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ON STABLE POLYNOMIALS

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Summary. The article is a survey on problems of the theorem of Hurwitz. The starting point of explanations is Schur's decomposition theorem for polynomials. It is showed how to obtain the well-known criteria on the distribution of roots of polynomials. The theorem on uniqueness of constants in Schur's decomposition seems to be new.

Keywords: stable polynomial, criterion of Routh, Hurwitz, Hermite, decomposition of Schur.

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1. PREFACE

We meet the criterion of Hurwitz, for example, when considering stability of solutions of ordinary differential equations with constant coefficients. As to the proof of the assertion, we are mostly refered to special literature. In addition to the usual lack of space, the reason for omitting the proof certainly is that it is by no means easy, involving facts from algebra as well as from the theory of complex variable. This is corroborated, for example, by R. Bellman's comment in [7] that there exists no simple proof of Hurwitz's theorem.

In the present paper we will show some aspects of the problem as well as one of the many possibilities how to explain it. Our starting point will be Schur's idea of decomposition. The theorem on invariance of constants in Schur's decomposition seems to be new.

2. TWO BASIC PROBLEMS

Consider a differential equation of degree n with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 .$$

The study of properties of solutions of the differential equation for $t \to +\infty$ induces the notion of stability. A solution of the differential equation is considered stable when, roughly speaking, for any small change of the initial values (for some t_0) the values of the new solution (as well as the values of its derivatives to degree $n - 1$) differ only little from the values of the original one for any $t \geqq t_0$. For equations with constant coefficients, the fundamental system of solutions is given by functions of the form e^{zt} or $t^k e^{zt}$ where k is a natural number and the (complex) number z is the root of the characteristic equation

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0 . \tag{2.1}$$

The requirement of asymptotic stability of solutions implies that the real part of each root of Equation (2.1) be negative. Such a polynomial is said to be stable.

Consider a difference equation

$$a_0 y(j + n) + a_1 y(j + n - 1) + \ldots + a_n y(j) = 0 , \quad j = 0, 1, 2, \ldots$$

with constant coefficients. The fundamental system of solutions is given by sequences of the form $\{w^j\}$ or $\{j^k w^j\}$ where k is a natural number and the (complex) number w fulfils the equation (2.1). The requirement of asymptotic stability of solutions implies that the absolute value of each root of Equation (2.1) be less than one. Using the function $w = (1 + z)/(1 - z)$ which transforms the half-plane $\operatorname{Re}(z) < 0$ of the complex plane one-to-one onto the domain $|w| < 1$ we obtain the condition for stability in the form that the real part of each root of the polynomial $Q(z) = (1 - z)^n P((1 + z)/(1 - z))$ is negative.

3. FROM THE HISTORY OF THE PROBLEM

We present here but some important data, more details can be found in [6]. Cauchy (1837) showed that the number of roots of a polynomial in a given region of the complex plane can be expressed by the index of a certain rational function. In the case of a half-plane, the index can be simply determined by the theorem of Sturm that was published in 1827. Thus the problem was in essence solved, but Cauchy did not give any effective criterion. Ch. Hermite (1856), see [1], showed that the stability condition for a polynomial is equivalent to the positive definiteness of a certain quadratic form (the famous "Hermite's forms"). Only twenty years later (1877) E. J. Routh [2] gave a very elegant and simple solution of the problem for polynomials with real coefficients. Later, the stability problem of polynomials was being solved again by some engineering specialists in special cases of polynomials up to the third degree. A. Stodola, a scientist of Slovak origin, an outstanding worker in the theory of turbines, formulated again the problem of finding a general criterion of stability; Stodola, evidently, did not know the result of Routh. Then, in 1895, A. Hurwitz [3] solved the problem of Stodola independently of Routh's work that he did not know, either. Using the results of Cauchy, Hermite, Sturm and the theory of quadratic forms (not long ago Frobenius had published the law of inertia of quadratic forms) as well as further advanced mathematical means he obtained a classical criterion of outstanding elegance in the form of certain inequalities with determinants, in the special case of polynomials with real coefficients. Later investigations showed additional connections leading both to simplifications of proofs and to the generalization of the results to polynomials with complex coefficients. (It should be noted that already Hermite's criterion was formulated for polynomials with complex coefficients.) It was shown, for example, that the criterion of Hurwitz differs not too much from the criterion of Routh, having only another form. For this reason, it is often called the criterion of Routh-Hurwitz, being usually applied in the form due to Hurwitz. In 1921, J. Schur [4] presented the method of related polynomials, by means of which it is possible to deduce the criterion of Hurwitz. Later on, the attraction of the topic was, time and again, demonstrated by a number of works that are listed in [6], [8]. The problem has been studied by means of quadratic forms of Hankel, continued fractions and many other methods.

In the meantime, the theory of retarded differential equations, difference differential equations and, finally, the theory of functional differential equations arose. From the end of the last century, the ideas of Liapunov were dominating in the theory of stability, especially the method of Liapunov's functions, see for example [9]. Thus, from this more general point of view, the classical theorem of Routh-Hurwitz or Hermite solves the problem of stability in a special, even though a very important case. In connection with stability of solutions of difference differential equations, the important work of Pontryagin [5] appeared in 1942 which generalized the classical results of Hurwitz.

4. THE INCREMENT OF ARGUMENT AND THE RELATED POLYNOMIALS

The notion of related polynomials introduced by J. Schur [4] offers a comparatively simple way of proving the Hurwitz-Routh theorem; such an approach was chosen, for example, in [9].

In what follows, N is the set of all natural numbers, N_0 the set of all nonnegative integers, R and C the sets of real and complex numbers, respectively. If $z = x + \mathrm{i}y$, $x \in R$, $y \in R$ and i is the imaginary unit, we denote $\operatorname{Conj}(z) = \bar{z} = x - \mathrm{i}y$, $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$. By M_n, $n \in N_0$ we denote the set of all polynomials P,

$$P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n\,, \quad a_0 \neq 0 \tag{4.1}$$

where $a_j \in C$, $j = 1, 2, \ldots, n$. A polynomial is called stable (or Hurwitz), if the real parts of all its roots are negative. By S_n, $n \in N_0$ we denote the set of all stable polynomials of degree n. For $n = 0$ we have, of course, $S_0 = M_0$.

Definition 4.1. *For $P \in M_n$ we denote by* $\mathrm{d}P$ *the increment of argument (angle) of the polynomial $P(z)$ when z passes through the imaginary axis $\{z: z = \mathrm{i}y, y \in R\}$ from $y = -\infty$ to $y = +\infty$.*

Lemma 4.1. *Let $n \in N$, let $P \in S_n$ be a polynomial of the form* (4.1). *Then* $\operatorname{Re}(a_1/a_0) > 0$, *which is equivalent to*

$$a_0\bar{a}_1 + \bar{a}_0 a_1 = 2 \operatorname{Re}(a_0\bar{a}_1) > 0 . \tag{4.2}$$

For $n = 1$ the condition (4.2) *implies $P \in S_1$.*

Proof. If $z_1, z_2, \ldots, z_n$ are the roots of P, then $-\Sigma z_j = a_1/a_0 = (\bar{a}_0 a_1)/(a_0\bar{a}_0)$. As $P \in S_n$ by assumption, we have $\operatorname{Re}(\Sigma z_j) < 0$, hence $\operatorname{Re}(a_0\bar{a}_1) > 0$, since $a_0\bar{a}_0$ is positive. The assertion for $n = 1$ is obvious.

