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UNIFORMLY ENCLOSING DISCRETIZATION METHODS AND GRID GENERATION FOR SEMILINEAR BOUNDARY VALUE PROBLEMS WITH FIRST ORDER TERMS

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Summary. The paper deals with uniformly enclosing discretization methods of the first order for semilinear boundary value problems. Some fundamental properties of this discretization technique (the enclosing property, convergence, the inverse-monotonicity) are proved. A feedback grid generation principle using information from the lower and upper solutions is presented.

Keywords: Discretization, boundary value problem, enclosure.

AMS Classification: 65L10.

1. INTRODUCTION

For numerical solution of differential equations it is useful to generate upper and lower bounds on the exact solution. In the present paper we consider the boundary value problem

$$(1)\qquad -u''(x) + b(x)\,u'(x) + g(x, u(x)) = 0 \quad \text{in} \quad \Omega = (a, b)\,,$$
$$u(a) = \alpha\,, \quad u(b) = \beta\,.$$

In [2] and [3] we presented uniformly enclosing discretization methods of the first order and of arbitrary orders for semilinear boundary value problems. While in [2] the case $b(x) \equiv 0$ was considered the more general approach in [3] allows to state some results for the case $b(x) \not\equiv 0$, too. Therefore, to describe some fundamental properties of enclosing discretization methods of the first order for the boundary value problem (1) it is necessary only to modify some proofs of [3]. We restrict ourselves to first order methods because in the second part we want to explain a grid generation process based on our enclosing discretization technique which is, unfortunately, proved only for first order methods. In [4] we presented this feedback grid generation method in the case $b(x) \equiv 0$. Some iteration schemes for solving the auxiliary problems generated by the monotone discretization technique were proposed in [3], [5], therefore we renounce the discussion of this question.

Let U be the Sobolev space $H_0^1(\Omega)$, i.e. the space of functions possessing square integrable generalized first order derivatives and vanishing at the endpoints. Furthermore, U^* and $\langle \cdot, \cdot \rangle$ denote the dual space $H^{-1}(\Omega)$ and the dual pairing, respectively. The norms in U and U^* are denoted by $\|\cdot\|$ and $\|\cdot\|_*$. We denote by $\|\cdot\|_0$ and $(\cdot, \cdot)$ the norm and the scalar product in the space $L^2(\Omega)$, by $\|\cdot\|_2$ the norm in the Sobolev space $H^2(\Omega)$, by $\|\cdot\|_C$ the norm in the space $C[a, b]$. Let us assume that the data of (1) fulfil

(b) $$b \in C^1[a, b], \quad b'(x) \leqq 0$$

and

(g) $$\text{(i)} \quad g(x, s) \leqq g(x, t) \quad \forall x \in [a, b], \quad s \leqq t,$$
$$\text{(ii)} \quad |g(x, s) - g(x, t)| \leqq l(r)(|x - y| + |s - t|),$$

where $l: R^+ \to R^+$ denotes some nondecreasing function. Let us define.

$$a(u, v) := (u', v') + (bu', v), \quad \langle Lu, v \rangle = a(u, v)$$

and

$$\langle Gu, v \rangle := (g(\cdot, u), v) \quad \forall v \in U.$$

We start from the following formulation: Find $u \in H^1(\Omega)$ with $u(a) = \alpha$, $u(b) = \beta$ and

(2) $$Lu + Gu = 0.$$

The coerciveness of the bilinear form and the monotonicity of g guarantee that a unique solution exists, moreover the smoothness property (g), (ii) of g ensures that the solution even is a classical one. We need the following apriori estimate:

Lemma 1. *The solution of* (2) *satisfies*

(i) $$\|u\|_C \leqq \max(|\alpha|, |\beta|) + c_1,$$

(ii) $$\|u'\|_C \leqq |c_2| + (4\|b\|_C + 2\|b'\|_C (b - a))\, c_1 + 2\|g(\cdot, u)\|_C$$

where

$$c_1 = \frac{(b - a)^2}{\gamma} [|c_2| \, \|b\|_C + \max(\|g(x, \alpha)\|_C, \|g(x, \beta)\|_C)],$$

$$c_2 = \frac{\beta - \alpha}{b - a}$$

(γ *denotes the coerciveness constant of* $a(\cdot, \cdot)$).

