Vladimír Haluška
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STABILITY OF A MODEL FOR THE BELOUSOV-ZHABOTINSKIJ REACTION

VLADIMÍR HALUŠKA

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Summary. The paper deals with the Field-Körös-Noyes' model of the Belousov-Zhabotinskij reaction. By means of the method of the Ljapunov function a sufficient condition is determined that the non-trivial critical point of this model be asymptotically stable with respect to a certain set.

Keywords: Belousov-Zhabotinskij reaction, equilibrium point, stability in the large, Ljapunov function.

AMS subject classification: 34D20.

The Belousov-Zhabotinskij reaction is an oscillating oxidation reaction. There are some mathematical models of that reaction. The best known ones have been given by Weisbuch-Salomon-Atlan [8], [1] or by Field-Körös-Noyes [2], [4], [6], [10]. In this paper, some stability properties of the Field-Körös-Noyes model are investigated.

The model of the reaction is of the form

$$\begin{aligned} \dot{X} &= s(Y - XY + X - gX^2) \\ \dot{Y} &= s^{-1}(fZ - Y - XY) \\ \dot{Z} &= w(X - Z) \end{aligned} \tag{1}$$

where f, s, w, g are real positive parameters representing kinetic constants and X, Y, Z are concentrations, all of them nonnegative.

The system (1) has exactly two critical equilibrium points which lie in the octant $X \geqq 0$, $Y \geqq 0$, $Z \geqq 0$ and hence they have a real meaning. These points are $a_0 = (0, 0, 0)$ and $a_1 = (x_0, y_0, z_0)$. The latter point satisfies the system

$$\begin{aligned} 0 &= s(y_0 - x_0 y_0 + x_0 - g x_0^2) \\ 0 &= s^{-1}(f z_0 - y_0 - x_0 y_0) \\ 0 &= w(x_0 - z_0) \end{aligned} \tag{2}$$

Clearly so does the former. The point $a_1(x_0, y_0, z_0)$ has the coordinates

(3)
$$x_0 = \frac{1 - f - g + \sqrt{((1 - f - g)^2 + 4g(1 + f))}}{2g}$$
$$y_0 = \frac{fx_0}{1 + x_0} = \frac{1}{2}(1 + f - gx_0)$$
$$z_0 = x_0 .$$

Definition 1. *A point (x_1, y_1, z_1) of the boundary of a region $B \subset R^3$ is said to be a strict ingress point of B with respect to (1) if for any solution (X, Y, Z) of (1) satisfying $X(t_0) = x_1$, $Y(t_0) = y_1$, $Z(t_0) = z_1$ there exists an $\varepsilon > 0$ such that the points $(X(t), Y(t), Z(t))$ for $t_0 - \varepsilon < t < t_0$ belong to $R^3 - \bar{B}$ ($\bar{B}$ is the closure of B), and for $t_0 < t < t_0 + \varepsilon$ they are from B.*

Lemma 1. *All boundary points of the region $P = \{(x, y, z) \in R^3: x > 0, y > 0, z > 0\}$ except the point a_0 are strict ingress points of P with respect to (1).*

Proof. The statement of the lemma follows: at points (x, y, z) of the boudary of P such that $x > 0$, $y > 0$, $z = 0$ from the inequality $\dot{Z} > 0$, at points $x = 0$, $y > 0$, $z = 0$ from the relations $\dot{X} > 0$, $\dot{Z} = 0$, $\ddot{Z} > 0$ and at points $x > 0$, $y = 0$, $z = 0$ from the inequalities $\dot{Y} = 0$, $\ddot{Y} > 0$, $\dot{Z} > 0$. In all other cases we get similar statements.

By Lemma 1 with respect to Lemma 8.1 [3] and to the uniqueness of a solution to the initial value problem for (1), the following theorem holds.

Theorem 1. *For each solution $(X(t), Y(t), Z(t))$ of the system (1) for which there is a t_0 such that $(X(t_0), Y(t_0), Z(t_0)) \in P$, its values for all $t \geqq t_0$ from the interval of its existence belong to P.*

Let us investigate the stability of the critical points. To that aim let us introduce new variables x, y, z by the relations

(4)
$$X = x_0 + x$$
$$Y = y_0 + y$$
$$Z = z_0 + z .$$

With respect to (3) and (2), the system (1) is transformed by means of (4) to the form

(5)
$$\dot{x} = s[y(1 - x_0) + x(1 - y_0 - 2gx_0)] + s(-xy - gx^2)$$
$$\dot{y} = s^{-1}[fz - y(1 + x_0) - xy_0] - s^{-1}xy$$
$$\dot{z} = w(x - z) .$$