Lemma 4.2. *Let $n \in N_0$, $P \in M_n$. Then*

$$\mathrm{d}P = (l - r)\,\pi \tag{4.3}$$

where l, r is the number of roots w of P such that $\operatorname{Re}(w) < 0$, $\operatorname{Re}(w) > 0$, *respectively. In particular, $P \in S_n$ if and only if $\mathrm{d}P = n\pi$. For $n = 0$ we have, of course,* $\mathrm{d}P = 0$.

Proof. 1. First we assume that none of the roots of the polynomial P lies on the imaginary axis. Then it suffices to realize that for the polynomial $P(z) = z - w$, $w \in C$, we have $\mathrm{d}P = \pi$ or $-\pi$ according to whether $\operatorname{Re}(w) < 0$ or $\operatorname{Re}(w) > 0$, and to use the properties of the argument in connection with the decomposition $P(z) = a_0 \prod_{j=1}^{n} (z - z_j)$.

2. If P has some roots on the imaginary axis, then $\mathrm{d}P$ is defined as follows. If, for example, $\mathrm{i}b$, $b \in R$ is the only pure imaginary root of P, we denote by $\mathrm{d}_1(P, b - v)$, $\mathrm{d}_2(P, b + v)$, $v > 0$ the increment of the argument of $P(\mathrm{i}y)$ when y passes from $-\infty$ to $b - v$ or from $b + v$ to $+\infty$, defining, finally, $\mathrm{d}P = \lim(\mathrm{d}_1 + \mathrm{d}_2)$ as $v \to 0+$. In the case of more than one pure imaginary roots, the generalization is obvious. For a polynomial P of the first degree, $P(z) = z - \mathrm{i}b$, $b \in R$ we obtain, of course, $\mathrm{d}P = 0$. If $P(z) = P_1(z)\,P_2(z)$ where P_1 has not roots on the imaginary axis and P_2 has only pure imaginary roots, then $\mathrm{d}P_2 = 0$ which implies $\mathrm{d}P = \mathrm{d}P_1$. We see that the formula (4.3) holds in general.

Remark. The increment of the argument of a polynomial P can be defined for any straight line parallel to the imaginary axis (and, more generally, for some other oriented curves). Denoting by $\mathrm{d}P(t)$, $t \in R$ the increment of the argument of P along the straight line $\{z: z = t + \mathrm{i}y,\ y \in R\}$ when y passes from $-\infty$ to $+\infty$, $\mathrm{d}P(t)$ is defined for all $t \in R$ except those for which the polynomial P has a root on the corresponding line. For this t we can complete the definition by using the argument given for the case $t = 0$ in the proof of the previous lemma. The function $\mathrm{d}P$ is then defined for all $t \in R$, it is nondecreasing, piecewise constant with jumps at those points t for which P has a root on the straight line $\operatorname{Re}(z) = t$. For such t we have $\mathrm{d}P(t) =$

$= (1/2) \lim_{h \to 0} (\mathrm{d}P(t + h) + \mathrm{d}P(t - h))$. Of course, the last equation is true for all $t \in R$. It can be used as a definition of $P(t)$ provided P has a root on the straight line $\operatorname{Re}(z) = t$.

Now we define the notion of the primary and the related polynomial. Having a polynomial P of the form (4.1) we define a polynomial P^* by

$$P^*(z) = (-1)^n \operatorname{Conj}(P(-\bar{z})) . \tag{4.4}$$

It is easy to find that

$$P^*(z) = \bar{a}_0 z^n - \bar{a}_1 z^{n-1} + \bar{a}_2 z^{n-2} - \dots + (-1)^n \bar{a}_n . \tag{4.5}$$

We see that $w \in C$ is a root of the polynomial (4.1) if and only if $-\bar{w}$ is a root of the polynomial (4.5). Thus the roots of these polynomials are mutually symmetric with respect to the imaginary axis.

Definition 4.2. *Let $n \in N_0$, $P \in M_n$, $Q \in M_{n+1}$. If there are $a \in C$, $a \neq 0$, $c \in R$, $c \neq 0$ such that*

$$a\, Q(z) = (z + c)\, P(z) + z\, P^*(z) , \tag{4.6}$$

we say that P is primary (with respect) to Q. A polynomial $P_1 \in M_n$ is called related to Q if there is a polynomial P primary to Q, and a constant $b \in C$, $b \neq 0$ such that $P_1 = bP$.

Lemma 4.3. *Let $n \in N_0$, let $P \in M_n$ be of the form (4.1), $Q \in M_{n+1}$. If (4.6) holds for $c \in R$, $c \neq 0$, $a \in C$, $a \neq 0$ (i.e., P is primary to Q), then* 1. $\operatorname{Re}(a_0) \neq 0$. 2. *If P has some roots on the imaginary axis, then Q has the same roots (including multiplicity) on the imaginary axis, and vice versa. Furthermore,*

$$\mathrm{d}Q = \mathrm{d}P + \pi \operatorname{sign} c .$$

In addition, if $P \in S_n$ and $c > 0$, then $Q \in S_{n+1}$.

Proof. 1. Assume that the polynomial P has the form (4.1). Since $Q \in M_{n+1}$, the equality $\operatorname{Re}(a_0) = 0$ cannot hold. In the opposite case we have $\bar{a}_0 = -a_0$, so we do not obtain a polynomial of degree $n + 1$ on the right-hand side of (4.6). Thus we have $\operatorname{Re}(\bar{a}_0/a_0) > -1$.

2. Rewriting (4.6) in the form

$$a\, Q(z) = (z + c)\, P(z)\, (1 + g(z)) , \quad g(z) = \frac{z}{z + c} P^*(z)/P(z)$$

we see that $\mathrm{d}Q = \pi \operatorname{sign} c + \mathrm{d}P + \mathrm{d}h$ where $h(z) = 1 + g(z)$. We shall show that $\mathrm{d}h = 0$. If $y \in R$, then $|g(\mathrm{i}y)| < 1$, because $|\mathrm{i}y/(\mathrm{i}y + c)| < 1$ and $|P^*(\mathrm{i}y)/P(\mathrm{i}y)| = 1$. The last equality is obvious in the case when $\mathrm{i}y$ is not a root of the polynomial P, since then it is an easy consequence of the formulas $P(z) = a_0 \Pi(z - z_j)$, $P^*(z) = \bar{a}_0 \Pi(z + z_j)$, as the roots of the polynomials P, P^* are mutually symmetric with respect to the imaginary axis. If $\mathrm{i}y$ is a root of P, it is at the same time a root of P^* with the same multiplicity. Hence, P^*/P can be defined at the point $\mathrm{i}y$ in such a way that it becomes continuous at this point (even holomorphic); then, by P^*/P we understand its continuous extension. At the same time we see that if P has some pure imaginary roots, then Q has the same roots including multiplicity, and vice versa. Consequently, for all $y \in R$ we have $\mathrm{Re}(1 + g(\mathrm{i}y)) > 0$. Therefore, the argument of the function $1 + g(\mathrm{i}y)$ can be taken (continuously) from the interval $(-\pi/2, \pi/2)$. Using (4.1), (4.5) we obtain for large z (in particular, for large $\mathrm{i}y$) the estimate $h(z) = 1 + g(z) = 1 + (1 + c/z)^{-1} \bar{a}_0(1 + \ldots)/(a_0(1 + \ldots)) = 1 + \bar{a}_0/a_0 + \ldots$, where the omitted quantities are of degree at least $1/z$. Therefore $\lim(1 + g(\mathrm{i}y)) = 1 + \bar{a}_0/a_0$ as $y \to +\infty$ or $y \to -\infty$. Hence, by virtue of $\mathrm{Re}(1 + \bar{a}_0/a_0) > 0$ we obtain $\mathrm{d}h = 0$, completing the proof.