Proof. Let p denote the affine function

$$p(x) := \alpha \frac{b - x}{b - a} + \beta \frac{x - a}{b - a}.$$

Then u can be represented by $u = p + w$, where $w \in H_0^1(\Omega)$ satisfies the variational equation

$$(w', v') + ((c_2 + w')\, b, v) + (g(\cdot, p + w), v) = 0 \quad \forall v \in U .$$

Using the monotonicity property of g and setting $v = w$ we obtain

$$\|w\| \leqq \frac{b - a}{\gamma} (|c_2| \, \|b\|_0 + \|g(\cdot, p)\|_0) .$$

Taking into account the embedding $H_0^1(\Omega) \subsetneq C[a, b]$ and the identity $\|p\|_C = \max(|\alpha|, |\beta|)$ we obtain the first estimate (i). The application of Green's function yields the identity

$$w'(x) = \frac{1}{b - a} \left(\int_a^x (t - a) \, [(b(t)\, w'(t) + g(t, u(t))] \, \mathrm{d}t + \right.$$

$$\left. + \int_x^b (b - t) \, [b(t)\, w'(t) + g(t, u(t)] \, \mathrm{d}t \right) .$$

Integration by parts eliminating $w'(t)$ immediately leads to the second assertion of Lemma 1.

Now, let a grid $Z[a, b] = \{x_i | \, i = 0, \ldots, N\}$ be given on the interval $[a, b]$, i.e.

$$a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b .$$

We denote the corresponding step sizes and subintervals by

$$h_i := x_i - x_{i-1} , \quad \Omega_i := (x_{i-1}, x_i) ,$$

respectively. The mesh width of the grid Z is characterized by

$$h = h(Z) := \max_{1 \leqq i \leqq N} h_i .$$

We define two special bounding operators $G_h^1, G_h^2 \colon U \to U^*$ corresponding to G by

(3) $$G_h^1 \, v(x) = \max_{\xi \in [x_{i-1}, x_i]} g(\xi, v(\xi)) \quad \text{for} \quad x \in \Omega_i ,$$

$$G_h^2 \, v(x) = \min_{\xi \in [x_{i-1}, x_i]} g(\xi, v(\xi)) \quad \text{for} \quad x \in \Omega_i , \quad i = 1(1)\, N .$$

These operators have the following basic properties:

(G_1) $G_h^1 v \geqq Gv \geqq G_h^2 v$ for all $v \in U$,

(G_2) $v, w \in U$ with $v \leqq w$ implies

$$G_h^1 v \leqq G_h^1 w \quad \text{a.e. in} \quad \Omega , \quad G_h^2 v \leqq G_h^2 w \quad \text{a.e. in} \quad \Omega .$$

The approximate problems corresponding to (2) are: Find $u_h^1, u_h^2 \in U$ such that

(4) $$\text{(i)} \quad L u_h^1 + G_h^1 u_h = 0$$

and

(4) (ii) $Lu_h^2 + G_h^2 u_h^2 = 0$.

These discrete problems seem to look rather unusual. However, (4) is equivalent to

(5) (i) $-u_h''(x) + b(x)\, u_h'(x) = -G_h(u_h)$ for $x \in \Omega_i$ $(i = 1(1)\,N)$

and

(5) (ii) $u_h \in C^1[a, b]$.

Thus, taking into account that $G_h(u_h)$ is constant on Ω_i we can note a finite-dimensional analogue for (4). To do this we define continuous trial functions $\varphi_i(x)$ and $\psi_i(x)$ by

(6) (i) $-\varphi_k''(x) + b(x)\, \varphi_k'(x) = 0$ for $x \in \Omega_i$ $(i = 1(1)\,N)$,
$$\varphi_k(x_j) = \delta_{kj} \quad (j = 0(1)\,N)$$

and

(6) (ii) $-\psi_k''(x) + b(x)\, \psi_k'(x) = \delta_{kj}$ for $x \in \Omega_j$ $(j = 1(1)\,N)$,
$$\psi_k(x_i) = 0 \quad (i = 0(1)\,N).$$