Introducing the notation

(6)
$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad f(\boldsymbol{x}) = \begin{pmatrix} s(-gx^2 - xy) \\ -s^{-1}xy \\ 0 \end{pmatrix},$$

$$B = \begin{pmatrix} s(1 - y_0 - 2gx_0), & s(1 - x_0), & 0 \\ -s^{-1}y_0, & -s^1(1 + x_0), & s^{-1}f \\ w, & 0, & -w \end{pmatrix}$$

we can write the system (5) as the vector equation

$$(7) \qquad \dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}),$$

Now, let us investigate the matrix $-\boldsymbol{B}$. We denote

$$-\boldsymbol{B} = \begin{pmatrix} a, & b, & 0 \\ c, & d, & e \\ 1, & 0, & r \end{pmatrix} = \begin{pmatrix} s(2gx_0 + y_0 - 1), & s(x_0 - 1), & 0 \\ s^{-1}y_0, & s^{-1}(x_0 + 1), & -s^{-1}f \\ -w, & 0, & w \end{pmatrix}.$$

The principal minors of $-\boldsymbol{B}$ are

$$M_1 = s(2gx_0 + y_0 - 1),$$
$$M_2 = (x_0 + 1)(2gx_0 + y_0 - 1) - y_0(x_0 - 1),$$
$$M_3 = (x + 1)(2gx_0 + y_0 - 1)w - wy_0(x_0 - 1) + wf(x_0 - 1).$$

At the point $a_0(0, 0, 0)$ we have

$$M_1 = -s, \quad M_2 = -1, \quad M_3 = -w - wf.$$

As $M_3 < 0$, the corresponding characteristic equation has at least one zero-point in the interval $(0, \infty)$ and thus the equilibrium point $a_0(0, 0, 0)$ is not stable. We shall now investigate the stability of the critical point $a_1(x_0, y_0, z_0)$. Similarly as in [9, p. 459] we shall use the following definition.

Definition 2. *Let $A = (a_{ij})$ be an $n \times n$ real matrix. We say that the matrix A is a P-matrix iff all its principal minors are positive.*

Lemma 2. *If*

$$(9) \qquad 2f + g < 1,$$

then matrix $-\boldsymbol{B}$ is a P-matrix for the point $a_1(x_0, y_0, z_0)$.

Proof. In view of (3) we have

$$M_1 = \tfrac{3}{4}s[\tfrac{1}{3} - \tfrac{1}{3}f - g + \sqrt{((1 - f - g)^2 + 4g(1 + f))}].$$

The inequality

$$3\sqrt{((1 - f - g)^2 + 4g(1 + f))} > f + 3g - 1$$

is valid. Indeed, in the case of nonnegative right-hand side, this inequality is equivalent to the inequalities

$$9(1 + f^2 + g^2 - 2f - 2g + 2fg + 4g + 4fg) > f^2 + 9g^2 + 1 - 2f - 6g + 6fg,$$

$$8f^2 - 16f + 24g + 48fg + 8 > 0,$$
$$8(f - 1)^2 + 24g + 48fg > 0.$$

Hence $M_1 > 0$. Further,

$$M_2 = (x_0 + 1)(2gx_0 + y_0 - 1) - y_0(x_0 - 1) = 2gx_0^2 + 2gx_0 + 2y_0 - x_0 - 1.$$

Similarly as when calculating M_1 we get from (3)

$$M_2 = \frac{1}{2g}\{(2gx_0)^2 + 2gx_0(g - 1) + 2fg\} >$$
$$> \frac{1}{2g}(1 - g - 2f)\{\sqrt{((1 - f - g)^2 + 4g(1 + f))} + 1 - g - f\}.$$

If $g + 2f < 1$, then $1 - g - 2f > 0$ and $1 - g - f > 0$. Hence $M_2 > 0$. Finally,

$$M_3 = (x_0 + 1)(2gx_0 + y_0 - 1)w - wy_0(x_0 - 1) + wf(x_0 - 1) =$$
$$= wM_2 + wf(x_0 - 1) > 0$$

because $x_0 > 1$, as can be easily shown.

Remark. $x_0 > 1$ *is equivalent to the inequality*

$$\frac{(1 - f - g) + \sqrt{((1 - f - g)^2 + 4g(1 + f))}}{2g} > 1$$

as well as to the inequality

$$(1 - f - g)^2 + 2(1 - f - g)\sqrt{((1 - f - g)^2 + 4g(1 + f))} +$$
$$+ (1 - f - g)^2 + 4g(1 + f) > 4g^2.$$

In view of (9), the last inequality is valid, because

$$4(1 - f - g)^2 + 4g(1 - g) + 4fg > 0.$$

Therefore $x_0 > 1$.

