristic function* $\exp\left(-\frac{1}{2}\pi\eta^2(\omega)\, t^2\right)$.

Eventually, we shall deal with Heyde's functional limit theorem. Let us abbreviate

$$X_n = f \circ T^n\,, \quad S_n = \sum_{j=1}^{n} X_j$$

(as in Theorem 4.A), and $\sigma_n^2 = \mathsf{E}(S_n^2 \mid \mathscr{I})$. Let $g = \sup\{n\colon \sigma_n^2 \leqq \mathrm{e}\}$. Notice that for μ ergodic the functions σ_n and g are constant almost surely. In Theorem 5.A we shall consider them as numbers.

We define

$$\begin{aligned}\theta_n(\mathrm{t}) &= \sigma_n^{-1}(S_k + (n\,\mathrm{t} - k)\, X_{k+1}) \quad \text{for} \quad \sigma_n^2 > 0\,,\\ &= \quad 0 \qquad\qquad\qquad \text{otherwise}\,,\\ &\qquad k \leqq n\,\mathrm{t} \leqq k + 1\,, \quad k = 0, 1, \ldots, n - 1\,;\\ \eta_n(\mathrm{t}) &= [\zeta(\sigma_n^2)]^{-1}\,(S_k + (n\,\mathrm{t} - k)\, X_{k+1}) \quad \text{if} \quad n > g\,,\\ &= \quad 0 \qquad\qquad\qquad \text{otherwise}\,,\end{aligned}$$

$$k \leqq n\,\mathrm{t} \leqq k + 1\,, \quad k = 0, 1, \ldots, n - 1\,, \quad \text{where} \quad \zeta(\mathrm{t}) = (2\mathrm{t} \log\log \mathrm{t})^{1/2}\,,$$

$$\mathrm{e} < \mathrm{t} < \infty\,.$$

The functions θ_n and η_n belong to $\mathsf{C} = \mathsf{C}[0, 1]$, the space of continuous functions on $[0, 1]$. By W we denote the Wiener measure on C as well as the standard Brownian motion process. Let K be the set of all absolutely continuous $\tau \in \mathsf{C}$ such that $\tau(0) = 0$, $\int_0^1 [(\tau'(\mathrm{t}))]^2\, \mathrm{dt} \leqq 1$ where τ' denotes the derivative of τ (which is determined almost everywhere with respect to the Lebesgue measure).

Theorem 5.A. (C. C. Heyde) *Let the measure* μ *be ergodic, let the* σ*-algebra* $\mathscr{M}$ *and the operator* P_0 *be defined as in Theorem 3.A. Let* $f \in L^1(\mathscr{A})$ *and* $x_l = P_0(f \circ T^l)$, $l \in \boldsymbol{Z}$. *If*

$$\text{(9)} \qquad \mathsf{E}(f \mid \mathscr{M}_\infty) = f \quad a.s. \quad (\mu)\,, \quad \mathsf{E}(f \mid \mathscr{M}_{-\infty}) = 0 \quad a.s. \quad (\mu)\,,$$

where $\mathscr{M}_{-\infty} = \cap\, T^i\mathscr{M}$ *and* $\mathscr{M}_\infty$ *is the smallest* σ*-algebra containing all* $T^i\mathscr{M}$, *and*

$$\text{(10)} \qquad \sum_{m=1}^{\infty} \{\limsup_{n\to\infty} \mathsf{E}(\sum_{l=m}^{n} x_l)^2 + \limsup_{n\to\infty} \mathsf{E}(\sum_{l=m}^{n} x_{-l})^2\} < \infty\,,$$

then $\lim_{n\to\infty} \sigma_n/n^{1/2} = \sigma$ *exists and* $0 \leqq \sigma < \infty$. *If* $\sigma > 0$, *then the measures* $\mu\theta_n^{-1}$ *weakly converge to* W. *Also,* $g < \infty$, $\{\eta_n\colon n = 1, 2, \ldots\}$ *is relatively compact, and the set of its limit points coincides with* K *almost surely.*

Theorem 5.B. *Let $f \in L^2(\mathscr{A})$ and let either* (9), (10) *or* (9), (11) *hold;*

$$(11) \qquad \sum_{m=1}^{\infty} \{\limsup_{n\to\infty} \mathsf{E}((\sum_{l=m}^{n} x_l)^2 \mid \mathscr{I}) + \limsup_{n\to\infty} \mathsf{E}((\sum_{l=m}^{n} x_{-l})^2 \mid \mathscr{I})\} < \infty \ a.s.\ (\mu)\,.$$

Then $\lim \sigma_n/n^{1/2} = \sigma$ exists with σ an $\mathscr{I}$-measurable function, $0 \leqq \sigma < \infty$. If $\sigma > 0$ a.s. (μ), then the measures $\mu\theta_n^{-1}$ weakly converge to W. Also, $g < \infty$ a.s. (μ), $\{\eta_n: n = 1, 2, \ldots\}$ is relatively compact, and the set of its limit points coincides with K a.s. (μ).

