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NONLINEAR ELLIPTIC PROBLEMS WITH INCOMPLETE DIRICHLET CONDITIONS AND THE STREAM FUNCTION SOLUTION OF SUBSONIC ROTATIONAL FLOWS PAST PROFILES OR CASCADES OF PROFILES

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Dedicated to Professor Jindřich Nečas on the occasion of his sixtieth birthday

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Summary. The paper is devoted to the solvability of a nonlinear elliptic problem in a plane multiply connected domain. On the inner components of its boundary Dirichlet conditions are known up to additive constants which have to be determined together with the sought solution so that the so-called trailing stagnation conditions are satisfied. The results have applications in the stream function solution of subsonic flows past groups of profiles or cascades of profiles.

Keywords: Nonviscous rotational flow past profiles and cascades of profiles, stream function, Kutta-Joukowski trailing stagnation condition, maximum principle, nonlinear elliptic problem, apriori estimates.

AMS subject classification: 35J25, 35Q20, 76G99, 76N10.

INTRODUCTION

In the study of plane, nonviscous, subsonic steady flows past profiles it is convenient to introduce the stream function which leads to a boundary value problem for a second order elliptic equation. However, the stream function is determined on each profile up to an unknown additive constant. In order to complete these boundary conditions, we can prescribe e.g. the velocity circulations along profiles or the mass fluxes per second between the components of the boundary. Nevertheless, it follows from physical considerations and experience that only the solutions satisfying the so-called trailing stagnation conditions model real flows in an appropriate way. It means that in the flow past a profile which is smooth except for a sharp trailing edge we demand the velocity to be bounded. (See e.g. [2].) In technology we often meet also smooth profiles. On the basis of experiments and numerical calculations, we have concluded that for obtaining physically reasonable flows it is sufficient to choose the trailing stagnation point (where the velocity is zero) as a point on the backward part of the profile (with respect to the direction of the flow) with the greatest curvature.

To similar problems we come in the stream function solution of plane flows past profiles in a layer of variable thickness, past cascades of profiles or past axially symmetric rings inserted into an axially symmetric channel, etc. In the papers [4, 5] we studied the solvability of incompressible (irrotational or rotational) flows and of subsonic compressible irrotational flows past smooth profiles with given trailing points. Here we extend the results to a general nonlinear equation which governs compressible rotational flows. We consider more general situation, when the incomplete Dirichlet conditions are combined both with the trailing conditions on some profiles and with prescribed velocity circulations on other profiles.

1. FORMULATION OF THE PROBLEM

By R^k we denote the k-dimensional Euclidean space. The distance of $x \in R^k$ and $x' \in R^k$ will be denoted by $|x - x'|$.

Let $\Omega \subset R^2$ be a bounded, $(r + 1)$-multiply connected domain $(r \geqq 1)$ with the boundary $\partial\Omega$ whose components $C_0, C_1, \ldots, C_r$ are geometric images of simple closed curves. Let $C_i \subset \operatorname{Int} C_0$ (= the bounded component of $R^2 - C_0$) for $i = 1, \ldots, r$. By $\bar{\Omega}$ we denote the closure of Ω. The curves $C_1, \ldots, C_r$ can be considered as profiles inserted into the domain $\operatorname{Int} C_0$.

1.1. **Boundary value problem.** Let functions $\psi_i: C_i \to R^1$, $i = 0, \ldots, r$, points $z_i \in C_i$, $i = 1, \ldots, m$ and constants $\gamma_i \in R^1$, $i = 1, \ldots, r$ be given. We seek a function $u: \bar{\Omega} \to R^1$ (sufficiently smooth) and constants $q_1, \ldots, q_r \in R^1$ satisfying the equation

$$(1.2) \qquad -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x) \quad \text{in} \quad \Omega$$

and the boundary conditions

$$(1.3) \qquad u \mid C_0 = \psi_0 \,,$$

$$(1.4) \qquad u \mid C_i = \psi_i + q_i \,, \quad i = 1, \ldots, r \,,$$

$$(1.5) \qquad \left(b(\cdot, u, \nabla u) \frac{\partial u}{\partial n}\right)(z_i) = \gamma_i \,, \quad i = 1, \ldots, m \leqq r \,,$$

$$(1.6) \qquad \int_{C_i} b(\cdot, u, \nabla u) \frac{\partial u}{\partial n} \, \mathrm{d}s = \gamma_i \,, \quad i = m + 1, \ldots, r \,.$$

Here $\partial/\partial n$ denotes the derivative in the direction of the outer normal to $\partial\Omega$, $\nabla u = (u_{x_1}, u_{x_2})$, $u_{x_i} = \partial u/\partial x_i$. Similarly, we write $u_{x_i x_j} = \partial^2 u/(\partial x_i \, \partial x_j)$. $z_i \in C_i$ are the so-called trailing points, $b = b(x, \xi_0, \xi_1, \xi_2)$: $\bar{\Omega} \times R^3 \to R^1$. We assume that the functions $a_i = a_i(x, \xi)$, $x \in \Omega$, $\xi = (\xi_0, \xi_1, \xi_2) \in R^3$, are continuous for $i = 0, 1, 2$

and continuously differentiable for $i = 1, 2$. Under this assumption equation (1.2) can be written in the form

$$\sum_{i,j=1}^{2} a_{ij}(x, u(x), \nabla u(x))\, u_{x_i x_j}(x) + \tag{1.7}$$
$$+ \sum_{i=1}^{2} b_i(x, u(x), \nabla u(x))\, u_{x_i}(x) = b_0(x, u(x), \nabla u(x)) \quad \text{in} \quad \Omega\,,$$

where

$$a_{ij}(x, \xi) = -\frac{\partial a_i}{\partial \xi_j}(x, \xi)\,, \quad i, j = 1, 2\,, \tag{1.8}$$

$$b_i(x, \xi) = -\frac{\partial a_i}{\partial \xi_0}(x, \xi)\,, \quad i = 1, 2\,,$$

$$b_0(x, \xi) = -a_0(x, \xi) + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x, \xi) + f(x)\,,$$

$$x \in \bar{\Omega}\,, \quad \xi = (\xi_0, \xi_1, \xi_2) \in R^3\,.$$

Because of the nonstandard discrete conditions (1.5) it is impossible to use the usual concept of a weak solution from the Sobolev space $H^1(\Omega)$ and therefore we will consider *classical solutions* of equation (1.7). With respect to this fact we introduce the following

1.9. **Assumptions.** The following conditions hold:

$$\alpha \in (0, 1)\,; \tag{1.10}$$

$$\partial\Omega \in C^{2,\alpha}\,; \tag{1.11}$$

$$\psi_i \in C^{2,\alpha}(C_i)\,, \quad i = 0, \ldots, r\,; \tag{1.12}$$

(1.13) the functions a_{ij} $(i, j = 1, 2)$ and b_i $(i = 0, 1, 2)$ are bounded, Hölder-continuous with respect to x with the exponent α and Lipschitz-continuous with respect to ξ:

$$|a_{ij}(x, \xi)|, |b_i(x, \xi)| \leqq c\,, \quad i, j = 1, 2\,, \quad |b_0(x, \xi)| \leqq c_0 \quad \forall x \in \bar{\Omega}\,, \quad \xi \in R^3\,,$$
$$|a_{ij}(x, \xi) - a_{ij}(y, \xi)| \leqq M_1 |x - y|^\alpha\,, \quad i, j = 1, 2\,,$$
$$|b_i(x, \xi) - b_i(y, \xi)| \leqq M_1 |x - y|^\alpha\,, \quad i = 1, 2\,,$$
$$|b_0(x, \xi) - b_0(y, \xi)| \leqq M_0 |x - y|^\alpha\,, \quad \forall x, y \in \bar{\Omega}\,, \quad \xi \in R^3\,,$$
$$|a_{ij}(x, \xi) - a_{ij}(x, \eta)| \leqq L |\xi - \eta|\,, \quad i, j = 1, 2\,,$$
$$|b_i(x, \xi) - b_i(x, \eta)| \leqq L |\xi - \eta|\,, \quad i = 1, 2\,,$$
$$|b_0(x, \xi) - b_0(x, \eta)| \leqq L_0 |\xi - \eta|\,, \quad \forall x \in \bar{\Omega}\,, \quad \xi, \eta \in R^3\,,$$

with constants c_0, c, M_0, M_1, L_0, L independent of x, y, ξ, η, i, j;

(1.14) there exist constants $\mu, \nu > 0$ such that

$$\mu(\eta_1^2 + \eta_2^2) \leqq \sum_{i,j=1}^{2} a_{ij}(x, \xi)\, \eta_i \eta_j \leqq \nu(\eta_1^2 + \eta_2^2)$$

$$\forall x \in \bar{\Omega}\,, \quad \xi \in R^3\,, \quad \eta_1, \eta_2 \in R^1\,;$$

(1.15) the function $b = b(x, \xi)$ $(x \in \bar{\Omega},\ \xi \in R^3)$ is continuous in $\bar{\Omega} \times R^3$ and

$$0 < \hat{c}_1 \leqq b \leqq \hat{c}_2 < +\infty \quad \text{in} \quad \bar{\Omega} \times R^3\,. \qquad \square$$

The spaces and classes $C, C^k, C^\alpha, C^{k,\alpha}$ etc., are defined e.g. in [1, 8, 9]. By the symbols $\|\cdot\|_{0,0,\bar{\Omega}}$, $\|\cdot\|_{k,0,\bar{\Omega}}$, $\|\cdot\|_{k,\alpha,\bar{\Omega}}$, $\|\cdot\|_{k,\alpha,\partial\Omega}$ we denote the norms in the spaces $C(\bar{\Omega})$, $C^k(\bar{\Omega})$, $C^{k,\alpha}(\bar{\Omega})$, $C^{k,\alpha}(\partial\Omega)$, respectively (here $k \geqq 0$ is integer, $\alpha \in (0, 1)$).