Lemma 3. *Let* (9) *be fulfilled and let the matrix* $\boldsymbol{W}$ *be of the form*

$$\boldsymbol{W} = \begin{bmatrix} 1, & 0, & 0 \\ 0, & \frac{b}{c}, & 0 \\ 0, & 0, & \frac{2r(ad - bc) + bel}{dl^2} \end{bmatrix}. \tag{10}$$

Then the matrix

$$\boldsymbol{C} = \boldsymbol{W}(-\boldsymbol{B}) + (-\boldsymbol{B})^{\mathrm{T}}\boldsymbol{W}, \tag{11}$$

where $-\boldsymbol{B}$ *is given by* (8) *and* $(-\boldsymbol{B})^{\mathrm{T}}$ *is the transpose of* $-\boldsymbol{B}$, *is a P-matrix.*

Proof. Let us calculate the matrix $\boldsymbol{C}$. With respect to (8) we have

$$(12)\qquad \boldsymbol{C} = \begin{bmatrix} 1, & 0, & 0 \\ 0, & \frac{b}{c}, & 0 \\ 0, & 0, & \frac{2r(ad-bc)+bel}{dl^2} \end{bmatrix} \cdot \begin{pmatrix} a, & b, & 0 \\ c, & d, & e \\ l, & 0, & r \end{pmatrix} +$$

$$+ \begin{pmatrix} a, & b, & 0 \\ c, & d, & e \\ l, & 0, & r \end{pmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} 1, & 0, & 0 \\ 0, & \frac{b}{c}, & 0 \\ 0, & 0, & \frac{2r(ad-bc)+bel}{dl^2} \end{bmatrix} =$$

$$= \begin{bmatrix} 2a, & 2b, & \frac{2r(ad-bc)+bel}{dl} \\ 2b, & \frac{2bd}{c}, & \frac{be}{c} \\ \frac{2r(ad-bc)+bel}{dl}, & \frac{eb}{c}, & \frac{2r(2r(ad-bc))+bel}{dl^2} \end{bmatrix} =: \begin{pmatrix} c_{11}, & c_{12}, & c_{13} \\ c_{21}, & c_{22}, & c_{23} \\ c_{31}, & c_{32}, & c_{33} \end{pmatrix}.$$

Now, let us calculate the principal minors of the matrix $\boldsymbol{C}$. Using the denotations from the proof of Lemma 2 and (8) we get

$$(13)\qquad \overline{M}_1 = 2a = 2M_1 > 0\,,$$

$$\overline{M}_2 = \frac{4abd}{c} - 4b^2 = 4b\,\frac{ad-bc}{c} = \frac{4bM_2}{c} > 0\,,$$

$$\overline{M}_3 = \frac{8abdr}{cdl^2}\,[2r(ad-bc)+bel] + \frac{4b^2e}{cdl}\,[2r(ad-bc)+bel] -$$

$$- \frac{2bd}{cd^2l^2}\,[2r(ad-bc)+bel]^2 - \frac{8b^2r}{dl^2}\,[2r(ad-bc)+bel] -$$

$$- \frac{2ab^2e^2}{c^2} = \frac{2b(ad-bc)}{c^2dl^2}\,\{4r^2c(ad-bc)+bel(4rc-el)\}\,.$$

Denote $L = 4rc - el$. With respect to (8) and (3)

$$(14)\qquad L = 4ws^{-1}y_0 - s^{-1}fw = s^{-1}w(4y_0 - f) = \frac{s^{-1}w}{1+x_0}f(3x_0 - 1) > 0\,,$$

because $x_0 > 1$. Then

$$\overline{M}_3 = \frac{2bM_2}{c^2dl^2}(4r^2cM_2 + belL) > 0$$

and hence the matrix $\boldsymbol{C}$ is a P-matrix.

Theorem 2. *If (9) is satisfied, then the equilibrium point $a_1(x_0, y_0, z_0)$ of the system (1) is exponentially asymptotically stable.*

Proof. By Lemma 3 there exists a positive definite diagonal matrix $\boldsymbol{W}$ such that the matrix $\boldsymbol{C}$ is a P-matrix and as it is symmetric, $\boldsymbol{C}$ is positive definite, too [7, p. 287]. Thus all conditions of Theorem 2 [9, p. 460] are fulfilled. Therefore the real parts of all eigenvalues of the matrix $-\boldsymbol{B}$ are positive, i.e., the real parts of the eigenvalues of $\boldsymbol{B}$ are all negative. This implies that the point $a_1(x_0, y_0, z_0)$ is exponentially asymptotically stable for the system (1).

In what follows we shall use this definition (compare with Definition 1 in [9, p. 454]).

Definition 3. *A positive equilibrium point a_1 of the system (1) is asymptotically stable in the large with respect to the set P if and only if*

1. the equilibrium point a_1 is stable with respect to P, namely, for every $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that if $\|(X(t_0), Y(t_0), Z(t_0)) - a_1\| < \delta$ and the solution $(X(t), Y(t), Z(t))$ is in P for $t \geqq t_0$, then $\|(X(t), Y(t), Z(t)) - a_1\| < \varepsilon$ for $t \geqq t_0$;

2. every solution $(X(t), Y(t), Z(t))$ of (1) such that $(X(t_0), Y(t_0), Z(t_0)) \in P$ approaches a_1 as $t \to +\infty$.