Theorem 5.C. *Let $(m_\omega; \omega \in \Omega)$ be an ergodic decomposition of μ, $f \in L^2(\mathscr{A})$, and let the assumptions of Theorem 5.B. be fulfilled. Then for almost all (μ) m_ω, the conclusions of Theorem 5.A hold for m_ω in the place of the measure μ.*

3. PROOFS

Let $\mathscr{M} \subset \mathscr{A}$ be an invariant σ-algebra and let P_i be the difference projection operator (d.p.o.) generated by $T^{-i}\mathscr{M}$ in $L^2(\mathscr{A}, \mu)$. For $f \in L^2(\mathscr{A}, \mu)$,

$$(12) \qquad P_i f = \mathsf{E}(f \mid T^{-i-1}\mathscr{M}) - \mathsf{E}(f \mid T^{-i}\mathscr{M}) \quad \text{a.s.} \quad (\mu)\,.$$

We can easily see that for an arbitrary σ-algebra $\mathscr{D} \subset \mathscr{A}$,

$$(13) \qquad \mathsf{E}(f \mid \mathscr{D}) \circ T = \mathsf{E}(f \circ T \mid T^{-1}\mathscr{D}) \quad \text{a.s.}\ (\mu)$$

(see [14]). So, taking $T^{-i}\mathscr{M}$ for $\mathscr{D}$, we have

$$(14) \qquad (P_i f) \circ T = P_{i+1}(f \circ T) \quad \text{a.s.}\ (\mu)\,, \quad i \in \boldsymbol{Z}\,.$$

The last equality implies that if $f = P_i f$ for some $i \in \boldsymbol{Z}$, then $f, f \circ T, \ldots$ is a sequence of martingale differences. On the other hand, the σ-algebra generated by $f \circ T^i$, $i < 0$, is invariant, so for each martingale difference sequence $(f \circ T^i)$ there exists a d.p. o. P_0 such that $f = P_0 f$.

Let an ergodic decomposition $(m_\omega; \omega \in \Omega)$ of measure μ exist. We shall show that if some relatively weak assumptions are fulfilled, the relations of the process $(f \circ T^i)$ to the σ-algebras $T^i\mathscr{M}$ are preserved in $L^2(\mathscr{A}, m_\omega)$ for almost all (μ) ergodic components m_ω of μ.

By $\mathscr{M}_\infty$ we denote the smallest σ-algebra containing all $T^i\mathscr{M}$, $i \in \boldsymbol{Z}$.

Lemma 1. *Let $\mathscr{M}$ be an invariant and separable σ-algebra and let $f \in L^1(\mathscr{M}_\infty, \mu)$. Then*

$$\mathsf{E}_{m_\omega}(f \mid \mathscr{M}) = \mathsf{E}(f \mid \mathscr{M}) \quad \text{a.s.} \quad (m_\omega)$$

for almost all (μ) ergodic components m_ω of μ.

Proof. By Birkhoff's pointwise ergodic theorem, $(1/n) \sum_{j=1}^{n} g \circ T^j \to \mathsf{E}(g \mid \mathscr{I})$ a.s. (μ) for each $g \in L^1(\mathscr{M}_\infty, \mu)$, hence $\mathsf{E}(g \mid \mathscr{I}) = \mathsf{E}(g \mid \mathscr{I} \cap \mathscr{M}_\infty)$ a.s. (μ). Let $\mathscr{G}$ be a countable algebra of sets from $\mathcal{M}$ which generates $\mathcal{M}$. Following [16], Theorem 3, we have