Let us remark that the following assertions hold:

a) *The imbedding* $C^{1,\alpha}(\bar{\Omega}) \subset C^\alpha(\bar{\Omega})$ $(\alpha \in (0, 1))$ *is continuous. Therefore, there exists a constant* $c^* = c^*(\Omega)$ *such that*

$$\|u\|_{0,\alpha,\bar{\Omega}} \leqq c^* \|u\|_{1,\alpha,\bar{\Omega}} \quad \forall u \in C^{1,\alpha}(\bar{\Omega})\,. \tag{1.16}$$

b) *The imbeddings* $C^{2,\alpha}(\bar{\Omega}) \subset C^{1,\alpha}(\bar{\Omega})$ *and* $C^\alpha(\bar{\Omega}) \subset C^\beta(\bar{\Omega})$ *with* $0 < \beta < \alpha < 1$ *are completely continuous. It means that from each sequence* u_n *bounded in* $C^{2,\alpha}(\bar{\Omega})$ $(C^\alpha(\bar{\Omega}))$ *we can choose a subsequence* u_{n_k} *convergent in* $C^{1,\alpha}(\bar{\Omega})$ $(C^\beta(\bar{\Omega}))$.

The subject-matter of this paper is the study of the following

Problem (P). *Find* $u \in C^{2,\alpha}(\bar{\Omega})$ *and constants* $q_1, \ldots, q_r \in R^1$ *satisfying equation* (1.7) *and conditions* (1.3)–(1.6). $\square$

2. ESTIMATES OF SOLUTIONS OF LINEARIZED PROBLEM

The main tools for proving the existence of a solution of Problem (P) are the strong maximum principle and estimates valid for solutions of elliptic equations.

Let us consider a linear elliptic equation

$$Lu \equiv \sum_{i,j=1}^{2} A_{ij}(x)\, u_{x_i x_j}(x) + \sum_{i=1}^{2} A_i(x)\, u_{x_i}(x) = g(x)\,, \quad x \in \Omega \tag{2.1}$$

with

$$A_{ij}, A_i, g \in C^\alpha(\bar{\Omega})\,, \quad \alpha \in (0, 1)\,, \quad A_{ij} = A_{ji}\,, \tag{2.2}$$

satisfying

$$\mu(\eta_1^2 + \eta_2^2) \leqq \sum_{i,j=1}^{2} A_{ij}(x)\, \eta_i \eta_j \leqq \nu(\eta_1^2 + \eta_2^2) \quad \forall x \in \bar{\Omega}\,, \quad \eta_1, \eta_2 \in R^1\,, \tag{2.3}$$

$(\mu, \nu > 0$ are independent of $x, \eta_1, \eta_2)$ and

$$\|A_{ij}\|_{0,\alpha,\bar{\Omega}},\ \|A_i\|_{0,\alpha,\bar{\Omega}} \leqq M\,, \tag{2.4}$$

$$\|g\|_{0,0,\bar{\Omega}} \leqq c_0\,. \tag{2.5}$$

In the following we shall consider *assumptions* (1.10)–(1.12), (2.2)–(2.5).

2.6. **Theorem** (the Schauder estimate). *There exists a constant k_1 dependent on μ, ν, α, M and Ω, i.e. $k_1 = k_1(\mu, \nu, \alpha, M, \Omega)$, such that any solution $u \in C^{2,\alpha}(\bar{\Omega})$ of equation* (2.1) *satisfies the estimate*

$$\|u\|_{2,\alpha,\bar{\Omega}} \leqq k_1[\|g\|_{0,\alpha,\bar{\Omega}} + \|u\|_{2,\alpha,\partial\Omega}] \,. \tag{2.7}$$

For proof see e.g. [1] or [9]. □

2.8. **Theorem.** *Let $u \in C^2(\bar{\Omega})$ be a solution of equation* (2.1) *satisfying*

$$u \mid \partial\Omega = \tilde{\psi} \mid \partial\Omega \tag{2.9}$$

with $\tilde{\psi} \in C^2(\bar{\Omega})$. Then

$$\|u\|_{1,\alpha,\bar{\Omega}} \leqq k_2(\mu, \nu, \alpha, c_0, \|\tilde{\psi}\|_{2,0,\bar{\Omega}}, \Omega) \,. \tag{2.10}$$

Proof is a consequence of results from [9, Ch. III, § 19 and § 1]. □

2.11. **Theorem** (on the solvability of a linear elliptic equation). *Under assumptions* (1.10), (1.11), (2.2), (2.3) *and $\tilde{\psi} \in C^{2,\alpha}(\bar{\Omega})$, problem* (2.1), (2.9) *has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$.*

Proof. See [9], Theorem 1.3 from Ch. III or [1], § 5.7. □

2.12. **Theorem** (strong maximum principle). *Let* (1.10), (1.11), (2.2), (2.3) *be satisfied and let $u \in C^2(\bar{\Omega})$ be a solution of the equation $Lu = 0$ in Ω. Then*:

1. *If u has maximum or minimum in Ω, then u is constant in $\bar{\Omega}$.*
2. *Let us assume that $\hat{x} \in \partial\Omega$ and u is not constant in $\bar{\Omega}$. Then, provided*

$$u(\hat{x}) = \max_{\bar{\Omega}} u \quad or \quad u(\hat{x}) = \min_{\bar{\Omega}} u \,,$$

we have

$$\frac{\partial u}{\partial n}(\hat{x}) > 0 \quad or \quad \frac{\partial u}{\partial n}(\hat{x}) < 0 \,, \quad respectively \,.$$

Proof. See [1] (where the theorem is proved under weaker assumptions). □

From (1.10)–(1.12) we get the existence of $\varphi_0, \ldots, \varphi_r \in C^{2,\alpha}(\bar{\Omega})$ such that

$$\begin{aligned} &\text{a)} \quad \varphi_0 \mid C_i = \psi_i \,, \quad i = 0, \ldots, r \,, \\ &\text{b)} \quad \varphi_i \mid C_j = \delta_{ij} \,, \quad i = 1, \ldots, r \,, \quad j = 0, \ldots, r \,. \end{aligned} \tag{2.13}$$

($\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$.) See e.g. [9].

Let us denote by u_i, $i = 0, \ldots, r$, solutions of the following problems:

$$\begin{aligned} &\text{a)} \quad Lu_0 = g \quad \text{in} \quad \Omega \,, \quad u_0 \mid \partial\Omega = \varphi_0 \mid \partial\Omega \,, \\ &\text{b)} \quad Lu_i = 0 \quad \text{in} \quad \Omega \,, \quad u_i \mid \partial\Omega = \varphi_i \mid \partial\Omega \,, \quad i = 1, \ldots, r \,, \end{aligned} \tag{2.14}$$

where $\varphi_0, \ldots, \varphi_r \in C^{2,\alpha}(\bar{\Omega})$ satisfy (2.13).

2.15. **Theorem.** *Problems* (2.14) *have unique solutions* $u_i \in C^{2,\alpha}(\bar{\Omega})$. *Moreover, there exist constants* $c_1 = c_1(\mu, \nu, \alpha, M, \Omega)$, $c_2 = c_2(\mu, \nu, \alpha, c_0, \|\varphi_0\|_{C^2(\bar{\Omega})}, \Omega)$ *and* $c_3 = c_3(\mu, \nu, \alpha, \Omega)$ *such that*

$$\|u_0\|_{2,\alpha,\bar{\Omega}} \leqq c_1\big[\|g\|_{0,\alpha,\bar{\Omega}} + \|\varphi_0\|_{2,\alpha,\partial\Omega}\big]\,, \tag{2.16}$$

$$\|u_i\|_{2,\alpha,\bar{\Omega}} \leqq c_1\,, \quad i = 1, \ldots, r\,,$$

$$\|u_0\|_{1,\alpha,\bar{\Omega}} \leqq c_2\,, \tag{2.17}$$

$$\|u_i\|_{1,\alpha,\bar{\Omega}} \leqq c_3\,, \quad i = 1, \ldots, r\,.$$

Proof is an immediate consequence of Theorems 2.6, 2.8 and 2.11. □

2.18. **Theorem.** *Let us consider operators*

$$L_n = \sum_{i,j=1}^{2} A^n_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{2} A_i \frac{\partial}{\partial x_i}\,, \quad n = 1, 2, \ldots\,, \tag{2.19}$$

$$L = \sum_{i,j=1}^{2} A_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{2} A_i \frac{\partial}{\partial x_i}$$

with $A^n_{ij}, A_{ij}, A^n_i, A_i \in C^\alpha(\bar{\Omega})$ *satisfying* (2.3) *with* $\mu, \nu > 0$ *independent of* n, *functions* $g_n, g \in C^\alpha(\bar{\Omega})$ *and* $\varphi \in C^{2,\alpha}(\bar{\Omega})$. *Let* $u_n, u \in C^{2,\alpha}(\bar{\Omega})$ *be solutions of problems*

$$\text{a)}\quad L_n u_n = g_n \quad \text{in} \quad \Omega\,, \quad u_n \mid \partial\Omega = \varphi \mid \partial\Omega\,, \tag{2.20}$$

$$\text{b)}\quad Lu = g \quad \text{in} \quad \Omega\,, \quad u \mid \partial\Omega = \varphi \mid \partial\Omega$$

and let

$$A^n_{ij} \to A_{ij}\,, \quad A^n_i \to A_i\,, \quad g_n \to g \quad \text{in} \quad C^\alpha(\bar{\Omega})\,. \tag{2.21}$$

Then $u_n \to u$ *in* $C^{2,\alpha}(\bar{\Omega})$.