We shall determine the set P by means of a Ljapunov function. Let us define the continuously differentiable function $V(x, y, z)$ by

$$V(x, y, z) = (x, y, z) \, . \, \boldsymbol{W} \, . \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\boldsymbol{W}$ is the matrix defined by (10). Then

$$V(x, y, z) \geqq 0 \quad \text{in} \quad R^3 \tag{15}$$

and $V(x, y, z) = 0$ holds only for the point $a_0 = (0, 0, 0)$. Let us calculate the time derivative of the function $V(x(t), y(t), z(t))$ along the solutions of the system (5). We get

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), y(t), z(t)) = \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{x}] = \dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}\dot{\boldsymbol{x}} \, . \tag{16}$$

As by (7) we have $\dot{\boldsymbol{x}}^{\mathrm{T}} = [\boldsymbol{B}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x})]^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} + \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x})$, we obtain from the definition of V, taking into account (10), (11), (12), the relation

(17) $$\frac{d}{dt} V(x, y, z) = [x^T B^T + f^T] Wx + x^T W[Bx + f] =$$

$$= x^T B^T W x + f^T W x + x^T W B x + x^T W f = x^T (B^T W + WB) x + 2 x^T W f =$$

$$= -x^T C x + 2(x, y, z) . \begin{bmatrix} 1, & 0, & 0 \\ 0, & \frac{b}{c}, & 0 \\ 0, & 0, & \frac{2r(ad - bc) + bel}{dl^2} \end{bmatrix} . \begin{pmatrix} s(-gx^2 - xy) \\ -s^{-1}xy \\ 0 \end{pmatrix} =$$

$$= -(x, y, z) . C . \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\left[sx^2(y + gx) + \frac{y^2 bx}{sc} \right] =$$

$$= -F_1(x(t), y(t), z(t)) - F_2(x(t), y(t), z(t)),$$

where

(18) $$F_1(x, y, z) = (x, y, z) . C . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sum_{i,j=1} c_{ij} x_i x_j ,$$

c_{ij} are defined by (12), $x_1 = x$, $x_2 = y$, $x_3 = z$, and

(19) $$F_2(x, y, z) = 2\left[sx^2(y + gx) + \frac{y^2 bx}{sc} \right].$$

Let us denote

(20) $$F(x, y, z) = F_1(x, y, z) + F_2(x, y, z) \quad ((x, y, z) \in R^3) .$$

The function F_2 does not depend on z. For fixed x, y, the function F_1 is a quadratic function of z and the coefficient c_{33} is positive at z^2. Hence for fixed x, y the function F attains its minimum. Let us calculate $\partial F(x, y, z)/\partial z$. We get

(21) $$\frac{\partial F(x, y, z)}{\partial z} = 2c_{33}z + 2c_{13}x + 2c_{23}y = 0$$

for $z = z_1 := (-c_{13}x - c_{23}y)/c_{33}$. Then

(22) $$\min_{z \in R} F(x, y, z) = F(x, y, z)|_{z=z_1} = c_{11}x^2 + 2c_{12}xy +$$

$$+ 2c_{13}x . \frac{-c_{13}x - c_{23}y}{c_{33}} + c_{22}y^2 + 2c_{23}y . \frac{-c_{13}x - c_{23}y}{c_{33}} + \frac{(c_{13}x + c_{23}y)^2}{c_{33}} +$$

$$+ F_2(x, y, z) = d_{11}x^2 + d_{22}y^2 + 2d_{12}xy + 2\left[s(x^2 y + x^3 g) + \frac{bxy^2}{sc} \right],$$

where

(23) $$d_{11} = c_{11} - \frac{c_{13}^2}{c_{33}}, \quad d_{22} = c_{22} - \frac{c_{23}^2}{c_{33}}, \quad d_{12} = c_{12} - \frac{c_{13}c_{23}}{c_{33}} .$$

Let us define a subset $M(\varrho)$ of R^3 for $\varrho > 0$ by

(24)
$$M(\varrho) = \{(x, y, z) \in R^3 : x \geqq 0,\ y \geqq 0,\ z \in R\} \cup$$
$$\cup \{(x, y, z) \in R^3 : x \geqq 0,\ y < 0,\ z \in R^3\} \cup$$
$$\cup \{(x, y, z) \in R^3 : 0 > x \geqq -(x_0^2 - 1) . \frac{y}{f x_0} - \varrho,\ y \leqq 0,\ z \in R\} \cup$$
$$\cup \{(x, y, z) \in R^3 : 0 > x \geqq -\varrho,\ 0 < y,\ z \in R\} .$$