Now we show how to find primary polynomials.

Lemma 4.4. *Let $n \in N$, let $P \in M_n$ be of the form (4.1). Then*

1. *If $a_0\bar{a}_1 + \bar{a}_0 a_1 = 0$, then there is no primary polynomial to P.*
2. *Let $a_0\bar{a}_1 + \bar{a}_0 a_1 \neq 0$. Define the (real) number c by*

$$c = a_1/a_0 + \bar{a}_1/\bar{a}_0 = (a_0\bar{a}_1 + \bar{a}_0 a_1)/(a_0\bar{a}_0) \tag{4.7}$$

and the polynomial P_1 by

$$P_1(z) = \bar{a}_0 c\, P(z) - z(\bar{a}_0\, P(z) - a_0\, P^*(z))\,. \tag{4.8}$$

Then $P_1 \in M_{n-1}$, P_1 is primary to P and

$$\mathrm{d}P = \mathrm{d}P_1 + \pi \operatorname{sign} c \tag{4.9}$$

where, in the last equation, P_1 can be replaced by an arbitrary polynomial related to P. In addition, if $P \in S_n$, then P_1 exists and $P_1 \in S_{n-1}$.

Proof. 1. Let $P_1 \in M_{n-1}$ be primary to P. Then Equation (4.6) holds for suitable a, c if we write P, P_1 instead of Q, P, respectively. Denoting $P_1(z) = a_0' z^{n-2} + a_1' z^{n-2} + \ldots$ we have $P_1^*(z) = \bar{a}_0' z^{n-1} - \bar{a}_1' z^{n-2} + \ldots$. Comparing the coefficients at the powers z^n, z^{n-1} in (4.6) we obtain $aa_0 = a_0' + \bar{a}_0'$, $aa_1 = a_1' - \bar{a}_1' + ca_0'$. Therefore, first, $a_0' + \bar{a}_0' \neq 0$. Multiplying aa_0 by $\bar{a}\bar{a}_1$ and doing the same with their conjugates we find by adding that $a\bar{a}(a_0\bar{a}_1 + \bar{a}_0 a_1) = (a_0' + \bar{a}_0')^2 c$. Now, using $a \neq 0$, $a_0' + \bar{a}_0' \neq 0$, $c \neq 0$ we get $a_0\bar{a}_1 + \bar{a}_0 a_1 \neq 0$, proving the first part of the theorem.

2. First, let $c \in R$, $c \neq 0$. With regard to the definition of c, the polynomial P_1 defined by (4.8) is of degree at last $n - 1$ (the coefficient at z^{n+1} is 0, and c is defined from the condition that the coefficient at z^n equals 0). For the coefficient a_0' at z^{n-1}

we get $a_0' = (a_1/a_0)(a_0\bar{a}_1 + \bar{a}_0 a_1) + a_0\bar{a}_2 - \bar{a}_0 a_2$. The assumption $a_0\bar{a}_1 + \bar{a}_0 a_1 \neq 0$ of the lemma is equivalent to $\operatorname{Re}(a_1/a_0) \neq 0$. Furthermore, $a_0\bar{a}_2 - \bar{a}_0 a_2$ is pure imaginary, hence $\operatorname{Re}(a_0') \neq 0$. Thus, the polynomial P_1 has degree $n - 1$. Now, we prove that

$$\bar{a}_0 c^2 P(z) = (z + c) P_1(z) + z P_1^*(z) . \tag{4.10}$$

Using (4.4) we get from (4.8) (P_1 is of degree $n - 1$)

$$P_1^*(z) = (-1)^{n-1} \operatorname{Conj}(P_1(-\bar{z})) = (-1)^{n-1} (a_0 c \operatorname{Conj}(P(-\bar{z})) + \\ + z a_0 \operatorname{Conj}(P(-\bar{z})) - z\bar{a}_0 \operatorname{Conj}(P^*(-\bar{z}))) .$$

Since $(-1)^n \operatorname{Conj}(P(-\bar{z})) = P^*(z)$ we have $\operatorname{Conj}(P^*(-\bar{z})) = (-1)^n P(z)$ and thefore $P_1^*(z) = -a_0 c P^*(z) - z a_0 P^*(z) + z\bar{a}_0 P(z)$. Using this relation and (4.8) we find by substituting on the right-hand side of (4.10) that Equation (4.10) is true, which means that P_1 is primary to P. Now, the remaining assertions of the lemma follow from Lemma 4.3.

The previous lemma shows the assumptions under which there exists a primary polynomial to the given one. Now we will discuss its uniqueness. We see from the definition that if P is primary to Q, then the polynomial kP where $k \in R$, $k \neq 0$ is also primary to Q. We shall show that in this way we obtain all primary polynomials. This is just the invariance assertion mentioned in the preface.

Lemma 4.5. *Let $n \in N_0$, $Q \in M_{n+1}$ and let $P_1, P_2 \in M_n$ be two primary polynomials to Q, i.e., for suitable $A_1, A_2 \in C$, $c_1, c_2 \in R$, $A_1 c_1 \neq 0$, $A_2 c_2 \neq 0$ we have*

$$A_1 Q(z) = (z + c_1) P_1(z) + z P_1^*(z) ,$$
$$A_2 Q(z) = (z + c_2) P_2(z) + z P_2^*(z) .$$

Then $c_1 = c_2$. Furthermore, there exists $k \in R$ such that $P_2 = kP_1$.

Proof. Eliminating Q from both equations we get

$$A_2 z(P_1 + P_1^*) + A_2 c_1 P_1 = A_1 z(P_2 + P_2^*) + A_1 c_2 P_2 . \tag{4.11}$$

Denoting $P_1(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$, $P_2(z) = b_0 z^n + b_1 z^{n-1} + \ldots + b_n$, expressing P_1^*, P_2^* as in (4.5) and then comparing the coefficients in (4.11) we obtain the system of $n + 2$ linear equations

$$\begin{aligned} (4.12) \quad & A_2(a_0 + \bar{a}_0) && = A_1(b_0 + \bar{b}_0) , \\ & A_2(a_{j+1} + (-1)^{j+1}\bar{a}_{j+1} + c_1 a_j) && = A_1(b_{j+1} + (-1)^{j+1} \bar{b}_{j+1} + c_2 b_j) , \\ & && \qquad j = 0, 1, \ldots, n - 1 , \\ & A_2 c_1 a_n && = A_1 c_2 b_n . \end{aligned}$$