Proof. Denoting $v_n = u - u_n$ and subtracting (2.20, a) from (2.20, b), we get

$$\sum_{i,j=1}^{2} A_{ij} v_{nx_ix_j} + \sum_{i=1}^{2} A_i v_{nx_i} = F_n\,, \quad v_n \mid \partial\Omega = 0\,, \tag{2.22}$$

where

$$F_n = \sum_{i,j=1}^{2} (A^n_{ij} - A_{ij})\, u_{nx_ix_j} + \sum_{i=1}^{2} (A^n_i - A_i)\, u_{nx_i} + g_n - g\,. \tag{2.23}$$

It is evident that $F_n \in C^\alpha(\bar{\Omega})$ for $n = 1, 2, \ldots$. From Theorem 2.6 and the boundedness of A^n_{ij}, A^n_i, g_n in $C^\alpha(\bar{\Omega})$ we see that $\|u_n\|_{2,\alpha,\bar{\Omega}} \leqq c$, where c is independent of n. This, (2.21) and (2.23) imply that $F_n \to 0$ in $C^\alpha(\bar{\Omega})$. Now, applying Theorem 2.6 to (2.22), we get the estimate

$$\|v_n\|_{2,\alpha,\bar{\Omega}} \leqq k_1(\mu, \nu, \alpha, \|A_{ij}\|_{0,\alpha,\bar{\Omega}}, \|A_i\|_{0,\alpha,\bar{\Omega}}, \Omega)\, \|F_n\|_{0,\alpha,\bar{\Omega}}\,.$$

This already implies that $v_n = u - u_n \to 0$ in $C^{2,\alpha}(\bar{\Omega})$. □

Now, let us consider the following linear problem:

Problem (L). Find $u \in C^{2,\alpha}(\bar{\Omega})$ and $q_1, \ldots, q_r \in R^1$ satisfying (2.1), (1.3), (1.4),

$$\frac{\partial u}{\partial n}(z_i) = v_i\,, \quad i = 1, \ldots, m\,, \tag{2.24}$$

and

$$\int_{C_i} b(x) \frac{\partial u}{\partial n}(x)\, \mathrm{d}s = v_i\,, \quad i = m + 1, \ldots, r\,, \tag{2.25}$$

with given $v_1, \ldots, v_r \in R^1$ and a given $b \in L^\infty(\bigcup_{i=m+1}^{r} C_i)$. Further, we assume that

$$0 < \hat{c}_1 \leqq b(x) \leqq \hat{c}_2 \quad \forall x \in \bigcup_{i=m+1}^{r} C_i\,. \tag{2.26}$$

2.27. **Lemma.** *Problem* (L) *has at most one solution.*

Proof. If Problem (L) has two different solutions, then the corresponding homogeneous problem, i.e. Problem (L) with $g = 0$, $\psi_0 = 0$, $\psi_i = 0$, $v_i = 0$ for $i = 1, \ldots, r$, has a nontrivial solution u. Hence, u is nonconstant in $\bar{\Omega}$ and by Theorem 2.12, $\max_{\bar{\Omega}} u = q_i = u \mid C_i$ for some $i \in \{1, \ldots, r\}$. Then $\partial u / \partial n > 0$ on C_i. If $i \in \{1, \ldots, m\}$, we have a contradiction with (2.24) (where $v_i = 0$). Let $i \in \{m + 1, \ldots, r\}$. Then, since $b > 0$, we get $\int_{C_i} b(\partial u / \partial n)\, \mathrm{d}s > 0$, which is a contradiction with (2.25). $\square$

The solution of Problem (L) will be sought in the form

$$u = u_0 + \sum_{j=1}^{r} q_j u_j\,. \tag{2.28}$$

It is easy to see that such $u \in C^{2,\alpha}(\bar{\Omega})$ is a solution of equation (2.1) and satisfies conditions (1.3) and (1.4). We shall seek the constants q_j to satisfy (2.24) and (2.25), i.e.

$$\sum_{j=1}^{r} \frac{\partial u_j}{\partial n}(z_i)\, q_j = v_i - \frac{\partial u_0}{\partial n}(z_i)\,, \quad i = 1, \ldots, m\,, \tag{2.29}$$

$$\sum_{j=1}^{r} \int_{C_i} b \frac{\partial u_j}{\partial n}\, \mathrm{d}s\, q_j = v_i - \int_{C_i} b \frac{\partial u_0}{\partial n}\, \mathrm{d}s\,, \quad i = m + 1, \ldots, r\,. \tag{2.30}$$

(2.29) and (2.30) form a system of linear equations

$$\mathbb{A} q = h\,, \tag{2.31}$$

where

$$\mathbb{A} = (\alpha_{ij})_{i,j=1}^{r}\,, \quad q = (q_1, \ldots, q_r)^{\mathrm{T}}\,, \quad h = (h_1, \ldots, h_r)^{\mathrm{T}}\,, \tag{2.32}$$

$$\alpha_{ij} = \frac{\partial u_j}{\partial n}(z_i) \quad \text{for} \quad i = 1, \ldots, m\,, \quad j = 1, \ldots, r\,,$$

$$\alpha_{ij} = \int_{C_i} \frac{\partial u_j}{\partial n} \, ds \quad \text{for} \quad i = m + 1, \ldots, r\,, \quad j = 1, \ldots, r\,,$$

$$h_i = v_i - \frac{\partial u_0}{\partial n}(z_i)\,, \quad i = 1, \ldots, m\,,$$

$$h_i = v_i - \int_{C_i} b \frac{\partial u_0}{\partial n} \, ds\,, \quad i = m + 1, \ldots, r\,.$$

2.33. **Lemma.** *The matrix $\mathbb{A}$ is regular.*

Proof. It is evident that provided $\mathbb{A}$ is singular, Problem (L) with $g = 0$, $\psi_0 = 0$, $\psi_i = 0$, $v_i = 0$, $i = 1, \ldots, r$ has a nonzero solution, which is a contradiction with Lemma 2.27. □

From the above results we get the following

2.34. **Theorem.** *Problem* (L) *has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$. This u has the form* (2.28), *where q is the unique solution of the linear system* (2.31). □

2.35. **Theorem.** *The solution u of Problem* (L) *satisfies the following estimates*:

$$\|u\|_{2,\alpha,\bar{\Omega}} \leqq c_1(1 + c_1 c_4 \|\mathbb{A}^{-1}\|_1) [\|g\|_{0,\alpha,\bar{\Omega}} + \|\varphi_0\|_{2,\alpha,\partial\bar{\Omega}}] + \\ + c_1 \|\mathbb{A}^{-1}\|_1 \|v\|_1\,, \tag{2.36}$$

$$\|u\|_{1,\alpha,\bar{\Omega}} \leqq c_2(1 + c_3 c_4 \|\mathbb{A}^{-1}\|_1) + c_3 \|\mathbb{A}^{-1}\|_1 \|v\|_1\,, \tag{2.37}$$

where

$$v = (v_1, \ldots, v_r)^{\mathrm{T}}\,, \quad \|v\|_1 = \sum_{i=1}^{r} |v_i|,$$

$\|\mathbb{A}^{-1}\|_1$ *is the norm of the matrix $\mathbb{A}^{-1}$ induced by the norm $\|\cdot\|_1$ in R^r. c_1, c_2, c_3 are the constants from Theorem* 2.15. $c_4 \equiv r + \hat{c}_2 \operatorname{meas}_1(\partial\Omega))$, *where $\hat{c}_2$ is the constant from* (2.26) *and* meas_1 *is the one-dimensional measure on $\partial\Omega$.*

Proof. Let us denote either $\|u\| = \|u\|_{2,\alpha,\bar{\Omega}}$ or $\|u\| = \|u\|_{1,\alpha,\bar{\Omega}}$. Then, by (2.28),

$$\|u\| \leqq \|u_0\| + \|q\|_1 \max_{i=1,\ldots,r} \|u_i\|\,. \tag{2.38}$$

We have $q = \mathbb{A}^{-1}h$ and

$$\|q\|_1 \leqq \|\mathbb{A}^{-1}\|_1 \|h\|_1\,. \tag{2.39}$$

Moreover,

$$\|h\|_1 \leqq \|v\|_1 + \sum_{i=1}^{m} \left|\frac{\partial u_0}{\partial n}(z_i)\right| + \sum_{i=m+1}^{r} \left|\int_{C_i} b \frac{\partial u_0}{\partial n} \, ds\right|.$$