First we show that under the assumption (9),

(25)
$$d_{11} > 0 , \quad d_{12} > 0 , \quad d_{22} > 0 .$$

In fact, since $x_0 > 1$, $M_2 > 0$, we have $c_{33} > 0$. Hence $d_{11} > 0$ iff $c_{11}c_{33} - c_{13}^2 > 0$. But

$$c_{11}c_{33} - c_{13}^2 = \frac{2r(ad - bc) + bel}{dl^2} \left[4ar - \frac{2r(ad - bc) + bel}{d} \right]$$

and the first factor is obviously positive, while the second is equal to

$$2ar + \frac{b}{d}(2rc - el) = 2ar + ws^{-1}(2y_0 - f) \frac{s^2(x_0 - 1)}{x_0 + 1} =$$
$$= 2ar + wsf \frac{(x_0 - 1)^2}{(x_0 + 1)^2} > 0 .$$

Thus $d_{11} > 0$.

Further, $d_{12} > 0$ iff $c_{12}c_{33} - c_{13}c_{23} > 0$. But

$$c_{12}c_{33} - c_{13}c_{23} = [2r(ad - bc) + bel] . \left[\frac{4br}{dl^2} - \frac{be}{cdl} \right] .$$

The first factor is positive, while the second is equal to

$$\frac{b}{dl}\left(-4 + \frac{f}{y_0} \right) = \frac{b}{dl}\left(-3 + \frac{1}{x_0} \right) > 0 ,$$

and hence $d_{12} > 0$.

Finally, by (23) and by $c_{33} > 0$ we have $d_{22} > 0$ iff $c_{22}c_{33} - c_{23}^2 > 0$. But this relation is equivalent to

$$\frac{b}{c^2 l^2} \{4rc[2r(a - b) + bel] - be^2 l^2\} > 0 .$$

As

$$\frac{b}{c^2 l^2} > 0, \quad 4rc[2r(ad - bc)] > 0$$

and by (14) also

$$4rcbel - be^2 l^2 = bel(4rc - el) = belL > 0 , \quad \text{we have} \quad d_{22} > 0 .$$

Put

(26) $$\varrho_1 = \frac{d_{11}}{2sg}\,.$$

Clearly $\varrho_1 > 0$.

Let us consider the equation

(27) $$2sg\,\frac{x_0^2 - 1}{fx_0}\,\varrho^2 - \varrho\left[d_{11}\,\frac{x_0^2 - 1}{fx_0} + 2sgd_{22}\right] + (d_{11}d_{22} - d_{12}^2) = 0\,.$$

(9) implies that $x_0 > 1$. In view of (23)

$$d_{11}d_{22} - d_{12}^2 = (c_{11}c_{22}c_{33} - c_{11}c_{23}^2 - c_{13}^2c_{22} - c_{12}^2c_{33} + 2c_{12}c_{13}c_{23})\,c_{33}^{-1}\,,$$

where c_{ij} are given by (12). By (9) we have $M_2 = ad - bc > 0$ and thus, (12) gives $c_{33} > 0$.

Therefore

(28) $$d_{11}d_{22} - d_{12}^2 > 0$$

iff $c_{11}(c_{22}c_{33} - c_{23}^2) - c_{13}^2c_{22} - c_{12}^2c_{33} + 2c_{12}c_{13}c_{23} > 0$.

The last expression can be written in the form

$$c_{33}(c_{11}c_{22} - c_{12}^2) + c_{23}(c_{12}c_{13} - c_{11}c_{23}) + c_{13}(c_{12}c_{23} - c_{13}c_{22}) =$$

and since the matrix $\boldsymbol{C}$ is symmetric,

$$= c_{13}(c_{21}c_{32} - c_{22}c_{31}) - c_{23}(c_{11}c_{32} - c_{12}c_{31}) + \\ + c_{33}(c_{11}c_{22} - c_{12}c_{21}) = \det \boldsymbol{C} = \overline{M}_3 > 0\,.$$

Thus (28) follows from (9). Then the equation (27) either has complex conjugate roots and the inequality

(29) $$2sg\,\frac{x_0^2 - 1}{fx_0}\,\varrho^2 - \varrho\left[d_{11}\,\frac{x_0^2 - 1}{fx_0} + 2sgd_{22}\right] + (d_{11}d_{22} - d_{12}^2) > 0$$

is true for each $\varrho > 0$, or it has two positive real roots or one double positive root. In all cases there is a ϱ_2, $0 < \varrho_2$ such that (29) is satisfied for all $0 < \varrho < \varrho_2$.