If we choose the representation

(7)
$$u_h^1(x) = \sum_{i=0}^{N} w_i\, \varphi_i(x) + \sum_{i=1}^{N} z_i\, \psi_i(x)$$

the problem (4) (i) is equivalent to

(8)
$$z_i + \max_{\xi \in [x_{i-1}, x_i]} (\xi, u_h^1(\xi)) = 0 \quad (i = 1(1)\,N),$$
$$u_h^1 \in C^1[a, b] \quad (\text{or } u_h'^1(x_i + 0) = u_h'^1(x_i - 0),\ i = 1(1)\,N - 1).$$

It is also possible to use C^1-trial functions from the beginning.

Lemma 2. *Let there exist solutions u_h^1, u_h^2 of (4). Then the solution u of the initial problem (2) is enclosed by*
$$u_h^1 \leqq u \leqq u_h^2\,;$$
furthermore, the error estimate

(9)
$$\|u - u_h\|_C \leqq \frac{(b - a)^{3/2}}{\gamma} \|G_h u - g(\cdot, u)\|_0$$

is valid.

Proof. We restrict our consideration to the subsolution u_h^1. The definitions of the continuous and the discrete problem lead to
$$a(u_h^1, v) + (g(\cdot, u_h^1), v) \leqq a(u_h^1, v) + (G_h^1 u_h^1, v) = 0 = a(u, v) + (g(\cdot, u), v)$$
for all $v \in H_0^1(\Omega)$ $(v \geqq 0)$. Thus the inverse-monotonicity of our original problem results in $u_h^1 \leqq u$.

To show (9) we start from

$$a(u - u_h, v) = (G_h u_h - G_h u, v) + (G_h u - g(\cdot, u), v) \quad \text{for all} \quad v \in H_0^1(\Omega)\,.$$

Choosing $v = u - u_h$ we have $v \geqq 0$ and the property (G_2) implies

$$\gamma \|u - u_h\|^2 \leqq (G_h u - g(\cdot, u), u - u_h)\,.$$

Thus, (9) immediately follows.

The property (g) (ii) enables us to estimate the right hand side of (9) in a simple way:

$$\|G_h u - g(\cdot, u)\|_0 \leqq (b - a)^{1/2}\, l(\|u\|_C)\,(1 + \|u'\|_C)\, h\,. \tag{10}$$

Combining (9) and (10) we obtain the final error estimate. To prove that a solution of the discrete problem exists is more difficult. Using the theory of pseudomonotone operators and an auxiliary variational inequality, respectively, we proved in [3] and [5] that there exists a number $h_0 > 0$ such that the discrete problem (4) for all $h \in (0, h_0]$ admits a solution u_h with $\|u_h\|_C \leqq \varrho$ if $\|u\|_C < \varrho$.

Theorem 1. *Let $h \leqq h_0$. Then the solution of the original problem (2) is enclosed by*

$$u_h^1 \leqq u \leqq u_h^2\,;$$

furthermore, the error estimate

$$\|u - u_h\|_C \leqq Ch \tag{11}$$

is valid.

It is possible to estimate the constant C in (11) by combining (9), (10) and the bounds from Lemma 1. Later it will be essential for us to know that

$$C \sim (b - a)^2\, l(\|u\|_C)\,(1 + \|u'\|_C)\,. \tag{12}$$

Remark. In [3] we proved

$$\|u - u_h\|_C \leqq C^* h^{1/2}$$

under the assumption

$$|g(x, s) - g(x, t)| \leqq l(r)\,(|x - y|^{1/2} + |s - t|)$$

without using the property (G_2). To obtain first order convergence it was necessary to introduce some additional assumptions about the partial derivatives of g. Taking into account the property (G_2) we can weaken the conditions on g to (g), (ii).