If we use Theorem 2.15 and assumption (2.26), we get

$$\left|\frac{\partial u_0}{\partial n}(z_i)\right| \leqq c_1[\|g\|_{0,\alpha,\bar\Omega} + \|\varphi_0\|_{2,\alpha,\partial\Omega}],$$

$$\left|\frac{\partial u_0}{\partial n}(z_i)\right| \leqq c_2, \quad i = 1, \ldots, m;$$

$$\left|\int_{C_i} b\frac{\partial u_0}{\partial n}\,\mathrm{d}s\right| \leqq c_1\hat{c}_2 \operatorname{meas}_1(C_i)\,[\|g\|_{0,\alpha,\bar\Omega} + \|\varphi_0\|_{2,\alpha,\partial\Omega}],$$

$$\left|\int_{C_i} b\frac{\partial u_0}{\partial n}\,\mathrm{d}s\right| \leqq c_1\hat{c}_2 \operatorname{meas}_1(C_i),$$

$$i = m+1, \ldots, r.$$

Hence,

$$\|h\|_1 \leqq \|v\|_1 + c_1(m + \hat{c}_2 \sum_{i=m+1}^{r} \operatorname{meas}_1(C_i))\,[\|g\|_{0,\alpha,\bar\Omega} + \|\varphi_0\|_{2,\alpha,\partial\Omega}] \leqq$$
$$\leqq c_1c_4[\|g\|_{0,\alpha,\bar\Omega} + \|\varphi_0\|_{2,\alpha,\partial\Omega}] + \|v\|_1. \tag{2.40}$$

Similarly we get

$$\|h\|_1 \leqq c_2c_4 + \|v\|_1. \tag{2.41}$$

Now, if we substitute (2.39), (2.40) and (2.41) into (2.38) and use (2.16), (2.17), we get (2.36)–(2.37). □

In the following let us consider given constants $\alpha \in (0, 1)$, $\mu, \nu, M, \hat{c}_1, \hat{c}_2 > 0$, $\mu \leqq \nu$, $\hat{c}_1 \leqq \hat{c}_2$ and functions $\varphi_1, \ldots, \varphi_r \in C^{2,\alpha}(\bar\Omega)$ with properties (2.13). Let us denote by $\mathscr{L}(\alpha, \mu, \nu, M)$ the set of all operators L from (2.1) with properties (2.2)–(2.4), and by $\mathscr{B}(\hat{c}_1, \hat{c}_2)$ the set of all $b \in L^\infty(\bigcup_{i=m+1}^{r} C_i)$ satisfying (2.26). Each operator $L \in \mathscr{L}(\alpha, \mu, \nu, M)$ can be associated with the functions $u_{iL} = u_i$ $(i = 1, \ldots, r)$ – solutions of (2.14, b). Hence, each pair (L, b), $L \in \mathscr{L}(\alpha, \mu, \nu, M)$ and $b \in \mathscr{B}(\hat{c}_1, \hat{c}_2)$, is associated with the regular matrix $\mathbb{A}_{L,b} = \mathbb{A}$ defined in (2.32). We shall prove

2.42. **Theorem.** *There exists a constant $K > 0$ such that*

$$\|\mathbb{A}_{L,b}^{-1}\|_1 \leqq K \quad \forall L \in \mathscr{L}(\alpha, \mu, \nu, M), \quad \forall b \in \mathscr{B}(\hat{c}_1, \hat{c}_2). \tag{2.43}$$

Proof. Let us denote $\mathscr{A} = \{\|\mathbb{A}_{L,b}^{-1}\|_1;\ L \in \mathscr{L}(\alpha, \mu, \nu, M),\ b \in \mathscr{B}(\hat{c}_1, \hat{c}_2)\}$ and $a = \sup \mathscr{A}$. Then there exist sequences $L_n \in \mathscr{L}(\alpha, \mu, \nu, M)$ (L_n has coefficients A_{ij}^n, A_i^n) and $b_n \in \mathscr{B}(\hat{c}_1, \hat{c}_2)$, $n = 1, 2, \ldots$ such that (denoting $\mathbb{A}_n = \mathbb{A}_{L_n,b_n}$)

$$\|\mathbb{A}_n^{-1}\|_1 \to a \quad \text{if} \quad n \to \infty. \tag{2.44}$$

By (2.32) we have

$$
(2.45)\qquad \mathbb{A}_n = \begin{pmatrix} \dfrac{\partial u_j^n(z_i)}{\partial n} \\ \displaystyle\int_{C_k} b_n \frac{\partial u_j^n}{\partial n}\,\mathrm{d}s \end{pmatrix} \quad \begin{matrix} i = 1, \dots, m,\ k = m + 1, \dots, r\,, \\ j = 1, \dots, r \end{matrix}
$$

where

$$
(2.46)\qquad \begin{gathered} u_i^n \in C^{2,\alpha}(\bar{\Omega})\,, \\ L_n u_i^n = 0 \text{ in } \Omega\,, \quad u_i^n \mid \partial\Omega = \varphi_i \mid \partial\Omega\,, \\ i = 1, \dots, r\,, \quad n = 1, 2, \dots\,. \end{gathered}
$$

Since the imbedding $C^{2,\alpha}(\bar{\Omega}) \subset C^{1,\alpha}(\bar{\Omega})$ is completely continuous, on the basis of (2.16) we can choose a subsequence of u_i^n (denoted again by u_i^n) such that $u_i^n \to u_i$ in $C^{1,\alpha}(\bar{\Omega})$ if $n \to \infty$ $(i = 1, \dots, r)$. Hence, $\partial u_j^n/\partial n\,(z_i) \to \partial u_j/\partial n\,(z_i)$ $(i = 1, \dots, m,$ $j = 1, \dots, r)$.

Moreover, in view of (2.26), the sequence b_n is bounded in

$$
L^2\Big(\bigcup_{i=m+1}^{r} C_i\Big)
$$

and thus it is possible to assume that $b_n \to b$ weakly in

$$
L^2\Big(\bigcup_{i=m+1}^{r} C_i\Big)\,.
$$

On the other hand, the set

$$
\Big\{b \in L^2\Big(\bigcup_{i=m+1}^{r} C_i\Big);\ 0 < \hat{c}_1 \leqq b \leqq \hat{c}_2\Big\}
$$

is convex and closed, and therefore, it is weakly closed. This implies that b satisfies (2.26), which means that $b \in \mathscr{B}(\hat{c}_1, \hat{c}_2)$. Further, it is evident that

$$
\partial u_i^n/\partial n \to \partial u_i/\partial n \quad \text{in} \quad L^2\Big(\bigcup_{i=m+1}^{r} C_i\Big)
$$

strongly, which yields

$$
\int_{C_i} b_n \frac{\partial u_i^n}{\partial n}\,\mathrm{d}s \to \int_{C_i} b \frac{\partial u}{\partial n}\,\mathrm{d}s\,, \quad i = m + 1, \dots, r\,.
$$

Hence,

$$
(2.47)\qquad \mathbb{A}_n \to \mathbb{A} = \begin{pmatrix} \dfrac{\partial u_j}{\partial n}(z_i) \\ \displaystyle\int_{C_k} b \frac{\partial u_j}{\partial n}\,\mathrm{d}s \end{pmatrix} \quad \begin{matrix} i = 1, \dots, m,\ k = m + 1, \dots, r\,, \\ j = 1, \dots, r\,. \end{matrix}
$$

Let $0 < \beta < \alpha$. As the imbedding $C^{\alpha}(\bar{\Omega}) \subset C^{\beta}(\bar{\Omega})$ is completely continuous and (2.4) is valid for all $n = 1, 2, \ldots$, we choose subsequences A^n_{ij}, A^n_i such that

$$(2.48) \qquad A^n_{ij} \to A_{ij}\,, \quad A^n_i \to A_i \quad \text{in} \quad C^{\beta}(\bar{\Omega})\,.$$

Of course, the coefficients A_{ij} satisfy (2.3). By Theorem 2.15, (where we substitute $\alpha := \beta$), there exist unique solutions $\tilde{u}_i \in C^{2,\beta}(\bar{\Omega})$ of the problems

$$(2.49) \qquad L\tilde{u}_i = 0 \quad \text{in} \quad \Omega\,, \qquad \tilde{u}_i \,|\, \partial\Omega = \varphi_i \,|\, \partial\Omega\,, \qquad i = 1, \ldots, r\,.$$

Let us define the matrix

$$(2.50) \qquad \mathbb{A}^{\sim} = \mathbb{A}_{L,b} = \begin{pmatrix} \dfrac{\partial \tilde{u}_j}{\partial n}(z_i) \\ \displaystyle\int_{C_i} b \frac{\partial \tilde{u}_j}{\partial n} \, \mathrm{d}s \end{pmatrix},$$

which is regular, as follows from Lemma 2.33.