Further, consider the equation

(30) $$s^2\varrho^2 + (2sgd_{22} - 2sd_{12})\,\varrho + (d_{12}^2 - d_{11}d_{22}) = 0\,.$$

In view of (28) there is a positive root ϱ_3 of (29). Then the inequality

(31) $$s^2\varrho^2 + (2sgd_{22} - 2sd_{12})\,\varrho + (d_{12}^2 - d_{11}d_{22}) \leqq 0$$

is valid for all $0 < \varrho < \varrho_3$.

Now, we can formulate a lemma.

Lemma 4. *Let* (9) *be satisfied and let* ϱ *be such that*

$$0 < \varrho < \varrho_4 = \min(\varrho_1, \varrho_2, \varrho_3) .$$

Then the function F *is positive definite in* $M(\varrho)$.

Proof. Since

$$F(x, y, z) \geqq G(x, y) \quad \text{for all} \quad (x, y, z) \in R^3 ,$$

where

$$G(x, y) = d_{11}x^2 + d_{22}y^2 + 2d_{12}xy + 2s\left(x^2y + x^3g + y^2x \frac{x_0^2 - 1}{fx_0}\right), \tag{32}$$

we have to show that $G(x, y) \geqq 0$ in $M(\varrho)$.

We shall investigate the following four cases.

1. $x \geqq 0$, $y \geqq 0$. Then in view of (25) and the remark after Lemma 2 all coefficients in the form G are positive and hence, $G(x, y) \geqq 0$ for all $x \geqq 0$, $y \geqq 0$. Moreover, $G(x, y) > 0$ for $x \geqq 0$, $y \geqq 0$, $(x, y) \neq (0, 0)$.

2. $x \geqq 0$, $y < 0$. By (25), (28) we have

$$d_{11}x^2 + d_{22}y^2 + 2d_{12}xy \geqq 0 \tag{33}$$

for all points in R^2 and

$$2s\left(x^2y + x^3g + y^2x \frac{x_0^2 - 1}{fx_0}\right) = 2sx^3\left(u^2 \frac{x_0^2 - 1}{fx_0} + u + g\right),$$

where $u = y/x$. We consider only the points $x > 0$, $y < 0$, since at $x = 0$, $y < 0$ we have $G(0, y) = d_{22}y^2 \geqq 0$. Then

$$u^2\left(\frac{x_0^2 - 1}{fx_0}\right) + u + g \geqq 0 \quad \text{for all } u$$

iff

$$1 - 4g\frac{x_0^2 - 1}{fx_0} \leqq 0 .$$

The last inequality is equivalent to

$$fx_0 \leqq 4g(x_0^2 - 1) .$$

For x_0 we have the equality

$$gx_0^2 + x_0(-1 + f + g) - (f + 1) = 0 ,$$

therefore

$$4gx_0^2 = 4(f + 1) + 4x_0(1 - f - g)$$

and hence

$$4gx_0^2 - fx_0 - 4g = x_0(4 - 5f - 4g) + 4(1 + f - g) > 0 .$$

The last inequality follows from (9).

3. $0 > x \geqq -(x_0^2 - 1)(y/fx_0) - \varrho,\ y \leqq 0$. Then

$$G(x, y) \geqq (d_{11} - 2sg\varrho)\, x^2 + d_{22}y^2 + 2(d_{12} - s\varrho)\, xy =$$
$$= x^2[d_{22}v^2 + 2(d_{12} - s\varrho)\, v + (d_{11} - 2sg\varrho)]\,,$$

where $v = y/x$. The last term is nonnegative iff (31) is valid. By the inequality $0 < \varrho < \varrho_3$ (31) is true and hence $G(x, y) \geqq 0$ for $0 > x \geqq -(x_0^2 - 1)(y/fx_0) - \varrho,\ y \leqq 0$.

4. $0 > x \geqq -\varrho,\ 0 < y$. Now we get that

$$G(x, y) \geqq (d_{11} - 2sg\varrho)\, x^2 + \left(d_{22} - \varrho\, \frac{x_0^2 - 1}{fx_0}\right) y^2 + 2d_{12}xy$$

for such points (x, y). By $0 < \varrho < \varrho_1$ we have $d_{11} - 2sg\varrho > 0$. If we put $w = x/y$, then

$$(d_{11} - 2sg\varrho)\, w^2 + 2d_{12}w + \left(d_{22} - \varrho\, \frac{x_0^2 - 1}{fx_0}\right) \geqq 0$$

iff

$$d_{12}^2 - (d_{11} - 2sg\varrho)\left(d_{22} - \varrho\, \frac{x_0^2 - 1}{fx_0}\right) \leqq 0\,.$$

The last inequality is equivalent to the nonstrict inequality (29). In view of $0 < \varrho < \varrho_2$, (29) is satisfied and hence $G(x, y) \geqq 0$ for $0 > x \geqq -\varrho,\ 0 < y$. The lemma is proved.