All results remain true if $b(x)$ is a piecewise constant function with $b(x) = b_i$ on Ω_i and $b_i \geqq b_{i+1}$. In this case the trial functions are

$$\varphi_i(x) = \begin{cases} (\exp(b_i x) - \exp(b_i x_{i-1}))\,(\exp(b_i x_i) - \exp(b_i x_{i-1}))^{-1} & \text{for } x \in [x_{i-1}, x_i] \\ (\exp(b_i x) - \exp(b_i x_{i+1}))\,(\exp(b_i x_i) - \exp(b_i x_{i+1}))^{-1} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi_i(x) = \begin{cases} h_i \dfrac{\exp(b_i x_{i-1}) - \exp(b_i x)}{\exp(b_i x_i) - \exp(b_i x_{i-1})} + x - x_{i-1} & \text{for} \quad x \in [x_{i-1}, x_i] \\ 0 & \text{otherwise} \end{cases}.$$

The nonlinear system (8) reads

$$z_i + \max_{\xi \in [x_{i-1}, x_i]} (\xi, u_h^1(\xi)) = 0 \quad (i = 1(1)\, N),$$

$$\frac{w_i - w_{i-1}}{e_i} - \frac{w_{i+1} - w_i}{f_i} - (h_i e_i - 1)\, z_i + (h_{i+1} f_i - 1)\, z_{i+1} = 0$$

$$(i = 1(1)\, N - 1), \quad w_0 = w_N = 0$$

with

$$e_i = \frac{b_i \exp(b_i x_i)}{\exp(b_i x_i) - \exp(b_i x_{i-1})}, \quad f_i = \frac{b_i \exp(b_i x_i)}{\exp(b_i x_{i+1}) - \exp(b_i x_i)}.$$

Numerical experiments confirm the estimate (11). For the model problem

$$-u'' + u' + u^3 - \sin^3(\pi x) - \pi^2 \sin(\pi x) - \pi \cos(\pi x) = 0,$$
$$u(0) = u(1) = 0$$

with the exact solution $u(x) = \sin(\pi x)$ we present the maximal difference of the upper and lower solutions at the gridpoints:

$N = 5$	$N = 10$	$N = 20$	$N = 50$	$N = 100$
0·373 962	0·179 253	0·083 035	0·035 272	0·017 422

Closing this chapter we remark that u_h' converges to u', too.

Lemma 3. The following estimate holds:

$$\|u' - u_h'\|_C \leqq Ch.$$

Proof. We start from

$$a(u - u_h, v) = (G_h u_h - G_h u, v) + (G_h u - g(\cdot, u), v). \tag{12}$$

Because the "right hand side" $G_h u_h - g(\cdot, u)$ belongs to the space $L^2(\Omega)$ there exists a constant κ such that

$$\|u - u_h\|_2 \leqq \kappa \|G_h u_h - g(\cdot, u)\|_0 \leqq$$
$$\leqq \kappa(\|G_h u_h - G_h u\|_\infty + \|G_h u - g(\cdot, u)\|_0) \leqq$$
$$\leqq \kappa(\|u - u_h\|_C + \|G_h u - g(\cdot, u)\|_0).$$

Thus the embedding $H^2 \subset C^1[a, b]$ and (10), (11) yield the assertion.

In this section we propose a grid generation principle using information from the lower and upper solutions. The basic idea is to subdivide those intervals where the difference between the upper and lower solutions is relatively large.

First we need some information about the behaviour of our upper and lower solutions if the grid becomes finer. This property is based on some kind of inverse-monotonicity of our discrete problems.

Lemma 4. *Let us assume*

(13) (i) $-u''(x) + b(x)\,u'(x) + G_h\,u(x) \leqq -v''(x) + b(x)\,v'(x) + G_h\,v(x)$ *for* $x \in \Omega_i$ $(i = 1(1)\,N)$,

(ii) $u, v \in C^1[a, b]$,

(iii) $u(a) \leqq v(a)$, $u(b) \leqq v(b)$.

Then $u(x) \leqq v(x)$ *for sufficiently small* h.