Now, by Theorem 2.18 we have

$$(2.51) \qquad u^n_i \to \tilde{u}_i \quad \text{in} \quad C^{2,\beta}(\bar{\Omega})\,.$$

On the other hand, $u^n_i \to u_i$ in $C^{1,\alpha}(\bar{\Omega})$. From this, (2.50), (2.51) and (2.47) we conclude that $u_i = \tilde{u}_i$ and therefore $\mathbb{A} = \mathbb{A}^{\sim}$. Hence,

$$\sup \mathscr{A} = a = \lim_{n \to \infty} \|\mathbb{A}_n^{-1}\|_1 = \|\mathbb{A}^{-1}\|_1 < +\infty\,,$$

which we wanted to prove. □

As a consequence of 2.35 and 2.42 we get

2.52. **Theorem.** *To given constants $\alpha, \mu, \nu, M, c_0, \hat{c}_1, \hat{c}_2 > 0$, $\alpha \in (0, 1)$, $\mu \leqq \nu$, $\hat{c}_1 \leqq \hat{c}_2$ there exist constants $c_1^*, c_2^* > 0$ such that*

$$(2.53) \qquad \|u\|_{2,\alpha,\bar{\Omega}} \leqq c_1^*[\|g\|_{0,\alpha,\bar{\Omega}} + \|\varphi_0\|_{2,\alpha,\partial\Omega} + \|v\|_1]$$

and

$$(2.54) \qquad \|u\|_{1,\alpha,\bar{\Omega}} \leqq c_2^*(1 + \|v\|_1)$$

for each solution u of Problem (L) *with an operator $L \in \mathscr{L}(\alpha, \mu, \nu, M)$, a right-hand side of equation* (2.1) *$g \in C^{\alpha}(\bar{\Omega})$ satisfying* (2.5), *and with a function $b \in \mathscr{B}(\hat{c}_1, \hat{c}_2)$ from conditions* (2.25).

Proof. From (2.36), (2.37) and (2.43) we see that it is sufficient to put $c_1^* = \max\{c_1(1 + c_1 c_4 K), c_1 K\}$ and $c_2^* = \max\{c_2(1 + K c_3 c_4), c_3 K\}$. □

We shall close this section by a theorem on continuous dependence of the solution to Problem (L) on the data:

2.55. **Theorem.** *Let us consider operators L_n, L form* (2.19) *with coefficients* *from* $C^\alpha(\bar{\Omega})$ *satisfying* (2.3). *Further, let* g_n, $g \in C^\alpha(\bar{\Omega})$, b_n, $b \in L^\infty(\bigcup_{i=m+1}^{r} C_i)$ *satisfy* (2.26), $\psi_i \in C^{2,\alpha}(C_i)$, $i = 1, \ldots, r$, v^n, $v \in R^r$. *We denote by* u_n *and* $u \in C^{2,\alpha}(\bar{\Omega})$ *the solutions of the problem*

(2.56)
$$L_n u_n = g_n \quad \text{in} \quad \Omega\,,$$
$$u_n \mid C_0 = \psi_0\,,$$
$$u_n \mid C_i = \psi_i + q_i^n\,, \quad q_i^n = \text{const}\,, \quad i = 1, \ldots, r\,,$$
$$\frac{\partial u_n}{\partial n}(z_i) = v_i^n\,, \quad i = 1, \ldots, m\,,$$
$$\int_{C_i} b_n \frac{\partial u_n}{\partial n}\,\mathrm{ds} = v_i^n\,, \quad i = m + 1, \ldots, r$$

and of Problem (L), *respectively.*

Then, provided

(2.57)
$$A_{ij}^n \to A_{ij}\,, \quad A_i^n \to A_i, \quad g_n \to g \ \ in \ \ C^\alpha(\bar{\Omega})\,,$$
$$b_n \to b \quad almost\ everywhere\ in \quad \bigcup_{i=m+1}^{r} C_i\,,$$
$$v^n \to v \quad in \quad R^r\,,$$

we have $u_n \to u$ *in* $C^{2,\alpha}(\bar{\Omega})$.

Proof is analogous to the proof of Theorem 2.18. From (2.57) it follows that there exists M such that L_n, $L \in \mathscr{L}(\alpha, \mu, \nu, M)$. Moreover, b_n, $b \in \mathscr{B}(\hat{c}_1, \hat{c}_2)$. By Theorem 2.52 and assumption (2.57),

(2.58)
$$\|u_n\|_{2,\alpha,\bar{\Omega}} \leqq c_1^*[\|g_n\|_{0,\alpha,\bar{\Omega}} + \|\varphi_0\|_{2,\alpha,\partial\Omega} + \|v^n\|_1] \leqq k_1^* = \text{const}\,,$$
$$\|u_n\|_{1,\alpha,\bar{\Omega}} \leqq c_2^*(1 + \|v^n\|_1) \leqq k_2^* = \text{const}\,, \quad n = 1, 2, \ldots\,.$$

Let us put $w_n = u - u_n \in C^{2,\alpha}(\bar{\Omega})$. Then w_n is a solution of the problem

(2.59)
$$Lw_n = F_n\,,$$
$$w_n \mid C_0 = 0\,, \quad w_n \mid C_i = q_i^n\,, \quad i = 1, \ldots, r\,,$$
$$\frac{\partial w_n}{\partial n}(z_i) = d_i^n\,, \quad i = 1, \ldots, m\,,$$
$$\int_{C_i} b \frac{\partial w_n}{\partial n}\,\mathrm{ds} = d_i^n\,, \quad i = m + 1, \ldots, r\,,$$

where F_n is given by (2.23) and

(2.60)
$$d_i^n = v_i - v_i^n\,, \quad i = 1, \ldots, m\,,$$
$$d_i^n = v_i - v_i^n - \int_{C_i} (b - b_n) \frac{\partial u_n}{\partial n}\,\mathrm{ds}\,, \quad i = m + 1, \ldots, r\,.$$

(2.57) and (2.58) imply

$$(2.61) \qquad F_n \to 0 \quad \text{in} \quad C^{\alpha}(\bar{\Omega}) \quad \text{and} \quad d^n \to 0 \quad \text{in} \quad R^r .$$

By Theorem (2.52) applied to problems (2.59) we get

$$\|w_n\|_{2,\alpha,\bar{\Omega}} \leqq c_1^*[\|F_n\|_{0,\alpha,\bar{\Omega}} + \|d^n\|_1] \to 0 ,$$

which concludes the proof. □

3. SOLVABILITY OF THE NONLINEAR PROBLEM

Now we shall study nonlinear Problem (P) (i.e. (1.7), (1.3)–(1.6)) and prove its solvability.

3.1. **Theorem.** *Let functions $\psi_i \in C^{2,\alpha}(C_i)$, $i = 0, \ldots, r$ and constants $\gamma_i \in R^1$, $i = 1, \ldots, r$ be given and let φ_i be functions satisfying* (2.13). *Further, let $M > 0$ and let* (1.10)–(1.15) *be satisfied with constants $\alpha, \mu, v, c, c_0, M_0, M_1, L_0, L, \hat{c}_1, \hat{c}_2$ such that*

$$(3.2) \qquad c + M_1 + L(1 + c^{*2})^{1/2}\, c_2^* \left(1 + \|\gamma\|_1 \max\left(1, \frac{1}{\hat{c}_1}\right)\right) \leqq M ,$$

where c_2^* and c^* are constants from (2.54) *and* (1.16), *respectively.*

Then Problem (P) *has at least one solution $u \in C^{2,\alpha}(\bar{\Omega})$.*

Proof. For each $u \in C^{1,\alpha}(\bar{\Omega})$ we shall consider the following *problem*: Find $w = w(u)$ and $q_i = q_i(u)$, $i = 1, \ldots, r$, such that

$$(3.3) \qquad L_u w = b_0(\cdot, u, \nabla u) \quad \text{in} \quad \Omega ,$$

$$(3.4) \qquad w \mid C_0 = \psi_0 ,$$

$$(3.5) \qquad w \mid C_i = \psi_i + q_i , \quad i = 1, \ldots, r ,$$

$$(3.6) \qquad \left[b(\cdot, u, \nabla u) \frac{\partial w}{\partial n}\right](z_i) = \gamma_i , \quad i = 1, \ldots, m ,$$

$$(3.7) \qquad \int_{C_i} b(\cdot, u, \nabla u) \frac{\partial w}{\partial n} \, ds = \gamma_i , \quad i = m + 1, \ldots, r ,$$

where

$$(3.8) \qquad L_u w = \sum_{i,j=1}^{2} a_{ij}(\cdot, u, \nabla u)\, w_{x_i x_j} + b_i(\cdot, u, \nabla u)\, w_{x_i} .$$

Now (1.13) implies for $x, y \in \bar{\Omega}$ the inequality

$$|a_{ij}(x, u(x), \nabla u(x)) - a_{ij}(y, u(y), \nabla u(y)| \leqq$$
$$\leqq M_1 \mid x - y|^{\alpha} + L(|u(x) - u(y)|^2 + |\nabla u(x) - \nabla u(y)|^2)^{1/2} \leqq$$
$$\leqq [M_1 + L(1 + c^{*2})^{1/2} \|u\|_{1,\alpha,\bar{\Omega}}] |x - y|^{\alpha} .$$

Similar estimates hold for b_i. Hence, $a_{ij}(\cdot, u, \nabla u)$, $b_i(\cdot, u, \nabla u) \in C^\alpha(\bar{\Omega})$ and

$$\begin{aligned}(3.9) \qquad & \|a_{ij}(\cdot, u, \nabla u)\|_{0,\alpha,\bar{\Omega}}, \|b_i(\cdot, u, \nabla u)\|_{0,\alpha,\bar{\Omega}} \leqq \\ & \leqq c + M_1 + L(1 + c^{*2})^{1/2} \|u\|_{1,\alpha,\bar{\Omega}}, \quad i, j = 1, 2, \\ & \|b_0(\cdot, u, \nabla u)\|_{0,0,\bar{\Omega}} \leqq c_0, \\ & \|b_0(\cdot, u, \nabla u)\|_{0,\alpha,\bar{\Omega}} \leqq c_0 + M_0 + L_0(1 + c^{*2})^{1/2} \|u\|_{1,\alpha,\bar{\Omega}}.\end{aligned}$$