Let us investigate the properties of the vector field defined by the system (1) for $0 \leqq X < \infty,\ 0 \leqq Y < \infty,\ 0 \leqq Z < \infty$.

a) $\dot{Z} = 0$ for $Z = X$ and $0 \leqq X < \infty,\ 0 \leqq Y < \infty$ and $\dot{Z} < 0$ $(\dot{Z} > 0)$ for $Z > X$ $(Z < X)$, $0 \leqq X < \infty,\ 0 \leqq Y < \infty$ and $0 \leqq Z < \infty$.

b) $\dot{X} = 0$ on the surface $Y - XY + X - gX^2 = 0$. This surface has two branches, a positive one and a negative one. Let us denote the positive branch as X_p. We have

$$X_p = \frac{1 - Y + \sqrt{((1 - Y)^2 + 4Yg)}}{2g}\,. \tag{34}$$

We have $\dot{X} > 0$ for $X < X_p$ and $\dot{X} < 0$ for $X > X_p$ and $0 \leqq X < \infty,\ 0 \leqq Y < \infty,\ 0 \leqq Z < \infty$.

Let us investigate X_p. Denoting $' = \mathrm{d}/\mathrm{d}Y$ we have

$$X_p' = \frac{-1}{2g} + \frac{-1(1 - Y) + 2g}{2g\sqrt{((1 - Y)^2 + 4Yg)}} = -\frac{1}{2g}\cdot\left[1 + \frac{1 - Y - 2g}{\sqrt{((1 - Y)^2 + 4Yg)}}\right]. \tag{35}$$

The inequality

$$\left(\frac{1 - Y - 2g}{\sqrt{((1 - Y)^2 + 4Yg)}}\right)^2 < 1 \tag{36}$$

is equivalent to the inequality $4g(g - 1) < 0$ and thus to the inequality $g < 1$ that is true by the assumption (9). Therefore (36) is valid, too. By (35) it follows that $X'_p < 0$.

Further we have

$$X''_p = -\frac{1}{2g}\left[-((1-Y)^2+4Yg)^{-1/2} - \frac{(1-Y-2g).(2g-(1-Y))}{\sqrt{((1-Y)^2+4Yg)^3}}\right] =$$

$$= -\frac{1}{2g}.[(1-Y)^2+4Yg]^{-3/2}.(4g^2-4g) =$$

$$= 2(1-g)[(1-Y)^2+4Yg]^{-3/2} > 0\,.$$

Hence X_p is decreasing and convex for $0 \leqq Y < \infty$, $0 \leqq Z < \infty$. Denote $k = X_p(0) = 1/g$, then $k - X_p(Y) > 0$ for all $Y \in (0, \infty)$. Further we put $h = \lim_{Y\to\infty} X_p(Y) = 1$, hence $h - X_p(Y) < 0$ for $Y \in (0, \infty)$.

c) For the Y-component we have $\dot{Y} = 0$ on the surface $fZ - Y - XY = 0$ and hence for $Y = fZ/(1+X)$. The intersection of this surface with the plane $X = \text{const}$ is a straight line, while with plane $Z = \text{const}$ it is a hyperbola. For $Y < fZ/(1+X)$ we have $\dot{Y} > 0$ while $\dot{Y} < 0$ for $Y > fZ/(1+X)$.

Lemma 5. *Let the assumption* (9) *be fulfilled and let the constants* X_i, Y_i, Z_i *for* $i = 1, 2$ *satisfy*

(37)
$$0 < X_1 \leqq h, \qquad k < X_2\,,$$
$$0 < Z_1 < X_1\,, \qquad X_2 < Z_2\,,$$
$$0 < Y_1 < \frac{fZ_1}{1+X_2}\,, \qquad \frac{fZ_2}{1+h} < Y_2\,.$$

Let $R_1 = \{(X, Y, Z) \in R^3 : X_1 \leqq X \leqq X_2,\ Y_1 \leqq Y \leqq Y_2,\ Z_1 \leqq Z \leqq Z_2\}$ *and let* R_1^0 *be the interior of* R_1. *Then the following statements are true*:

1. *Each solution of* (1) *passing through a point of* R_1 *enters* R_1^0 *and remains in* R_1^0.

2. *The system* (1) *has a unique equilibrium point in* R_1 *namely the point* $a_1(x_0, y_0, z_0)$.

Proof. 1. The set R_1 is constructed in such a way that each solution of (1) which arrives at a point of the boundary of R_1 goes to R_1^0, which follows from the signs of $\dot{X}$, $\dot{Y}$, $\dot{Z}$ at that point.

2. The system (1) has only two equilibrium points, $a_0(0, 0, 0)$ and $a_1(x_0, y_0, z_0)$. The point a_0 does not belong to R_1, hence we investigate the point $a_1(x_0, y_0, z_0)$ where the values x_0, y_0, z_0 are determined by (3).