Since $a_0 + \bar{a}_0 \neq 0$, $b_0 + \bar{b}_0 \neq 0$ (see Lemma 4.3) we can write the first equation of the system in the form

$$\operatorname{Re}(b_0) = k \operatorname{Re}(a_0) \tag{4.13}$$

where the number $k = A_2/A_1$ is necessarily non-zero and real. Defining $q = c_1/c_2$ we have $q \in R$, $q \neq 0$. The last equation of (4.12) can be rewritten in the form

$$b_n = kqa_n\,. \tag{4.14}$$

Assuming n to be odd (for an even n the calculations proceed in an analogous way) we take the equation from (4.12) with $j = n - 1$. Using (4.14) we get

$$k(a_n - \bar{a}_n)(1 - q) + kc_1a_{n-1} = c_2b_{n-1}\,.$$

As the first member on the left-hand side is pure imaginary we conclude that

$$\operatorname{Re}(b_{n-1}) = kq \operatorname{Re}(a_{n-1})\,.$$

Now we take the equation of (4.12) with $j = n - 2$. Using the result just obtained we get

$$k(a_{n-1} + \bar{a}_{n-1})(1 - q) + kc_1a_{n-2} = c_2b_{n-2}\,.$$

As the first member on the left-hand side is real, we conclude that

$$\operatorname{Im}(b_{n-2}) = kq \operatorname{Im}(a_{n-2})\,.$$

In this way we get, for $j = n - 1, n - 2, \ldots, 0$:

$$\begin{aligned} \operatorname{Re}(b_j) &= kq \operatorname{Re}(a_j) \quad \text{if } j \text{ is even}\,,\\ \operatorname{Im}(b_j) &= kq \operatorname{Im}(a_j) \quad \text{if } j \text{ is odd}\,. \end{aligned}$$

In particular, for $j = 0$ we have $\operatorname{Re}(b_0) = kq \operatorname{Re}(a_0)$. Comparing this result with (4.13) we get $q = 1$, i.e., $c_1 = c_2$. Now, from the above relations we obtain $b_j = ka_j$, $j = 0, 1, \ldots, n$, completing the proof.

Remark. The previous lemma on the invariance of the constant c has important consequences. Comparing the notions of the primary and related polynomials (see Definition 4.1) we see that the related polynomial (or the set of all related polynomials to a given one) is determined by the position of its roots. On the other hand, in Equation (4.6), P cannot be an arbitrary related polynomial to Q but only the primary one to Q. Of course, if P is primary to Q, then P is also related to Q. In order to reproduce the polynomial Q knowing some P related to Q, it is necessary to give $c \in R$ (which is uniquely determined) and, in addition, to choose some polynomial from the set of all related polynomials to be primary to Q. According to Lemma 4.5, the argument of the coefficient at the last power is uniquely prescribed for such a polynomial. This is precisely the reason for introducing two notions, namely those of the set of primary polynomials and the set of related polynomials.

The above assertions provide the possibility to find out the distribution of roots of polynomials with respect to the imaginary axis. Let $P \in M_n$ be of the form (4.1). Set $P_n = P$ and define the number $[0, 1]_n$ by

$$[0, 1]_n = a_0\bar{a}_1 + \bar{a}_0 a_1 = 2 \operatorname{Re}(a_0\bar{a}_1) \tag{4.15}$$

where the index n indicates that the number is constructed for a polynomial of degree n. For the case $n = 0$ we define $[0, 1]_0 = 0$. The number $[0, 1]_n$ is always real. If $[0, 1]_n \neq 0$, then $n \geqq 1$ and, by Lemma 4.4, there is a polynomial $P_{n-1} \in M_{n-1}$ related (possibly primary) to P_n. Denoting, in accordance with (4.7),

$$c_n = \frac{1}{a_0\bar{a}_0} [0, 1]_n , \tag{4.16}$$

we have $\operatorname{sign} c_n = \operatorname{sign} [0, 1]_n$ and from Lemma 4.4 we get $\mathrm{d}P_n = \mathrm{d}P_{n-1} + \pi \operatorname{sign} [0, 1]_n$ where P_{n-1} may be an arbitrary polynomial related to P_n. Now we define $[0, 1]_{n-1}$, c_{n-1} (for the polynomial P_{n-1}) analogously as we have defined $[0, 1]_n$ for P_n. Lemma 4.5 implies that c_{n-1} is independent of the polynomial P_{n-1} chosen from the set of all related polynomials to P_n. If $[0, 1]_{n-1} \neq 0$ (i.e., if $c_{n-1} \neq 0$), then there is $P_{n-2} \in M_{n-2}$ related (in particular, it can be primary) to P_{n-1} and, furthermore, $\mathrm{d}P_{n-1} = \mathrm{d}P_{n-2} + \pi \operatorname{sign} [0, 1]_{n-1}$. In this way we construct the sequence

$$P_n, P_{n-1}, \ldots, P_s , \quad s \in N_0 , \quad 0 \leqq s \leqq n \tag{4.17}$$

of polynomials such that $P_j \in M_j$, $[0, 1]_j \neq 0$, P_{j-1} is related (in particular, it can be primary) to P_j, $j = s + 1, s + 2, \ldots, n$ and, in addition, $[0, 1]_s = 0$. With regard to the last equation, the sequence (4.17) cannot be continued, i.e., P_{s-1} cannot be constructed. In the case $s = 0$, the sequence is closed by a polynomial of degree zero, i.e., by a nonzero constant. We summarize the results in the following theorem.

Theorem 4.1. *Let $n \in N$, $s \in N_0$, $0 \leqq s \leqq n$, let the sequence (4.17) be such that P_{j-1} is related to P_j, $[0, 1]_j \neq 0$, $j = s + 1, s + 2, \ldots, n$, $[0, 1]_s = 0$. For $j = s$, $s + 1, \ldots, n$ we denote by l_j, o_j, r_j the number of roots of P_j with negative, zero, positive real parts, respectively ($l_j + o_j + r_j = j$, each root being counted with its multiplicity). Then $o_s = o_{s+1} = \ldots = o_n$. Moreover, if for the sequence*

$$[0, 1]_n, [0, 1]_{n-1}, \ldots, [0, 1]_{s+1} \tag{4.18}$$

p and q denote the number of positive and negative numbers, respectively, then

$$l_n = l_s + p , \quad r_n = r_s + q . \tag{4.19}$$

In the case $s = 0$ we have $l_n = p$, $r_n = q$. In particular, P_n is a Hurwitz polynomial, i.e., $P_n \in S_n$ if and only if $s = 0$ and each member in the sequence (4.18) is positive.

Remark. Instead of the sequence (4.18) we can take, of course, the sequence

(4.20) $$c_n, c_{n-1}, \dots, c_{s+1},$$

which is independent of the choice of the related polynomials in (4.17).