By (1.15), $b(\cdot, u, \nabla u) \mid \partial\Omega \in C(\partial\Omega)$ and

$$(3.10) \qquad \hat{c}_1 \leqq b(\cdot, u, \nabla u) \leqq \hat{c}_2 \quad \text{on} \quad \partial\Omega.$$

Let us put $v_i = \gamma_i$ for $i = m + 1, \ldots, r$ and $v_i = \gamma_i b(z_i, u(z_i), \nabla u(z_i))^{-1}$ for $i = 1, \ldots, m$. Then

$$(3.11) \qquad \|v\|_1 \leqq \|\gamma\|_1 \max\left(1, \frac{1}{\hat{c}_1}\right).$$

We see that (3.3)–(3.7) form a linear Problem (L) with a differential operator $L = L_u$ satisfying (2.2), (2.3). By results from Section 2, there exists a unique solution $w = w(u) \in C^{2,\alpha}(\bar{\Omega})$ to (3.3)–(3.7). Hence, we can define the mapping $\Phi: C^{1,\alpha}(\bar{\Omega}) \to \to C^{2,\alpha}(\bar{\Omega})$ by

$$(3.12) \qquad \Phi(u) = w(u), \quad u \in C^{1,\alpha}(\bar{\Omega}).$$

In view of the imbedding $C^{2,\alpha}(\bar{\Omega}) \subset C^{1,\alpha}(\bar{\Omega})$ we have also the mapping $F: C^{1,\alpha}(\bar{\Omega}) \to \to C^{1,\alpha}(\bar{\Omega})$:

$$(3.13) \qquad F(u) = w(u), \quad u \in C^{1,\alpha}(\bar{\Omega}).$$

That is $F = J \circ \Phi$, where J is the imbedding operator of $C^{2,\alpha}(\bar{\Omega})$ into $C^{1,\alpha}(\bar{\Omega})$. It is obvious that *$u$ is a solution of Problem* (P) *if and only if u is a fixed point of the mapping F.*

Now let us put

$$(3.14) \qquad \mathfrak{M} = \left\{u \in C^{1,\alpha}(\bar{\Omega});\ \|u\|_{1,\alpha,\bar{\Omega}} \leqq c_2^*\left(1 + \|\gamma\|_1 \max\left(1, \frac{1}{\hat{c}_1}\right)\right)\right\}.$$

This set is nonempty, bounded, convex and closed in $C^{1,\alpha}(\bar{\Omega})$. If $u \in \mathfrak{M}$, then by (1.14), (3.2) and (3.9),

$$\begin{aligned}(3.15) \qquad & \|a_{ij}(\cdot, u, \nabla u)\|_{0,\alpha,\bar{\Omega}}, \quad \|b_i(\cdot, u, \nabla u)\|_{0,\alpha,\bar{\Omega}} \leqq M, \quad i, j = 1, 2, \\ & L_u \in \mathscr{L}(\alpha, \mu, \nu, M).\end{aligned}$$

Moreover,

$$\begin{aligned}(3.16) \qquad & b(\cdot, u, \nabla u)| \bigcup_{i=m+1}^{r} C_i \in \mathscr{B}(\hat{c}_1, \hat{c}_2), \\ & g_u = b_0(\cdot, u, \nabla u) \in C^\alpha(\bar{\Omega}), \quad \|g_u\|_{0,\alpha,\bar{\Omega}} \leqq \tilde{M} =\end{aligned}$$

$$= c_0 + M_0 + L_0(1 + c^{*2})^{1/2}\left(1 + \|\gamma\|_1 \max\left(1, \frac{1}{\hat{c}_1}\right)\right), \quad |g_u| \leqq c_0 .$$

By Theorem 2.52 and (3.11),

$$\|w\|_{1,\alpha,\Omega} \leqq c_2^*\left(1 + \|\gamma\|_1 \max\left(1, \frac{1}{\hat{c}_1}\right)\right) \tag{3.17}$$

and thus, $w \in \mathfrak{M}$. This means that $F: \mathfrak{M} \to \mathfrak{M}$.

In order to complete the proof by applying the well-known Schauder fixed point theorem it is sufficient to prove

3.18. **Lemma.** *The operator F is completely continuous in $\mathfrak{M}$.*

Proof. a) By (3.15), (3.16) and Theorem 2.52,

$$\|\Phi(u)\|_{2,\alpha,\bar{\Omega}} = \|w(u)\|_{2,\alpha,\bar{\Omega}} \leqq c_1^*[\tilde{M} + \|\varphi_0\|_{2,\alpha,\bar{\Omega}} + \|v\|_1] < +\infty .$$

Hence, the set $\Phi(\mathfrak{M})$ is bounded in $C^{2,\alpha}(\bar{\Omega})$. Since the operator $J: C^{2,\alpha}(\bar{\Omega}) \to C^{1,\alpha}(\bar{\Omega})$ is completely continuous, the set $F(\mathfrak{M}) = J(\Phi(\mathfrak{M}))$ is compact in $C^{1,\alpha}(\bar{\Omega})$. This implies the compactness of F.

b) Let us show that F is continuous. Let $\gamma \in (0, \alpha)$. The operator F can be written in the form $F = \hat{J} \circ \hat{\Phi}$, where $\hat{J}$ is the imbedding of $C^{2,\gamma}(\bar{\Omega})$ into $C^{1,\alpha}(\bar{\Omega})$ and the mapping $\hat{\Phi}: C^{1,\alpha}(\bar{\Omega}) \to C^{2,\gamma}(\bar{\Omega})$ is defined by $\hat{\Phi}(u) = w(u)$ for $u \in C^{1,\alpha}(\bar{\Omega})$. The operator $\hat{J}$ is continuous and therefore, it is sufficient to prove the continuity of $\hat{\Phi}$.

Let $u_n \in C^{1,\alpha}(\bar{\Omega})$, $w_n = \hat{\Phi}(u_n)$, $n = 1, 2, \ldots$, $u_n \to u$ in $C^{1,\alpha}(\bar{\Omega})$. We shall prove that

$$\begin{aligned} a_{ij}(\cdot, u_n, \nabla u_n) &\to a_{ij}(\cdot, u, \nabla u) , \\ b_i(\cdot, u_n, \nabla u_n) &\to b_i(\cdot, u, \nabla u) \quad \text{in} \quad C^\gamma(\bar{\Omega}) . \end{aligned} \tag{3.19}$$

If $x \in \bar{\Omega}$, we denote $\xi_n(x) = (u_n(x), \nabla u_n(x))$, $\xi(x) = (u(x), \nabla u(x))$. For arbitrary $x, y \in \bar{\Omega}$, $x \neq y$, we have

$$\begin{aligned} h_n(x, y) := {} & |a_{ij}(x, \xi_n(x)) - a_{ij}(y, \xi_n(y)) - \\ & - a_{ij}(x, \xi(x)) + a_{ij}(y, \xi(y))| \, |x - y|^{-\gamma} \leqq \\ \leqq {} & \min \{[M_1 + L(\|u_n\|_{1,\alpha,\bar{\Omega}} + \|u\|_{1,\alpha,\bar{\Omega}})] \, |x - y|^{\alpha-\gamma} , \\ & L[|\xi_n(x) - \xi(x)| + |\xi_n(y) - \xi(y)|] \, |x - y|^{-\gamma}\} . \end{aligned} \tag{3.20}$$

There exists $k > 0$ such that $\|u\|_{1,\alpha,\bar{\Omega}}, \|u_n\|_{1,\alpha,\bar{\Omega}} \leqq k$ $(n = 1, 2, \ldots)$. Further, $\xi_n \to \xi$ uniformly in $\bar{\Omega}$ and thus, $a_{ij}(\cdot, \xi_n) \to a_{ij}(\cdot, \xi)$ in $C(\bar{\Omega})$. We need to prove that

$$\lim_{n \to \infty} \Big[\sup_{\substack{x, y \in \bar{\Omega} \\ x \neq y}} h_n(x, y) \Big] = 0 . \tag{3.21}$$

Let $\varepsilon > 0$; we find $\delta > 0$ and n_0 such that

(3.22) $$(M_1 + 2Lk)\,\delta^{\alpha-\gamma} < \varepsilon\,,$$
$$2L|\xi_n(x) - \xi(x)|\,\delta^{-\gamma} < \varepsilon \quad \forall x \in \bar{\Omega}\,, \quad \forall n > n_0\,.$$

Now, if $n > n_0$, then $|h_n(x, y)| < \varepsilon$, as follows from (3.20) and (3.22). This yields (3.21). Similar results hold for $b_i(\cdot, u, \nabla u)$, $i = 0, 1, 2$.

Further, from the continuity of b we have $b(\cdot, u_n, \nabla u_n) \to b(\cdot, u, \nabla u)$ and therefore, $v^n \to v$, where $v_i^n = \gamma_i / b(z_i, u_n(z_i), \nabla u_n(z_i))$, $v_i = \gamma_i / b(z_i, u(z_i), \nabla u(z_i))$, $i = 1, \ldots, m$ and $v_i^n = v_i = \gamma_i$ for $i = m + 1, \ldots, r$.