Proof. We set $w(x) = v(x) - u(x)$ and $w(x^*) = \min_{x\in[a,b]} w(x)$. In the case $x^* \in \{a, b\}$, $w(x) \geqq 0$ follows immediately. Otherwise we have

$$w'(x^*) = 0\,, \quad w''(x^*) \geqq 0 \quad \text{for} \quad x^* \in D_i\,,$$
$$w'(x^*) = 0\,, \quad w''(x^* + 0) \geqq 0\,. \quad w''(x^* - 0) \geqq 0 \quad \text{for} \quad x = x_i\,.$$

Taking into account (13), (i) we obtain

$$0 \leqq w''(x^*) \leqq [G_h v]\,(x^*) - [G_h u]\,(x^*)\,.$$

Thus there exists $\xi \in \Omega_i$ with

$$0 \leqq [Gv]\,(\xi) - [Gu]\,(\xi)\,.$$

Taking into account the monotonicity of g we obtain

$$w(\xi) = v(\xi) - u(\xi) \geqq 0\,.$$

Now we choose

$$w(x^+) := \max_{x\in[x_{i-1},x_i]} w(x) \geqq 0\,.$$

Then

$$w(x^*) = w(x^+) - \int_{x^*}^{x^+}\int_{x^+}^{t} w''(\tau)\,\mathrm{d}\tau\,\mathrm{d}t = w(x^+) - w''(\eta)\left(\frac{x^+ - x^*}{2}\right)^2 \tag{14}$$

where $w''(\eta) \geqq 0$. Now we estimate $w''(\eta)$. From (13), (i) we obtain

$$\|w''(x)\|_C \leqq \|b\|_C\,\|w'(x)\|_C + \|w\|_C\,.$$

Further,

$$\|w'\|_{C[x_{i-1},x_i]} \leqq \frac{2}{h_i}\,\|w\|_{C[x_{i-1},x_i]} + \frac{h_i}{2}\,\|w''\|_{C[x_{i-1},x_i]}\,.$$

Assuming $h_i \|b\|_{C[x_{i-1}, x_i]} \leqq \frac{1}{2}$ we conclude

$$\|w''\|_{C[x_{i-1}, x_i]} \leqq \delta \|w\|_{C[x_{i-1}, x_i]} \quad \text{with} \quad \delta = 2 + \frac{4}{h_i}. \tag{15}$$

There are two possibilities:

(a) $$w''(\eta) \leqq -\delta\, w(x^*) \quad \text{if} \quad w(x^*) < 0\,,$$

or

(b) $$w''(\eta) \leqq \delta\, w(x^+)\,.$$

In the case (a) the identity (14) implies

$$w(x^*)\left(1 - \delta\left(\frac{x^+ - x^*}{2}\right)^2\right) \geqq w(x^+)\,.$$

This inequality cannot hold for sufficiently small $|x^+ - x^*|$ because $w(x^*) < 0$ and $w(x^+) \geqq 0$. In the case (b) we have

$$w(x^*) \geqq \left(1 - \delta\left(\frac{x - x^*}{2}\right)^2\right) w(x^+)\,.$$

Thus, $w(x^*) \geqq 0$ for sufficiently small $|x^+ - x^*|$.

It is obvious that our bounding operator has the following property:

(G_3) For any finer grid Z_f, i.e. when $Z \subset Z_f$, the estimates $G_Z^1 v \geqq G_{Z_f}^1 v$ a.e. in Ω, $G_Z^2 v \leqq G_{Z_f}^2 v$ a.e. in Ω hold for all $v \in C[a, b]$.

Henceforth we change the notation a little $(G_h := G_Z)$ in order to characterize the dependence on the actual grid.

Lemma 5. *Given a grid Z_f finer than Z, then the related solutions $u_{Z_f}^1$ and $u_{Z_f}^2$ improve the two-sided inclusion $u_Z^1 \leqq u \leqq u_Z^2$, that is,*

$$u_Z^1(x) \leqq u_{Z_f}^1(x) \leqq u(x) \leqq u_{Z_f}^2(x) \leqq u_Z^2(x) \quad \textit{for all} \quad x \in [a, b]\,.$$

The proof follows immediately from Lemma 4 if we write our discrete problems in the form (5) and take into account the property (G_3) of our bounding operators.