5. THE CRITERION OF ROUTH

In order to find an effective criterion for the distribution of roots of polynomials with respect to the imaginary axis it suffices to show how to compute the numbers $[0, 1]_j$. To this end, the coefficients of the polynomials P_j are needed. Of course, it suffices to express the coefficients of P_{n-1} in terms of those of P_n. Let

$$P_n(z) = \sum_{k=0}^{n} a_k z^{n-k}, \quad P_{n-1}(z) = \sum_{k=0}^{n-1} a'_k z^{n-1-k}.$$

Define the numbers $[j, k]$, $\{j, k\}$ (we should write a subscript n as in (4.15) but for simplicity we omit it) by

(5.1) $$\begin{aligned}[j, k] &= a_j\bar{a}_k + \bar{a}_j a_k = 2\,\mathrm{Re}\,(a_j\bar{a}_k),\\ \{j, k\} &= a_j\bar{a}_k - \bar{a}_j a_k = 2\mathrm{i}\,\mathrm{Im}\,(a_j\bar{a}_k),\end{aligned}$$

$j, k = 0, 1, \dots, n$; $[0, 1]$ coincides, of course, with $[0, 1]_n$ defined in (4.15). Substituting in (4.8) P_{n-1}, P_n instead of P_1, P, respectively, we get by comparing the coefficients

(5.2) $$\begin{aligned} a'_k &= \frac{1}{a_0}[0, 1]\, a_{k+1} + \{0, k+2\}, \quad k \text{ even}, \quad 0 \leqq k \leqq n-1,\\ a'_k &= \frac{1}{a_0}[0, 1]\, a_{k+1} - [0, k+2], \quad k \text{ odd}, \quad 0 \leqq k \leqq n-1.\end{aligned}$$

If we put $a_k = a'_k = 0$ for $k < 0$ and $a_k = 0$ for $k > n$, $a'_k = 0$ for $k > n-1$, the relations (5.2) hold for all integers k. The formulas (5.2) give the coefficients of some primary polynomial. The coefficients of any other related polynomial can be obtained by multiplying the right-hand sides of (5.2) by a nonzero complex constant. Choosing this constant equal to $a_0/[0, 1]$ we get the coefficients of a special related polynomial

(5.3) $$\begin{aligned} a'_k &= a_{k+1} + a_0\{0, k+2\}/[0, 1], \quad k \text{ even},\\ a'_k &= a_{k+1} - a_0[0, k+2]/[0, 1], \quad k \text{ odd}.\end{aligned}$$

In this relations, the coefficients of P_n as well as P_{n-1} have the same "dimension". Note that, if a_0 is real, then (5.3) gives the coefficients of a polynomial which is even primary. In the case of polynomials with real coefficients the calculation is simpler. Namely, we have $[j, k] = 2a_j a_k$, $\{j, k\} = 0$, the coefficients a'_k are, of course, also real and from (5.3) we get

(5.4) $$a'_k = a_{k+1}\,, \quad k \text{ even}\,,$$
$$a'_k = a_{k+1} - a_0 a_{k+2}/a_1\,, \quad k \text{ odd}\,.$$

In this case, the algorithm can be arranged in a scheme, see Table 5.1 where the first two rows represent the coefficients of the polynomial P_n, the second and third rows represent the coefficients of P_{n-1}, etc. Each row in the scheme results from the two previous ones in the manner which is demonstrated by the third row in the scheme. The coefficients in the first column are called Routh's testing functions. The members of (4.18) can be obtained from the testing functions of Routh. In particular, $(1/2)\,[0, 1]_n$ is the product of the testing functions in the first two rows, $(1/2)\,[0, 1]_{n-1}$ is the product of the testing functions from the second and third row, etc. It is easy to see that the polynomial is stable if and only if all testing functions have the same sign. It was precisely in this way that the classical result of Routh was formulated.

$$\begin{array}{llll}
a_0 & a_2 & a_4 & \dots \\
a_1 & a_3 & a_5 & \dots \\
(a_1a_2 - a_0a_3)/a_1 & (a_1a_4 - a_0a_5)/a_1 & (a_1a_6 - a_0a_7)/a_1 & \dots \\
\vdots & & &
\end{array}$$

Table 5.1. The algorithm of Routh

6. THE CRITERION OF HURWITZ

In this part we consider polynomials with real coefficients. The coefficients of the related (even primary) polynomials calculated by (5.4) will be real as well.

The formulas (5.4) have a vectorial character. Introducing vectors

$$u = (a_0, a_2, \dots)\,, \quad v = (a_1, a_3, \dots)$$

we easily find out that by (5.4) the new vectors are formed, namely

(6.1) $$u' = v\,, \quad v' = u - (a_0/a_1)\,v\,,$$

their coordinates being the coefficients of the related polynomial. Let

(6.2) $$A_n = \det\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & \dots & 0 \\ a_2 & a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_4 & a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ & & & & & \dots & a_n \end{pmatrix}$$

be the determinant of order $n + 1$ such that its first column is the vector u, the second and third columns are the vectors v, u with shifted coordinates, the following two columns are given by shifting the previous two columns, etc.; the numbers in the main diagonal are $a_0, a_1, \dots, a_n$. If a_k with $k > n$ occurs in the determinant, then, of course, $a_k = 0$. Expanding the determinant with respect to the first row we get

$$A_n = a_0 \det \begin{pmatrix} a_1 & a_0 & 0 & 0 & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & & & & \end{pmatrix}.$$

The formulas (6.1) which express the relations (5.4) are reflected in the last determinant as follows: the odd columns remain the same but from each even column we subtract a multiple of its left neighbour such that we obtain zero at place of a_0. Of course, the determinant does not change the value. Hence, using the notation from (5.4) we obtain

$$A_n = a_0 \det \begin{pmatrix} a'_0 & 0 & 0 & 0 & \dots & 0 \\ a'_2 & a'_1 & a'_0 & 0 & \dots & 0 \\ a'_4 & a'_3 & a'_2 & a'_1 & \dots & 0 \\ \vdots & & & & & \vdots \\ & & & & & a'_{n-1} \end{pmatrix}. \tag{6.3}$$

So we get a determinant of the same form as in (6.2), now of degree n, whose elements are the coefficients of the polynomial P_{n-1} related to P_n by the formulas (5.4). Denoting this determinant in accordance with (6.2) by A_{n-1} we have $A_n = a_0 A_{n-1}$. Moreover, denoting the coefficients of the related polynomial P_j (see (4.17)) by $a_k^{(j)}$, $j = n, n-1, \dots$, so that $P_j(z) = a_0^{(j)} z^j + a_1^{(j)} z^{j-1} + \dots$, the relation (6.3) can be written in the form $A_n = a_0^{(n)} A_{n-1}$. Thus in an analogous way we get $A_j = a_0^{(j)} A_{j-1}$ provided the related polynomial P_{j-1} exists. Assume in the rest of the section that the sequence (4.17) ends by the polynomial $P_0 = a_0^{(0)}$ of degree 0. Then we have

$$A_n = a_0^{(n)} a_0^{(n-1)} \dots a_0^{(1)} a_0^{(0)} .$$

Notice that the numbers in this product are Routh's functions placed in the first column in the scheme 5.1.

Furthermore, denoting by B_j, $j = 0, 1, \dots, n$ the determinant of degree $j + 1$ which we obtain from the determinant A_n by keeping the first $j + 1$ rows and columns, i.e.

$$B_j = \det \begin{pmatrix} a_0 & 0 & 0 & \dots & 0 \\ a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & & & & \vdots \\ & & & & a_j \end{pmatrix},$$

then in an analogous way we get

$$B_j = a_0^{(n)} a_0^{(n-1)} \dots a_0^{(n-j)} , \quad j = 0, 1, \dots, n . \tag{6.4}$$

In this product the first $j + 1$ Routh's functions occur. With regard to (5.4) we have $a_0^{(j-1)} = a_1^{(j)}$, $j = 1, 2, \dots, n$. Therefore we can express $a_0^{(j)}$, $[0, 1]_j$, c_j in terms of B_j. Defining, in addition, $B_{-1} = 1$ we get the formulas

$$
\begin{aligned}
(6.5) \qquad a_0^{(j)} &= B_{n-j}/B_{n-j-1}\,, \\
[0, 1]_j &= 2B_{n-j+1}/B_{n-j-1}\,, \\
c_j &= 2B_{n-j+1}\, B_{n-j-1}/B_{n-j}^2\,,
\end{aligned}
$$

$j = 0, 1, \ldots, n$. Thus, we can extract the desired information on the distribution of roots from the determinants B_j. For instance, for $s = 0$ Theorem 4.1 yields the following assertion.