Now, by the application of Theorem 2.55, where we substitute $\alpha := \gamma$, we find out that $\hat{\Phi}(u_n) = w(u_n) \to \hat{\Phi}(u) = w(u)$ in $C^{2,\gamma}(\bar{\Omega})$, which we wanted to prove. □

3.23. **Remark.** We have proved the solvability of nonlinear Problem (P) under the restrictive condition (3.2). This is satisfied, e.g., if the constant L of the Lipschitz-continuity of a_{ij} and b_i is sufficiently small. This is caused by the fact that we are not able to estimate the constants K and c_2^* from (2.43) and (2.54), respectively, in dependence on the constant M. Therefore we have obtained the solvability result for a model of rotational compressible flows with a small velocity (i.e., with a small Mach number). As a special case of our results we get the solvability of a rotational incompressible flow studied in [4]. The general case of rotational compressible flows past profiles with trailing conditions, when the velocity is high, remains open.

4. APPLICATIONS

Let us investigate a *steady, plane, compressible, subsonic, adiabatic, barotropic flow*. It is described by the following equations:

(4.1) $$p = C\varrho^{\varkappa}\,, \quad C > 0\,, \quad \varkappa > 1 \quad \text{are constants}\,,$$

(4.2) $$\mathscr{P}(\varrho) = \int_{\varrho_0}^{\varrho} \frac{\mathrm{d}p}{\mathrm{d}\varrho}(\tau)\frac{1}{\tau}\,\mathrm{d}\tau\,, \quad \varrho_0 > 0 \quad \text{is a constant}\,,$$

(4.3) $$H = \mathscr{P}(\varrho) + \tfrac{1}{2}|\mathbf{v}|^2\,,$$

(4.4) $$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\varrho v_i) = 0\,,$$

(4.5) $$\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\,,$$

(4.6) $$\text{a)} \quad \omega v_2 = \frac{\partial H}{\partial x_1}\,,$$

$$\text{b)} \quad -\omega v_1 = \frac{\partial H}{\partial x_2}\,,$$

which are considered in the domain Ω filled by the fluid. We use the following notation: p – pressure, ϱ – density, H – generalized enthalpy, $\mathscr{P}$ – pressure function, $\mathbf{v} = (v_1, v_2)$ – velocity vector, ω – vorticity. (4.1) is the adiabatic barotropic state equation ,(4.4) is the continuity equation and (4.6, a–b) are the Euler equations of motion. We neglect the outer volume force.

On the basis of (4.4) we introduce the stream function $u: \bar{\Omega} \to R^1$ satisfying the relations

$$\frac{\partial u}{\partial x_1} = -\varrho v_2 \,, \quad \frac{\partial u}{\partial x_2} = \varrho v_1 \,. \tag{4.7}$$

Since both u and H are constant along an arbitrary streamline, we introduce the assumption that H is a function of u. It means that there exists a function $A: R^1 \to R^1$ (we assume that A is sufficiently smooth and bounded) such that

$$H = A(u) = A \circ u \,. \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.5)–(4.6, a–b) we derive the stream function equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{1}{\varrho}\frac{\partial u}{\partial x_i}\right) = \varrho \frac{\mathrm{d}A}{\mathrm{d}u}(u) \,. \tag{4.9}$$

With the use of (4.1)–(4.3) and (4.7) we derive the implicit equation

$$\varrho = \varrho_0 \left[1 + \frac{\varkappa - 1}{a_0^2}\left(A(u) - \frac{1}{2}\frac{1}{\varrho^2}|\nabla u|^2\right)\right]^{1/(\varkappa-1)} , \quad 0 < a_0 = \text{const} \,, \tag{4.10}$$

for the density. If we introduce the speed of sound $a = (\mathrm{d}p/\mathrm{d}\varrho)^{1/2}$ and the Mach number $\mathcal{M} = |\mathbf{v}|/a$, we can prove that for a subsonic flow, i.e. $\mathcal{M} < 1$, equation (4.10) has exactly one solution $\varrho = \varrho(u, |\nabla u|^2) > 0$. Hence, if we put

$$\beta = \frac{1}{\varrho} \quad \text{and} \quad f = \varrho \frac{\mathrm{d}A}{\mathrm{d}u} \,, \tag{4.11}$$

equation (4.9) assumes the form

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\beta(u, |\nabla u|^2)\frac{\partial u}{\partial x_i}\right) = f(u, |\nabla u|^2) \,. \tag{4.12}$$

After differentiation we get a special case of equation (1.7).

It is not difficult to prove that for each fixed $\mathcal{M}^* \in (0, 1)$ it is possible to modify b and f in such a way that the coefficients a_{ij}, b_i from (1.7) satisfy assumptions 1.9 and the following assertion holds: If u is a solution of (1.7), $\varrho = 1/\beta$, v_1, v_2 are given by (4.7), p by (4.1) and the corresponding Mach number satisfies the condition $\mathcal{M} \leqq \mathcal{M}^*$, then v_1, v_2, p, ϱ represent a real subsonic flow satisfying equations (4.1)–(4.6).

Further, the constant L of the Lipschitz-continuity of the functions a_{ij}, b_i depends on M^* in such a way that

$$L = L(M^*) \to 0 \quad \text{if} \quad M^* \to 0 \,. \tag{4.13}$$

A similar result holds for axially symmetric flows (cf. [3, 10]).

In the following we shall study two patterns of a plane flow.

I) **Plane rotational flow past a group of profiles.** Let $\Omega \subset R^2$ be a bounded domain with $\partial\Omega \in C^{2,\alpha}(\alpha \in (0, 1))$, $\partial\Omega = C_0 \cup C_1 \cup \ldots \cup C_r$. The curves $C_1, \ldots, C_r$ represent fixed and impermeable profiles, see Fig. 1. The problem of a flow past these profiles is described by equation (4.12) with the boundary conditions

$$u \mid C_0 = \psi_0 \,, \tag{4.14}$$

$$u \mid C_i = q_i \,, \quad i = 1, \ldots, r \,, \tag{4.15}$$

$$\frac{\partial u}{\partial n}(z_i) = 0 \,, \quad i = 1, \ldots, r \,. \tag{4.16}$$

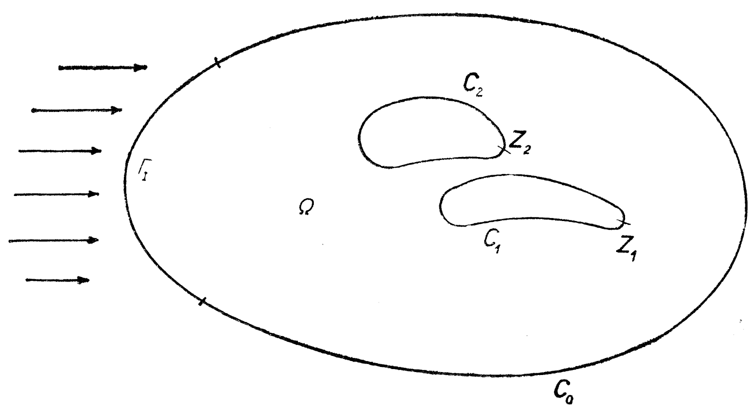

Fig. 1.

The stream function $u \in C^{2,\alpha}(\bar{\Omega})$ and constants $q_1, \ldots, q_r \in R^1$ are unknown. The function ψ_0 is obtained by integrating the quantity $\varrho v_n \mid C_0$ past C_0 ($v_n = \mathbf{v} \,.\, \boldsymbol{n}$, where $\boldsymbol{n} = (n_1, n_2)$ is a unit outer normal to $\partial\Omega$). On the inlet $\Gamma_I \subset C_0$, i.e. $\Gamma_I = \{x \in C_0;\ \varrho v_n(x) < 0\}$ (see Fig. 1), we prescribe the distribution of H, and this determines the function A. (We do not go into details. The situation is quite analogous to [4].) $z_i \in C_i$ are given trailing stagnation points, where the velocity is zero.

If we consider the trailing conditions (4.16) on the profiles $C_1, \ldots, C_m$ ($m < r$) only and prescribe the velocity circulations $(-\gamma_i)$ past the profiles $C_{m+1}, \ldots, C_r$, i.e.

$$\int_{C_i} \mathbf{v} \,.\, \mathbf{t} \, \mathrm{d}s = -\gamma_i \,, \quad i = m + 1, \ldots, r \,, \tag{4.17}$$

where $\mathbf{t} = (-n_2, n_1)$ is the unit tangent to C_i, then by (4.7) and (4.11) we get the conditions

$$\int_{C_i} \beta(u, |\nabla u|^2) \frac{\partial u}{\partial n} \, ds = \gamma_i \,, \quad i = m + 1, \ldots, r \,. \tag{4.18}$$

We see that the rotational compressible flow past profiles $C_1, \ldots, C_r$ can be formulated as Problem (P). On the basis of the above results and (4.13), to a prescribed constant $M > c$ (the constant $M_1 = 0$ in (1.13) in this case) we can choose $M^* \in (0, 1)$ in such a way that the solvability condition (3.2) is satisfied and hence, by Theorem 3.1, our model problem (4.12), (4.14), (4.15), (4.16) ($i = 1, \ldots, m$) and (4.18) has at least one solution.