$$\chi_A = \mathsf{E}(\chi_A \mid \mathcal{M}) \quad \text{a.s.} \quad (\mu)$$

for each $A \in \mathcal{I} \cap \mathcal{M}_\infty$. Hence, for $B \in \mathcal{G}$, we have $\mathsf{E}(\chi_B\, \mathsf{E}(f \mid \mathcal{M}) \mid \mathcal{I}) = \mathsf{E}(\mathsf{E}(\chi_B f \mid \mathcal{M}) \mid \mathcal{I} \cap \mathcal{M}_\infty) = \mathsf{E}(\mathsf{E}(\chi_B f \mid \mathcal{M}) \mid \mathcal{I} \cap \mathcal{M}) = \mathsf{E}(\chi_B f \mid \mathcal{I} \cap \mathcal{M}) = \mathsf{E}(\chi_B f \mid \mathcal{I} \cap \mathcal{M}_\infty) = \mathsf{E}(\chi_B f \mid \mathcal{I})$ a.s. (μ).

Thus,

$$\int_B \mathsf{E}(f \mid \mathcal{M})\, \mathrm{d}m_\omega = \int_B f\, \mathrm{d}m_\omega\,, \quad B \in \mathcal{G}\,,$$

for a.e. (μ) ergodic component m_ω of μ, q.e.d.

As a corollary to Lemma 1 and to (12) we get:

Lemma 2. *Let $\mathcal{M} \subset \mathcal{A}$ be a separable and invariant σ-algebra, let P_i be the d.p.o. generated by $T^{-i}\mathcal{M}$ in $L^2(\mathcal{A}, \mu)$, and P_i^ω the d.p.o. generated by $T^{-i}\mathcal{M}$ in $L^2(\mathcal{A}, m_\omega)$, $i \in \mathbf{Z}$. For $f \in L^2(\mathcal{M}_\infty, \mu)$ we have*

$$P_i f = P_i^\omega f \quad \text{a.s.} \quad (m_\omega)$$

for a.e. (μ) ergodic component m_ω of μ.

Proof of Theorem 1.C. Let $\mathcal{M}$ be the smallest σ-algebra with respect to which the functions $f \circ T^{-n}$, $n = 1, 2, \ldots$ are measurable. $\mathcal{M}$ is invariant and separable. If f satisfies the assumptions of Theorem 1.C, then $f = P_0 f$ a.s. (μ), where P_0 is the d.p.o. generated by $\mathcal{M}$. According to Lemma 2, $f = P_0^\omega f$ a.s. (m_ω) for almost all (μ) m_ω, so $(f \circ T^i)$ is a martingale difference sequence in almost every (μ) space $L^2(\mathcal{A}, m_\omega)$. Now, the theorem follows from Theorem 1.A.

In the proofs of Theorems 4.C, 5.C, we shall replace an invariant σ-algebra $\mathcal{M}$ by a separable and invariant σ-algebra $\mathcal{M}^\wedge \subset \mathcal{M}$. This will be possible by virtue of Lemma 3.

Lemma 3. *Let $\mathcal{C} \subset \mathcal{A}$ be a separable σ-algebra, $T^{-1}\mathcal{C} = \mathcal{C}$, and let $\mathcal{M} \subset \mathcal{A}$ be an invariant σ-algebra. Then there exists a separable and invariant algebra $\mathcal{M}^\wedge \subset \mathcal{M}$ such that for each $g \in L^1(\mathcal{C}, \mu)$ and $i \in \mathbf{Z}$,*

$$\mathsf{E}(g \mid T^i\mathcal{M}^\wedge) = \mathsf{E}(g \mid T^i\mathcal{M}) \quad a.s. \quad (\mu)\,.$$

Proof. Let $\mathcal{G} \subset \mathcal{C}$ be a countable algebra of sets which generates $\mathcal{C}$. We define $\mathcal{M}^\wedge$ as the smallest invariant σ-algebra with respect to which the functions $\mathsf{E}(\chi_A \mid \mathcal{M})$, $A \in \mathcal{G}$, are measurable. $\mathcal{M}^\wedge$ is separable and with respect to each $\mathcal{G}$-measurable simple function h, $\mathsf{E}(h \mid \mathcal{M}^\wedge) = \mathsf{E}(h \mid \mathcal{M})$ a.s. (μ). Each $h \in L^1(\mathcal{C}, \mu)$ is a limit of $\mathcal{G}$-measurable simple functions in $L^1(\mathcal{A}, \mu)$, hence $\mathsf{E}(h \mid \mathcal{M}^\wedge) = \mathsf{E}(h \mid \mathcal{M})$ a.s. (μ) for each $h \in L^1(\mathcal{C}, \mu)$. For every $i \in \mathbf{Z}$, the function $h \circ T^i$ is $\mathcal{C}$-measurable (and integrable) as well. Hence, according to (13),

$$\mathsf{E}(h \mid T^i \mathcal{M}^\wedge) = \mathsf{E}(h \circ T^i \mid \mathcal{M}^\wedge) \circ T^{-i} = \mathsf{E}(h \circ T^i \mid \mathcal{M}) \circ T^{-i} =$$
$$= \mathsf{E}(h \mid T^i \mathcal{M}) \quad \text{a.s.} \quad (\mu)\,.$$

This completes the proof.