Theorem 6.1. (*Hurwitz*). *The polynomial P with the real coefficients is stable if and only if all determinants $B_1, B_3, B_5, \ldots$ are positive and all determinants $B_0, B_2, B_4, \ldots$ are nonzero and have the same sign.*

Remark. If $a_0 > 0$ (the coefficient at the leading power of the polynomial), then the condition for $P \in S_n$ is the positiveness of all B_j, $j = 1, 2, \ldots, n$. Instead of the determinants B_j, the more familiar formulation uses the determinants $\hat{B}_j$ which we get from B_j by removing the first row and column.

The connection between the criteria of Hurwitz and Routh is expressed by (6.4) or by the first equation in (6.5). The reason why the form of Hurwitz is almost exclusively preferred is the fact that the conditions are expressed by coefficients of P explicitly, whereas in the form of Routh they are not. As concerns the necessary operations the criterion of Routh is more advantageous to that of Hurwitz, as the complete algorithm requires only $n^2/4$ multiplications and the same number of divisions and additions where n is the degree of the polynomial considered. On the other hand, applying the criterion of Hurwitz we have to evaluate n determinants and, if we do not use (5.4) to compute the determinants simultaneously, the number of operations will increase considerably. Of course, under suitable organization (following, in fact, the algorithm of Routh), the number of operations will be the same as in Routh's procedure. Thus, from the numerical point of view, it is just the algorithm of Routh which is more economical. In applications the number n is often not too large, the coefficients of the polynomial depend, as a rule, on some parameters, and the problem to be solved is to determine the domain of the parameters for which the polynomial is stable. In such a case, the explicit form of the criterion of Hurwitz is preferable.

7. THE SINGULAR CASE

By a singularity we understand the case in which the sequence (4.17) cannot be continued to the polynomial of degree 0, i.e. the sequence ends by a polynomial P_s, $s \geqq 1$ where $[0, 1]_s = 0$. Of course, in such a case the polynomial is not stable. Being interested in the distribution of roots of the polynomial with respect to the imaginary axis we can use the method explained in the remark following after Lemma

4.2. Assuming $t \in R$, $t \neq 0$ we apply Routh's algorithm to the polynomial $Q_t(z) = P(t + z)$ (the variable in the polynomial is z). Relations (5.3) imply that the coefficients of all related polynomials to Q_t as well as the numbers c_j, $[0, 1]_j$ are rational functions of the variable t. Evidently, the polynomial Q_t is stable for sufficiently large t which implies $c_j > 0$ for this t. This implies that no c_j equals identically zero. Therefore, there exists $h > 0$ such that for all t, $0 < |t| < h$ all the functions c_j are nonzero. For such t the algorithm of Routh may be performed without restrictions to a polynomial of degree 0. Denoting by $l(t)$, $r(t)$ the number of roots of Q_t lying in the half-plane $\operatorname{Re}(z) < 0$, $\operatorname{Re}(z) > 0$, respectively, we conclude that the equation $l(t) + r(t) = n$ holds for all t with $0 < |t| < h$. This implies the existence of the limit $l(0+) = \lim l(t)$ as $t \to 0+$ and, analogously, of $r(0+)$, $l(0-)$, $r(0-)$. Obviously $l(0-)$, $r(0+)$ is the number of roots of P lying in the half-plane $\operatorname{Re}(z) < 0$, $\operatorname{Re}(z) > 0$, respectively, and, moreover, the number of roots lying on the imaginary axis equals $l(0+) - l(0-) = r(0-) - r(+)$. Thus, using this method we can solve the problem of distribution of roots also in the singular case. As the polynomials $P(t + z)$ are considered for small t only, it usually suffices, for n not too large, to look only for some small degrees of the variable t.

8. THE CRITERION OF HERMITE

As we have seen in the case of polynomials with real coefficients, it is possible to determine the distribution of roots with respect to the imaginary axis by determinants constructed in a suitable way from the coefficients of the polynomial under investigation. It can be shown that even in the case of polynomials with complex coefficients a similar characterization is possible. Suppose that $P_n \in M_n$ is a polynomial with complex coefficients which possesses a primary polynomial. Take P_{n-1} related to P_n by (5.3). In particular, we have

$$a'_0 = a_1 + a_0\{0, 2\}/[0, 1]\,,$$
$$a'_1 = a_2 - a_0[0, 3]/[0, 1]$$

where $[0, 1] = [0, 1]_n = a_0\bar{a}_1 + \bar{a}_0 a_1$ according to (5.1). Express $[0, 1]_{n-1} = a'_0\bar{a}'_1 + \bar{a}'_0 a'_1$ by the coefficients of P_n. Remember that the numbers $[j, k]$ are real and $\{j, k\}$ pure imaginary, see (5.1). We have

$$a'_0\bar{a}'_1 = a_1\bar{a}_2 - \bar{a}_0 a_1[0, 3]/[0, 1] + a_0\bar{a}_2\{0, 2\}/[0, 1] - a_0\bar{a}_0\{0, 2\}\,[0, 3]/[0, 1]^2\,;$$
$$\bar{a}'_0 a'_1 = \bar{a}_1 a_2 - a_0\bar{a}_1[0, 3]/[0, 1] - \bar{a}_0 a_2\{0, 2\}/[0, 1] + \bar{a}_0 a_0\{0, 2\}\,[0, 3]/[0, 1]^2\,.$$

Adding and using (5.1) we get

$$[0, 1]_{n-1} = \frac{1}{[0, 1]}\left([0, 1]\,[1, 2] - [0, 1]\,[0, 3] + \{0, 2\}^2\right). \tag{8.1}$$

The expression in the brackets can be written as a determinant of the matrix $\boldsymbol{U}_2$

where

$$\boldsymbol{U}_2 = \begin{pmatrix} [0,1] & \{0,2\} \\ -\{0,2\} & [1,2]-[0,3] \end{pmatrix}.$$