Now we proceed to describing the grid generation principle in detail. Let an initial grid $Z^1 = \{x_i^1 \mid i = 0, \ldots, N_1\}$ be given. We denote the related subintervales by

$$\Omega_i^1 := (x_{i-1}^1, x_i^1) \quad (i = 1(1)\, N_1)\,.$$

We assume $h(Z^1)$ to be small enough so that each of the discrete problems possesses a solution and the inverse-isotonicity of the discrete problems holds. The algorithm under consideration generates a sequence (Z^k) of grids

$$\Omega_i^k := (x_{i-1}^k, x_i^k) \quad (i = 1(1)\, N_k)\,.$$

We denote the bounding operators on the grids by $G^{1,k}$ and $G^{2,k}$.

Algorithm

Step 1: Let an initial grid Z^1 be given. Choose $\varrho \in (0, 1)$ and set $k = 1$.

Step 2: Determine $u^{1,k}, u^{2,k} \in H^1(\Omega)$ with

$$u^{1,k}(a) = u^{2,k}(a) = \alpha\,, \quad u^{1,k}(b) = u^{2,k}(b) = \beta$$

such that

$$a(u^{1,k}, v) + (G^{1,k}u^{1,k}, v) = a(u^{2,k}, v) + (G^{2,k}u^{2,k}, v) = 0 \quad \text{for all} \quad v \in H_0^1(\Omega)\,.$$

Step 3: Introducing the notation

$$d_i^k := \max_{x \in \Omega_i^k} (u^{2,k}(x) - u^{1,k}(x)) \; (i = 1(1)\,N_k)\,, \quad D^k := \max_{1 \leqq i \leqq N_k} d_i^k\,,$$

$$I_k := \{i \in \{1, \ldots, N_k\} : d_i^k \geqq \varrho D^k\}\,,$$

$$y_i^k := \tfrac{1}{2}(x_{i-1}^k + x_i^k) \quad (i \in I_k)$$

define a new grid Z^{k+1} by

$$Z^{k+1} = Z^k \cup \{y_i^k : i \in I^k\}\,.$$

Set $N_{k+1} = N_k + \operatorname{card} I_k$ and denote the grid points contained in Z^{k+1} by x_i^{k+1} $(i = 0, 1, \ldots, N_{k+1})$. Reset $k^{\text{new}} := k^{\text{old}} + 1$ and go to Step 2.

Theorem 2. *The functions* $u^{1,k}, u^{2,k}$ *generated by the above algorithm satisfy*

$$u^{1,k}(x) \leqq u^{1,k+1}(x) \leqq u(x) \leqq u^{2,k+1}(x) \leqq u^{2,k}(x) \quad \textit{for} \quad x \in [a, b]\,,$$

and

$$\lim_{k \to \infty} \|u^{2,k} - u^{1,k}\|_C = 0\,.$$

Proof. Due to the properties of the algorithm the grids are successively refined. Thus Lemma 5 immediately yields the monotone behaviour of the sequences $\{u^{1,k}\}$, $\{u^{2,k}\}$.

The sequence (D^k) satisfies $0 \leqq D^{k+1} \leqq D^k$. Thus this sequence is convergent. Let us assume that $\lim\limits_{k \to \infty} \|u^{2,k} - u^{1,k}\|_C \neq 0$. We select sequences (a_k) and (b_k) with $a_k < b_k$ possessing the following properties:

$$(16) \qquad [a_k, b_k] \subset \bigcup_{i \in I_k} [x_{i-1}^k, x_i^k]\,, \quad a_k, b_k \in \partial \bigcup_{i \in I_k} [x_{i-1}^k, x_i^k]\,,$$

$$\|u^{2,k} - u^{1,k}\|_{C[a_k, b_k]} = D^k\,.$$

Let us denote

$$(17) \qquad \alpha_{1,k} := u^{1,k}(a_k)\,, \quad \beta_{1,k} := u^{1,k}(b_k)\,, \quad \alpha_{2,k} := u^{2,k}(a_k)\,, \quad \beta_{2,k} := u^{2,k}(b_k)\,.$$