Proof of Theorem 4.C. Let (7), (8) hold. By Lemma 3, we can suppose that $\mathcal{M}$ is a separable σ-algebra. Hence, by Lemma 1, (7) and (8) are fulfilled in a.s. (μ) all spaces $L^1(\mathscr{A}, m_\omega)$. Now, the conclusion of Theorem 4.C follows from the ergodicity of m_ω and from Theorem 4.A.

Let us start from (5), (6). Using the proof from [7] for the ergodic version of the theorem, we can see that the function f can be expressed as

$$f = h + g - g \circ T\,, \tag{15}$$

where $g, h \in L^1(\mathscr{A}, \mu)$, and $(h \circ T^i)$ is a martingale difference sequence. In the proof of the ergodic case it is shown that $h \in L^2(\mathscr{A}, \mu)$, and that the limit behaviour of $s_n(f)$ is the same as that of $s_n(h)$, where $\mu s_n^{-1}(h)$ converge by Theorem 1.A, see [7], [4]. We shall show that in the non ergodic case, the same development is possible, with respect to the ergodic components m_ω of μ.

We shall prove that $h \in L^2(\mathscr{A}, m_\omega)$ for a.e. (μ) ergodic component m_ω of μ. (Equality (15) is preserved as well.) Birkhoff's pointwise ergodic theorem implies that there exists a $[0, \infty]$-valued function h',

$$(1/n) \sum_{j=1}^{n} h^2 \circ T^j \to h' \quad \text{a.s.} \quad (\mu)\,.$$

From Burkholder's inequality (compare [4]) we get

$$\mu\{(1/n) \sum_{j=1}^{n} h^2 \circ T^j > \lambda\} \leqq (c/\lambda)\, \mathsf{E}|S_n/\sqrt{n}| \quad \text{for} \quad n = 1, 2, \ldots, \quad \lambda > 0\,;$$

c is a positive constant. Hence, $\mu\{h' > \lambda\} \leqq (c/\lambda) \limsup_{n\to\infty} (1/\sqrt{n})\, \mathsf{E}|S_n|$. Using $\mu\{h' > \lambda\} = \int m_\omega\{h' > \lambda\}\, \mathrm{d}\mu(\omega)$ we get $m_\omega\{h' = \infty\} = 0$ for a.e. (μ) measure m_ω. The measures m_ω are ergodic and $m_\omega\{h' = \infty\} = 0$ yields $h \in L^2(\mathscr{A}, m_\omega)$; if h^2 were not integrable w.r. to m_ω, then from Birkhoff's theorem $(1/n) \sum_{j=0}^{n} h^2 \circ T^j \to \infty = h'$ a.s. (m_ω). According to Lemma 3 we can suppose that $\mathcal{M}$ is a separable σ-algebra. By Lemma 2, $(h \circ T^i)$ is a sequence of square integrable martingale differences in $L^2(\mathscr{A}, m_\omega)$. Using this fact, (15) and the ergodicity of the measures m_ω, we can derive the limit theorem in the same way as in [7].

Proof of Theorem 5.C. According to Lemma 3 we can suppose that $\mathcal{M}$ is a separable σ-algebra.

Let the conditions (9), (11) be fulfilled. By Lemma 1 and Lemma 2, Conditions (9), (10) are fulfilled in a.e. (μ) space $L^2(\mathscr{A}, m_\omega)$, with m_ω replacing the measure μ. The conclusion of Theorem 5.C. thus follows from the ergodicity of m_ω and from Theorem 5.A.