Observe that the numbers on the main diagonal are real and the others pure imaginary. The matrix $\boldsymbol{U}_2$ is thus Hermitian (a matrix is Hermitian when after transposing and subsequently replacing all entries by their complex conjugates we obtain the original matrix). The important fact is that $\boldsymbol{U}_2$ contains all the information needed for the evaluation of $[0, 1]_n$, $[0, 1]_{n-1}$. Denote by H_1 the first main determinant of $\boldsymbol{U}_2$ (the first row, the first column), by H_2 the second main determinant of $\boldsymbol{U}_2$ (the first two rows and columns, so $H_2 = \det \boldsymbol{U}_2$). Then using (8.1) we can write

$$[0,1]_n = H_1\,, \quad [0,1]_{n-1} = H_2/H_1\,. \tag{8.2}$$

Thus the numbers $[0, 1]_n$, $[0, 1]_{n-1}$, are, indeed, determined by H_1, H_2. For example, in the case $n = 2$ we obtain from (8.2) this special result: a polynomial of the second degree is stable if and only if the matrix $\boldsymbol{U}_2$ is positive definite. Indeed, P_2 is stable if and only if both the numbers ($n = 2$) $[0, 1]_2$, $[0, 1]_1$ are positive which is equivalent to the positiveness of the determinants H_1, H_2 and, consequently, to the positive definiteness of $\boldsymbol{U}_2$. The result can be generalized, but so far we do not know how to construct the matrix $\boldsymbol{U}_n$ in general. To find it, further steps of induction should be carried out. We give here the final result without proving it. For a given polynomial we construct the Hermitian matrix $\boldsymbol{U}_n$ of order n, $\boldsymbol{U}_n = [u_{jk}]$, where

$$\begin{aligned} u_{jk} &= [j-1, k] - [j-2, k+1] + [j-3, k+2] - \dots\,, \\ &\qquad j \leqq k\,, \quad j+k \quad \text{even}\,; \\ u_{jk} &= \{j-1, k\} - \{j-2, k+1\} + \{j-3, k+2\} - \dots\,, \\ &\qquad j \leqq k\,, \quad j+k \quad \text{odd}\,. \end{aligned} \tag{8.3}$$

In fact, the sums are finite, for obviously $[j, k] = \{j, k\} = 0$ if $j < 0$ or $k < 0$ or $j > n$ or $k > n$. Thus the matrix $\boldsymbol{U}_n$ has the the form

$$\boldsymbol{U}_n = \begin{pmatrix} [0,1] & \{0,2\} & [0,3] & \{0,4\} & \dots \\ & [1,2]-[0,3] & \{1,3\}-\{0,4\} & [1,4]-[0,5] & \\ & . & [2,3]-[1,4]+[0,5] & \{2,4\}-\{1,5\}+\{0,6\} & \\ & & & \vdots & \end{pmatrix}.$$

It is Hermitian, therefore we do not write the entries below the main diagonal. If H_k denotes the main subdeterminant of $\boldsymbol{U}_n$ constructed from the first k rows and columns, then it can be shown by induction that

$$H_k = [0,1]_n\,[0,1]_{n-1} \dots [0,1]_{n-k+1} \tag{8.4}$$

holds as long as the related polynomials of degree $n-1, \dots, n-k+1$ exist. As a special result we have the following assertion.

Theorem 8.1 (*Hermite*). *The polynomial $P \in M_n$ is stable if and only if all the main subdeterminants of the matrix $\mathbf{U}_n$ are positive.*

Remark. Note that the condition of positiveness of all main subdeterminants H_k is equivalent to the positive definiteness of the matrix $\mathbf{U}_n$.

9. THE GENERALIZATION OF THE CRITERION OF HURWITZ TO THE CASE OF COMPLEX COEFFICIENTS

Deducing the criterion of Hurwitz we started from the transformation formulas (5.3). We stated that, in the case of polynomials with real coefficients, the formulas have a vectorial character, which implies the special structure of Hurwitz's matrix. Thus, the natural question arises about the existence of an analogue of Hurwitz's matrix for the case of complex coefficients. First, from (5.3) we can see that it is not possible to form the corresponding matrix directly in terms of the coefficients of the polynomial P_n. So we try to separate the real and imaginary parts. Writing the coefficients of P_n in the form

$$a_k = p_k + \mathrm{i}q_k\,, \quad p_k, q_k \in R\,, \quad k = 0, 1, \ldots, n$$

and, analogously, $a_k' = p_k' + \mathrm{i}q_k'$ for the coefficients of the related polynomial P_{n-1} given by (5.3) we get

$$\begin{aligned} p_k' &= p_{k+1} - \frac{2q_0^2}{[0,1]}\, p_{k+2} + \frac{2p_0q_0}{[0,1]}\, q_{k+2}\,, \quad k \text{ even} \\ q_k' &= q_{k+1} + \frac{2p_0q_0}{[0,1]}\, p_{k+2} - \frac{2p_0^2}{[0,1]}\, q_{k+2}\,, \end{aligned} \tag{9.1}$$

and

$$\begin{aligned} p_k' &= p_{k+1} - \frac{2p_0^2}{[0,1]}\, p_{k+2} - \frac{2p_0q_0}{[0,1]}\, q_{k+2}\,, \quad k \text{ odd} \\ q_k' &= q_{k+1} - \frac{2p_0q_0}{[0,1]}\, p_{k+2} - \frac{2q_0^2}{[0,1]}\, q_{k+2} \end{aligned} \tag{9.2}$$

where $0 \leqq k \leqq n - 1$. Note that the formulas remain true for all integers if we set $p_k = q_k = 0$ for $k < 0$ and $k > n$ (in connection with P_n) as well as $p_k' = q_k' = 0$ for $k < 0$ and $k > n - 1$ (in connection with P_{n-1}). Without troubling the reader with details we immediately show the corresponding matrix in which the relations (9.1), (9.2) have a "vectorial" character. The matrix is of the form

$$\mathbf{W} = \begin{pmatrix} p_0 & q_0 & & & & & \\ -q_1 & p_1 & p_0 & q_0 & & & \\ -p_2 & -q_2 & -q_1 & p_1 & p_0 & q_0 & \cdots \\ q_3 & -p_3 & -p_2 & -q_2 & -q_1 & p_1 & \cdots \\ p_4 & q_4 & q_3 & -p_3 & -p_2 & -q_2 & \cdots \\ \vdots & & & & & & \end{pmatrix}. \tag{9.3}$$

It is a square matrix of order $2n$, the omitted entries being zeros. It suffices to write the first two columns (the distribution of p_k, q_k including the signs has some regularity, it is "periodical" with respect to k with the period 4), as all further pairs of columns are obtained by shifting the first one.

Now we calculate det $\mathbf{W}$. We carry out the following transformations in $\mathbf{W}$: we let the first pair of columns without any change; in every further pair we exchange the two columns, change the sign in the right column and then add to them linear combinations of the previous two columns following the rule

$$(9.4) \qquad (2k+1)' = (2k+2) + \frac{2q_0^2}{[0,1]}(2k-1) - \frac{2p_0q_0}{[0,1]}(2k),$$

$$(2k+2)' = -(2k+1) - \frac{2p_0q_0}{[0,1]}(2k-1) + \frac{2p_0^2}{[0,1]}(2k),$$

$k = n-1, n-2, \ldots, 2, 1$ (we apply the rules in the matrix from the right to the left). In (9.4), the symbol (k) denotes the k-th column of the matrix $\mathbf{W}$ and the symbol $(k)'$ denotes the k-th column of the new matrix $\mathbf{V}$. Using (9 1), (9.2) we get

$$\mathbf{V} = \begin{pmatrix} p_0 & q_0 & & & & & \\ -q_1 & p_1 & & & & & \\ -p_2 & -q_2 & p_0' & q_0' & & & \\ q_3 & -p_3 & -q_1' & p_1' & p_0' & q_0' & \\ p_4 & q_4 & -p_2' & -q_2' & -q_1' & p_1' & \cdots \\ \vdots & & & & & & \end{pmatrix}$$