We define $w^{1,k}, w^{2,k}$ by the boundary conditions (17) and the variational equations

$$Lw^{1,k} + Gw^{1,k} = 0\,,$$

$$Lw^{2,k} + Gw^{2,k} = 0\,.$$

By virtue of the inverse-monotonicity of our original problem our definitions result in

$$w^{1,k}(x) \leqq u(x) \leqq w^{2,k}(x) \quad \text{for all} \quad x \in [a_k, b_k]$$

and

$$\|w^{2,k} - w^{1,k}\|_{C[a_k,b_k]} < \varrho D^k . \tag{18}$$

On the other hand, our discretization principle implies

$$u^{1,k}(x) \leqq w^{1,k}(x) , \quad w^{2,k}(x) \leqq u^{2,k}(x) \quad \text{for all} \quad x \in [a_k, b_k] .$$

Theorem 1 yields the estimates

$$\|w^{1,k} - u^{1,k}\|_{C[a_k,b_k]} \leqq C_k(w^{1,k})\, h(Z^k) ,$$

$$\|w^{2,k} - u^{2,k}\|_{C[a_k,b_k]} \leqq C_k(w^{2,k})\, h(Z^k) .$$

Using the estimate (12) for C_k and the a-priori estimates for $w^{1,k}$, $w^{2,k}$ due to Lemma 1 it is not difficult to see that there exists a constant C independent of $w^{1,k}$, $w^{2,k}$ and $b_k - a_k$ such that

$$\|w^{1,k} - u^{1,k}\|_{C[a_k,b_k]} \leqq Ch(Z^k), \ \|w^{2,k} - u^{1,k}\| \leqq Ch(Z^k) . \tag{19}$$

Combining (18), (19) we obtain via the triangle inequality

$$\|u^{2,k} - u^{1,k}\|_{C[a_k,b_k]} \leqq \varrho D^k + Ch(Z^k) . \tag{20}$$

For sufficiently large k, (20) contradicts one of the conditions (16) defining the sequences (a_k), (b_k). Thus we have proved $\lim_{k \to \infty} D^k = 0$ which is equivalent to the second assertion of Theorem 2.

It is essential to note that we do not need any assumption on the mesh width.

In [3] we developed uniformly enclosing discretization methods of arbitrary orders. When we combined these methods and the proposed grid generation algorithm, numerical experiments resulted in improved error bounds in comparison to an equidistributed grid of the same cardinality, especially for problems with boundary layers. However, the theoretical foundation of the feedback grid generation cannot be carried out similarly to the first-order technique, because the first step of the convergence proof of Theorem 2 consists in the conclusion

$$0 \leqq D^{k+1} \leqq D^k$$

based on Lemma 5, and it is not clear whether or not Lemma 5 holds for higher order methods. Namely, the corresponding bounding operators, in general, do not satisfy (G_3) and therefore the investigation of the statement of Lemma 5 leads to significant difficulties.

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Souhrn

DISKRETIZAČNÍ METODY VÝPOČTŮ STEJNOMĚRNÝCH OBOUSTRANNÝCH APROXIMACÍ A ADAPTIVNÍ VYTVÁŘENÍ SÍTÍ PRO SEMILINEÁRNÍ OKRAJOVÉ ÚLOHY S ČLENY PRVNÍHO ŘÁDU

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Článek se zabývá metodami diskretizace prvního řádu pro semilineární okrajové úlohy, které dávají stejnoměrné oboustranné aproximace přesného řešení. Jsou dokázány některé základní vlastnosti této diskretizační techniky (vlastnosti dolních a horních aproximací, konvergence, inverzní monotonie). Je popsán princip adaptivního vytváření sítě, který používá informace získané z dolních a horních aproximací.

Резюме

МЕТОДЫ ДИСКРЕТИЗАЦИИ ВЫЧИСЛЕНИЙ РАВНОМЕРНЫХ ДВУСТОРОННИХ АППРОКСИМАЦИЙ И АДАПТИВНОЕ ФОРМИРОВАНИЕ СЕТЕЙ ДЛЯ ПОЛУЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ С ЧЛЕНАМИ ПЕРВОГО ПОРЯДКА

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В статье изучаются методы дискретизации первого порядка для полулинейной краевой задачи, которые дают равномерные двусторонние аппроксимации точного решения. Доказаны некоторые основные свойства этой техники дискретизации (свойства нижних и верних аппроксимаций, сходимость, обратная монотонность) и описан принцип адаптивного формирования сети, использующей информацию извлеченную из нижних и верхних аппроксимаций.

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