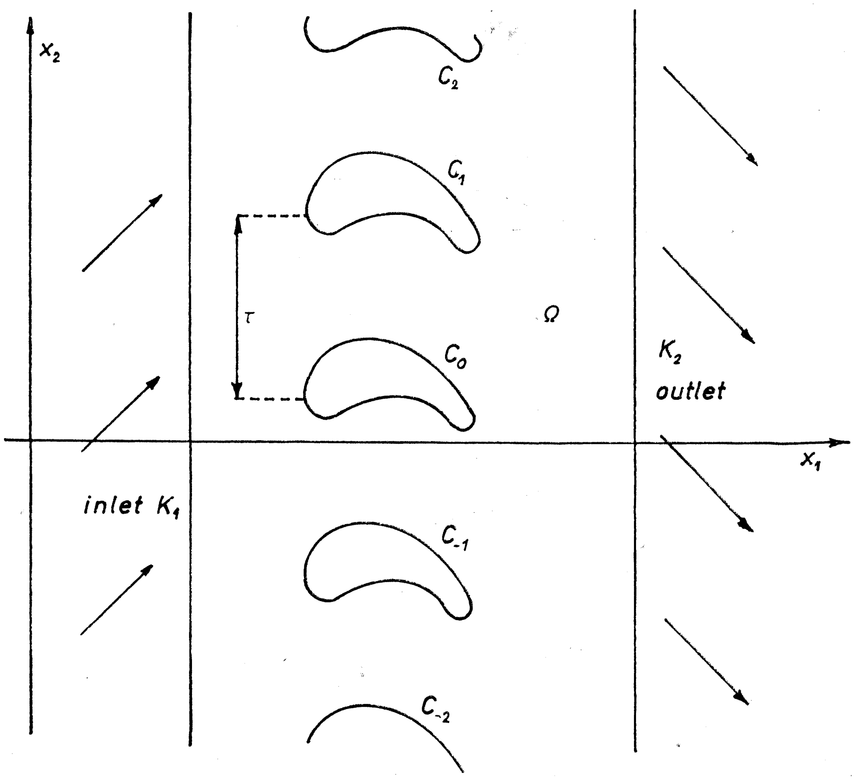

Fig. 2.

II) Cascade flow problem. We consider a domain $\Omega \subset R^2$ with the boundary $\partial\Omega \in C^{2,\alpha}(\alpha \in (0, 1))$ formed by two straight lines $K_i = \{(x_1, x_2); \, x_1 = d_i, \, x_2 \in R^1\}$, $i = 1, 2$, $d_1 < d_2$, and disjoint simple closed curves C_k, $k = 0, \pm 1, \pm 2, \ldots$, periodically spaced in the direction x_2 with a period $\tau > 0$ and contained in the strip $\Omega^* = [d_1, d_2] \times R^1$. The curve C_k is obtained by translating C_0 in the direction x_2 by $k\tau$. The curves C_k form the so-called *cascade of profiles*. K_1 and K_2 represent the *inlet* and *outlet* of the cascade, respectively. See Fig. 2. The domain Ω is periodic in the direction x_2 with the period τ:

$$(x_1, x_2) \in \bar{\Omega} \Leftrightarrow (x_1, x_2 + \tau) \in \bar{\Omega} . \tag{4.19}$$

We shall consider equations (4.1)–(4.6, a–b) in the domain Ω combined with the following boundary conditions (cf. [4, 6]):

$$\varrho v_n = 0 \text{ on } C_k , \quad k = 0, \pm 1, \ldots , \tag{4.20}$$

$$\varrho v_n(d_i, x_2) = \tilde{\varphi}_i(x_2) \text{ on } K_i , \quad i = 1, 2 , \tag{4.21}$$

$$H(d_1, x_2) = h(x_2) \text{ on } K_1 , \tag{4.22}$$

$$\frac{1}{\tau} \int_{x_2}^{x_2+\tau} v_t(d_1, \xi_2) \, d\xi_2 = \bar{\mu}_1 , \tag{4.23}$$

$$v_t(z_k) = 0 , \quad k = 0, \pm 1, \pm 2, \ldots . \tag{4.24}$$

Here $\tilde{\varphi}_1, \tilde{\varphi}_2 \in C^{1,\alpha}(R^1)$, $h \in C^{2,\alpha}(R^1)$ are given τ-periodic functions, $\tilde{\varphi}_1 \geqq \tilde{\varphi} > 0$, $\tilde{\varphi} = \text{const}$,

$$\int_{x_2}^{x_2+\tau} \tilde{\varphi}_1(\xi) \, d\xi = \int_{x_2}^{x_2+\tau} \tilde{\varphi}_2(\xi) \, d\xi = Q \quad \forall x_2 \in R^1 . \tag{4.25}$$

$\bar{\mu}_1 \in R^1$ is a given constant which represents a mean value of the tangential velocity component over the segment of the inlet K_1 of the length τ. $z_k = z_0 + (0, k\tau) \in C_k$ are the trailing points. We assume that $v_1, v_2, p, \varrho, \omega$ are τ-periodic in the direction x_2.

Introducing the stream function u satisfying (4.7) we come to the following *problem*: Find $u \in C^{2,\alpha}(\bar{\Omega})$ and constants $q_0, q_1 \in R^1$ satisfying equation (4.12) in Ω and the conditions

$$u(x_1, x_2 + \tau) = u(x_1, x_2) + Q , \quad (x_1, x_2) \in \bar{\Omega} , \tag{4.26}$$

$$u \mid C_k = q_0 + kQ , \quad k = 0, \pm 1, \pm 2, \ldots , \tag{4.27}$$

$$u(d_i, x_2) = \psi_i(x_2) + q_i , \quad i = 1, 2 , \quad x_2 \in R^1 , \quad \text{with} \quad q_2 = 0 , \tag{4.28}$$

$$\int_{x_2}^{x_2+\tau} \left[\beta(u, |\nabla u|^2) \frac{\partial u}{\partial n} \right] (d_1, \xi) \, d\xi = -\tau \bar{\mu}_1 , \quad x_2 \in R^1 , \tag{4.29}$$

$$\frac{\partial u}{\partial n}(z_k) = 0 , \quad k = 0, \pm 1, \pm 2, \ldots . \tag{4.30}$$

Here $\psi_i \in C^{2,\alpha}(R^1)$ is a primitive to $\tilde{\varphi}_i$, $i = 1, 2$. Hence, by (4.25), $\psi_i(x_2 + k\tau) = \psi_i(x_2) + kQ$, $x_2 \in R^1$. The function A from (4.8) is determined on the basis of (4.21), (4.22) (cf. [4]) and is Q-periodic in R^1.

By the use of the transformation proposed in [4] (to an incompressible rotational cascade flow) we transform the cascade flow problem (4.12), (4.26)–(4.30) to Problem (P) in a bounded domain. Then, using Theorem 3.1 and assertion (4.13), we get the solvability of our cascade flow problem under the assumption that the maximum Mach number $\mathbb{M}^* \in (0, 1)$ is sufficiently small.

Let us remark that by following the process used in Sections 2 and 3, we can prove the solvability of the cascade flow problem (4.12), (4.26)–(4.30) directly by confining our considerations to one period of the cascade (cf. e.g. [6, 7]). In this way we avoid the necessity to apply the above mentioned transformation from [4].

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Souhrn

NELINEÁRNÍ ELIPTICKÝ PROBLÉM S NEÚPLNÝMI DIRICHLETOVÝMI PODMÍNKAMI A ŘEŠENÍ ZAVÍŘENÉHO OBTÉKÁNÍ PROFILŮ A PROFILOVÝCH MŘÍŽÍ

MILOSLAV FEISTAUER

Článek se zabývá řešitelností nelineární eliptické úlohy ve vícenásobně souvislé oblasti. Na vnitřních komponentách hranice uvažujeme Dirichletovy podmínky, známé až na aditivní konstanty, které je třeba určit spolu s hledaným řešením tak, aby byly splněny tzv. Kutta-

Žukovského odtokové podmínky. Výsledky byly získány pomocí silného principu maxima a vhodných apriorních odhadů řešení a mají aplikace v úlohách obtékání profilů a profilových mříží, formulovaných pomocí proudové funkce.

Резюме

НЕЛИНЕЙНАЯ ЭЛЛИПТИЧЕСКАЯ ЗАДАЧА С НЕПОЛНЫМИ КРАЕВЫМИ УСЛОВИЯМИ ДИРИЦХЛЕ И РЕШБНИЕ ДОЗВУКОВЫХ ЗАВИХРЕННЫХ ОБТЕКАНИЙ ПРОФИЛЕЙ И РЕШЕТОК ПРОФИЛЕЙ ПРИ ПОМОЩИ ФУНКЦИИ ТОКА

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Статья посвящена разрешимости нелинейной эллиптической задачи в многосвязной области. На внутренних компонентах границы условие Дирихле известно только до аддитивных постоянных, которые надо определить вместе с неизвестным решением таким образом, чтобы были выполнены так называемые условия Кутта-Жуковского. Результаты получены при использовании строгого принципа максимума и априорных оценок решения и имеют приложения в задачах для функции тока, описивающих обтекание профилей и решеток продилей.

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