where p_k', q_k' are the coefficients of the polynomial P_{n-1} related to P_n by (5.3). Of course, the changes performed in the matrix $\mathbf{W}$ do not change the value of det $\mathbf{W}$, i.e. det $\mathbf{W}$ = det $\mathbf{V}$. Expanding det $\mathbf{V}$ with respect to the first two rows (the theorem of Laplace) we get det $\mathbf{V} = (p_0p_1 + q_0q_1)$ det $\mathbf{W}'$ where $\mathbf{W}'$ is a square matrix of order $2n-2$ formed in the same manner as $\mathbf{W}$ but from the coefficients of the polynomial P_{n-1}. Since $p_0p_1 + q_0q_1 = (1/2)[0,1]_n$, we have det $\mathbf{W} = (1/2)$ . . $[0,1]_{n-1}$ det $\mathbf{W}'$. This formula can be generalized. Eventually we obtain

$$\det \mathbf{W} = (1/2^n)[0,1]_n [0,1]_{n-1} \ldots [0,1]_1$$

assuming that all the numbers $[0,1]_k$ are nonzero. Denoting by W_k the main subdeterminant of the matrix $\mathbf{W}$ consisting of the first $2k$ rows and columns we analogously have

$$(9.5) \qquad W_k = (1/2^k)[0,1]_n [0,1]_{n-1} \cdots [0,1]_{n-k+1}$$

assuming that all the numbers $[0,1]_k$ in the equation are nonzero (the corresponding related polynomials exist). Comparing (9.5), (8.4) we obtain

$$H_k = 2^k W_k .$$

The criterion for the distribution of roots of polynomials with respect to the imaginary axis can be, therefore, formulated in terms of W_k. In particular, polynomial of degree n is stable if and only if $W_k > 0$ for all $k = 1, 2, \ldots, n$.

Now we show the way of forming the matrix $\mathbf{W}$ which can be easily remembered. Let

$$\mathrm{i}^n P_n(-\mathrm{i}z) = \sum_{k=0}^{n} u_k z^{n-k} + \mathrm{i} \sum_{k=0}^{n} v_k z^{n-k}$$

where $u_k, v_k \in R$, $k = 0, 1, \ldots, n$. Then we find out that

$$\mathbf{W} = \begin{pmatrix} u_0 & v_0 & & & & & \\ u_1 & v_1 & u_0 & v_0 & & & \\ u_2 & v_2 & u_1 & v_1 & u_0 & v_0 & \\ u_3 & v_3 & u_2 & v_2 & u_1 & v_1 & \ldots \\ \vdots & & & & & & \end{pmatrix}.$$

Of course, we can form the matrix $\mathbf{W}$ also for a polynomial with real coefficients $(q_k = 0)$. In this case we get, by comparing (9.5), (6.5),

$$W_k = (1/a_0)\, B_{k-1} B_k\,.$$

10. THE GENERALIZATION OF PONTRYAGIN

We will present some of the results which can be found in [10]. In [5] Pontryagin investigated the distribution of roots with respect to the imaginary axis for the characteristic equation $P(z, \mathrm{e}^z) = 0$ where $P(x, y)$ is a polynomial in x, y. Let

$$P(z, w) = \sum_{m=0}^{r} \sum_{n=0}^{s} a_{mn} z^m w^n\,. \tag{10.1}$$

We call $a_{rs} z^r w^s$ the principal term of the polynomial if $a_{rs} \neq 0$ and, for each term $a_{mn} z^m w^n$ with $a_{mn} \neq 0$ we have either $r > m$, $s > n$ or $r = m$, $s > n$ or $r > m$, $s = n$. Of course, not every polynomial has a principal term.

If $w = \mathrm{e}^z$, then $P(z, \mathrm{e}^z) = 0$ is the characteristic equation for the differential-difference equation

$$\sum_{m=0}^{r} \sum_{n=0}^{s} a_{mn} \frac{\mathrm{d}^m}{\mathrm{d}t^m}\, x(t + n) = 0\,. \tag{10.2}$$

Theorem 10.1. *If the polynomial $P(z, w)$ has no principal term, then the equation $P(z, \mathrm{e}^z) = 0$ has infinite number of zeros with arbitrarily large real parts.*

Theorem 10.2. *Denote $d(z) = P(z, \mathrm{e}^z)$ and suppose $P(z, w)$ has a principal term $a_{rs} z^r w^s$. All zeros of $d(z)$ have negative real parts if and only if*

(i) *the complex vector $d(\mathrm{i}y)$ rotates in the positive direction with a positive velocity for y ranging in $(-\infty, +\infty)$;*

(ii) *for* y *ranging in* $\langle -2k\pi, 2k\pi \rangle$, $k \geqq 0$ *an integer, there is* ε_k *such that* $\varepsilon_k \to 0$ *as* $k \to +\infty$ *and* $d(\mathrm{i}y)$ *subtends the angle* $4k\pi s + \pi r + \varepsilon_k$.

Theorem 10.3. *Let* $d(z) = P(z, \mathrm{e}^z)$ *where* $P(z, w)$ *is a polynomial with a principal term. Suppose that* $d(\mathrm{i}y)$, $y \in R$ *is separated into its real and imaginary parts,* $d(\mathrm{i}y) = F(y) + \mathrm{i}G(y)$. *If all zeros of* $d(z)$ *have negative real parts, then the zeros of* $F(y)$ *and* $G(y)$ *are real, simple, alternate and*

$$G'(y)\, F(y) - G(y)\, F'(y) > 0 \tag{10.3}$$

for $y \in R$. *Conversely, all zeros of* $d(z)$ *are in the left half-plane provided that at least one of the following conditions is satisfied:*

(i) *all zeros of* $F(y)$ *and* $G(y)$ *are real, simple, alternate and the inequality* (10.3) *is satisfied for at least one* y;

(ii) *all zeros of* $F(y)$ *are real and, for each zero, the relation* (10.3) *is satisfied;*

(iii) *all zeros of* $G(y)$ *are real and, for each zero, the relation* (10.3) *is satisfied.*

Using the previous theorems we can prove "algebraical" criteria in some simple cases. We present one of the results of this type, proved, for example, in [10].

Theorem 10.4. *All roots of the equation* $(z + a)\, \mathrm{e}^z + b = 0$, *where* a *and* b *are real, have negative real parts if and only if*

(i) $a > -1$,

(ii) $a + b > 0$,

(iii) $b < t \sin t - a \cos t$

where t *is the root of the equation* $t = -a \operatorname{tg} t$, $0 < t < \pi$ *if* $a \neq 0$, *and* $t = \pi/2$ *if* $a = 0$.

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Souhrn

O STABILNÍCH POLYNOMECH

MILOSLAV NEKVINDA

Jde o přehledný článek o problematice Hurwitzovy věty. Vychází se v něm z Schurova rozkladu polynomu a uvádí se, jak lze dospět k běžně známým kritériím. Zdá se, že věta o jednoznačnosti konstant v Schurově rozkladu není dosud známa.

Резюме

ОБ УСТОЙЧИВЫХ МНОГОЧЛЕНАХ

MILOSLAV NEKVINDA

В статье дается обзор проблематики теоремы Гурвица. Отправным пунктом изложения является теорема Шура о разложении полиномов. По казано, как можно получить хорошо известные критерии для распределений корней полиномов. По видимому, теорема об однозначности констант в разложении Шура пока не известна.

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