Let us suppose that Condition (9), (10) are fulfilled. By [7], pp. 141 – 142, there exist functions $g, h \in L^2(\mathscr{A}, \mu)$ such that

$$f = h + g - g \circ T, \tag{16}$$

where $(h \circ T^i)$ is a martingale difference sequence. In the ergodic case, the theorem can be derived from (16), see [10], [7]. (The difference between (15) and (16) is in the square integrability of g.) In the non ergodic case, according to Lemma 3 and Lemma 2, $(h \circ T^i)$ is a martingale difference sequence in $L^2(\mathscr{A}, m_\omega)$ for a.e. (μ) measure m_ω (see the proof of Theorem 1.C), and the theorem can be derived from (16) as in the previous case.

When deriving Theorems 1.B, 4.B and 5.B we first reduce the situation to a case where an ergodic decomposition of measure μ exists. The desired results then follow from Theorems 1.C, 4.C, 5.C, and Propositions 2, 3.

Proposition 2. *Let an ergodic decomposition $(m_\omega;\ \omega \in \Omega)$ of measure μ exist. Let $f_1, f_2, \dots$ be measurable functions, and let for a.e. (μ) measure m_ω, $m_\omega f_n^{-1}$ weakly converge to a distribution with a characteristic function $\varphi_\omega(t)$. Then the measures μf_n^{-1} weakly converge to a distribution with the characteristic function $\varphi(t) = \int \varphi_\omega(t)\, d\mu(\omega)$.*

Remark. Due to Theorem 2.C and Theorem 3.C, the opposite of Proposition 2 does not hold.

Proof. By the Lebesgue dominated converegence theorem we have $\mathsf{E} \exp(\imath t f_n) = \iint \exp(\imath t f_n)\, dm_\omega\, d\mu(\omega) \xrightarrow[n\to\infty]{} \int \varphi_\omega(t)\, d\mu(\omega)$.

Proposition 3. *Let $g_1, g_2, \dots$ be $\mathsf{C}[0, 1]$-valued random variables. Let for a.e. (μ) measure m_ω, the measures $m_\omega g_n^{-1}$ weakly converge to the Wiener measure W. Then μg_n^{-1} weakly converge to W.*

Proof. We shall view g_n as a function $g_n(t, \omega)$ of t and ω. It is sufficient to prove the convergence of finite dimensional distributions and the tightness of the set $\{\mu g_n^{-1} : n = 1, 2, \dots\}$. (See [7].) A proof of the convergence of finite dimensional distributions of g_n to those of the Brownian motion can be done in the same way in which we proved Proposition 2 (see [18] for details). Tightness of μg_n^{-1} can be expressed by (17), (18), compare [7]:

$$\sup_n \mu\{|g_n(0, \cdot)| > \lambda\} \xrightarrow[\lambda\to\infty]{} 0\,. \tag{17}$$

(18) For each $\varepsilon > 0$, $\sup_n \mu\{\sup_{|s-t|<\delta} |g_n(s, \cdot) - g_n(t, \cdot)| > \varepsilon\} \xrightarrow[\delta \downarrow 0]{} 0\,.$

Let $\varepsilon > 0$ be fixed. For $\tau > 0$ we put

$$A(\lambda, \tau) = \mu\{\omega\colon \sup_n m_\omega\{|g_n(0, \cdot)| > \lambda\} < \tau\}\,,$$

and

$$B(\delta, \tau) = \mu\{\omega: \sup_n m_\omega\{ \sup_{|s-t|<\delta} |g_n(s, \cdot) - g_n(t, \cdot)| > \varepsilon\} < \tau\} .$$

By the assumption and by [7], the measures $m_\omega h_n^{-1}$ are tight (for a.e. (μ) measure m_ω), hence $A(\lambda, \tau) \xrightarrow[\lambda \to \infty]{} 1$ and $B(\delta, \tau) \xrightarrow[\delta \downarrow 0]{} 1$ for each $\tau > 0$. For arbitrarily small $\tau > 0$ we have $\mu\{|g_n(0, \cdot)| > \lambda\} \leqq \tau + 1 - A(\lambda, \tau)$ and $\mu\{ \sup_{|s-t|<\delta} |g_n(s, \cdot) - g_n(t, \cdot)| > > \varepsilon\} \leqq \tau + 1 - B(\delta, \tau)$. Hence, (17) and (18) are fulfilled for the measure μ, which completes the proof.

Proofs of Theorems 1.B, 4.B and 5.B. In Theorems 4.B and 5.B we can (according to Lemma 3) find a separable invariant σ-algebra $\mathscr{M}^\wedge \subset \mathscr{M}$ such that the assumptions are fulfilled with $\mathscr{M}^\wedge$ replacing $\mathscr{M}$. Hence, there exists a separable σ-algebra $\mathscr{C} \subset \mathscr{A}$ such that $T\mathscr{C} = \mathscr{C}$, $\mathscr{M}^\wedge \subset \mathscr{C}$, and f is $\mathscr{C}$-measurable. From Birkhoff's pointwise ergodic theorem it follows that $\mathsf{E}(g \mid \mathscr{I})$ is $\mathscr{C}$-measurable for each integrable and $\mathscr{C}$-measurable function g. In Theorem 1.B. we can take the σ-algebra generated by $f \circ T^i$, $i \in \boldsymbol{Z}$ for $\mathscr{C}$. As we have seen in the first section, there exists a mapping $\psi: \Omega \to \boldsymbol{R}^Z$ such that $\mathscr{C} = \psi^{-1}(\mathscr{B}^Z)$ and $S \circ \psi = \psi \circ T$, where S is the shift on $\boldsymbol{R}^Z$ defined by $(Sx)_i = x_{i+1}$, $i \in \boldsymbol{Z}$, $x \in \boldsymbol{R}^Z$, $\mathscr{B}^Z$ is the Borel σ-algebra on $\boldsymbol{R}^Z$. Let $\nu = \mu\psi^{-1}$. Then $\nu S^{-1} = \nu$, hence $(\boldsymbol{R}^Z, \mathscr{B}^Z, S, \nu)$ is a dynamical system. As we have already stated in Section 1, for each measurable function f on Ω there exists a $\mathscr{B}^Z$-measurable function f' on $\boldsymbol{R}^Z$ such that (1) $f = f' \circ \psi$.

It can be easily seen that for $g \in L^1(\mathscr{A}, \mu)$, $g = g' \circ \psi$, where g' is a $\mathscr{B}^Z$-measurable function, and a σ-algebra $\mathscr{D}' \subset \mathscr{B}^Z$, we have

$$\mathsf{E}(g \mid \psi^{-1}(\mathscr{D}')) = \mathsf{E}_\nu(g' \mid \mathscr{D}') \circ \psi \quad \text{a.s.} \quad (\mu) .$$

Hence, if the function f fulfills the assumptions of Theorem 1.B, Theorem 4.B, Theorem 5.B, respectively, then f' fulfills the same assumptions. If the limit theorems hold for f', then they hold for f as well. According to Proposition 1, there exists an ergodic decomposition of the measure ν. Theorems 1.B and 4.B thus follow from Theorems 1.C and 4.C, and from Proposition 2. The functional log log law in Theorem 5.B is a pointwise property, so it follows from Theorem 5.C. The invariance principle is a consequence of Theorem 5.C and Proposition 3.

Theorem 2.B. can be derived from Theorem 1.B via approximating the sums $s_n(f)$ by the sums $s_n(g)$ of square integrable sequences of martingale differences (see [15], [14]); no ergodic decomposition is needed there (except in the proof of Theorem 1.B, of course).

In [7], C. C. Heyde proved that (2) follows from (3), (4). In this way, Theorem 3.A was derived from Theorem 2.A; Theorem 3.B can be derived from Theorem 2.B in the same way.

Counterexamples needed for proofs of Theorem 2.C and Theorem 3.C can be constructed by employing properties of difference projection operators and ergodic decompositions, see [17].

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Souhrn

O NEERGODICKÝCH VERZÍCH LIMITNÍCH VĚT

DALIBOR VOLNÝ

V práci jsou studovány neergodické verze několika známých limitních vět pro striktně stacionární procesy. Ukazuje se, že v některých případech splnění předpokladů věty pro obecnou invariantní míru má za následek splnění předpokladů i vzhledem ke skoro všem jejím ergodickým složkám. V jiných případech toto neplatí a je ukázáno, že věta může pro všechny ergodické složky této míry selhat.

Резюме

О НЕЭРГОДИЧЕСКИХ ВАРИАНТАХ ПРЕДЕЛЬНЫХ ТЕОРЕМ

DALIBOR VOLNÝ

В работе исследованы неэргодические варианты некоторых известных предельных теорем для строго стационарных процессов. В некоторых случаях выполнение предположений теоремы для общей инвариантной меры влечет их выполнение и по отношению к почти всем ее эргодическим компонентам. Однако в других случаях теорема может оказаться неверной для всех эргодических компонент.

Author's address: RNDr. *Dalibor Volný*, CSc., Matematický ústav UK, Sokolovská 83, 186 00 Praha 8.