We have to show that

(38)
$$X_1 \leqq h = 1 < x_0 < k = \frac{1}{g} < X_2\,,$$

that is

$$1 < \frac{1 - f - g + \sqrt{((1 - f - g)^2 + 4g(1 + f))}}{2g} < \frac{1}{g},$$

which becomes

$$(39) \qquad 9g^2 - 6g + 1 + 6gf - 2f + f^2 < 1 + f^2 + g^2 - 2f - 2g + \\ + 2fg + 4g + 4fg < 1 + f^2 + g^2 + 2f + 2g + 2fg\,.$$

The relation (39) represents the system of inequalities

$$(40) \qquad \begin{aligned} 8g(g - 1) &< 0\,, \\ 4f(g - 1) &< 0 \end{aligned}$$

which is valid because $g + 2f < 1$ and hence $X_1 < x_0 < X_2$.

As $z_0 = x_0$, we have

$$(41) \qquad Z_1 < X_1 < x_0 = z_0 < X_2 < Z_2$$

and by the strict monotonicity of the function $fx/(1 + x)$ the inequalities (37), (38), (41), imply that the inequalities

$$(42) \qquad \begin{aligned} &\frac{fx_0}{1 + x_0} < \frac{fX_2}{1 + X_2} < \frac{fZ_2}{1 + h} < Y_2\,, \\ &\frac{fx_0}{1 + x_0} > \frac{fX_1}{1 + X_1} > \frac{fZ_1}{1 + X_2} > Y_1 \end{aligned}$$

are true. The inequalities (38), (41), (42) show that the point $a_1(x_0, y_0, z_0)$ lies in R_1^0.

Lemma 6. *Let the assumption* (9) *be satisfied, let the constants* $X_i, Y_i, Z_i, i = 1, 2$, *satisfy* (37) *and let* K *be such that* $k < K < X_2$.

Further let $P(K)$ *be the set*

$$(43) \quad P(K) = \left\{(X, Y, Z) \in R^3 : h \leqq X \leqq K, \frac{fh}{1 + K} \leqq Y \leqq \frac{fK}{1 + h}, h \leqq Z \leqq K\right\}.$$

Then $a_1 \in P(K)$ *and each solution* f (1) *remains in* $P(K)$ *for all* $t \geqq t_0$ *if the initial value of that solution at* t_0 *belongs to* $P(K)$.

Proof. By the inequalities (37), (38), (41) as well as by the estimate for $fx_0/(1 + x_0)$ it follows that $a_1 \in P(K)$ and $P(K) \subset R_1$.

By the construction of the set $P(K)$ as well as by Lemma 5 it follows that for every $\varepsilon > 0$ the trajectory of the solution of (1) mentioned in the statement of the lemma remains in the ε-neighbourhood of $P(K)$ and hence it lies in $P(K)$.

The transformation (4) maps the set $P(K)$ to the set

$$(44) \qquad \tilde{P}(K) = \{(x, y, z) \in R^3 : h - x_0 \leqq x \leqq K - x_0\,,$$

$$f\left(\frac{h}{1+K} - \frac{x_0}{1+x_0}\right) \leqq y \leqq f\left(\frac{K}{1+h} - \frac{x_0}{1+x_0}\right),\ h - x_0 \leqq z \leqq K - x_0\Big\}\,.$$

Suppose that $K > 1$ is such that

$$\frac{1 - x_0^2 + 2x_0}{2(1+x_0)} \leqq \frac{1}{1+K}\,. \tag{45}$$

Then the inequality

$$\frac{1 - x_0}{2} \leqq \frac{1}{1+K} - \frac{x_0}{1+x_0}$$

is true and hence, under the assumption (45), $\tilde{P}(K) \subset P_1(K)$ where

$$P_1(K) = \Big\{(x, y, z) \in R^3 \colon 1 - x_0 \leqq x \leqq K - x_0\,, \tag{46}$$

$$\frac{f(1 - x_0)}{2} \leqq y \leqq f\left(\frac{K}{1+h} - \frac{x_0}{1+x_0}\right),\ 1 - x_0 \leqq z \leqq K - x_0\Big\}\,.$$

If

$$1 - x_0 > -(x_0^2 - 1)\,\frac{1 - x_0}{2x_0} - \varrho\,, \tag{47}$$

then $x > (-(x_0^2 - 1)\,(y/fx_0) - \varrho)$ in $P_1(K)$ and thus $P_1(K) \subset M(\varrho)$ where $M(\varrho)$ is defined by (24). The condition (47) is equivalent to the relation

$$\varrho > (x_0 - 1)\,(x_0^2 + 2x_0 - 1)/2x_0\,. \tag{48}$$

Lemma 7. *If*

$$f = 0{\cdot}06\,,\quad g = 0{\cdot}64\,,\quad w = 1\,,\quad s = 1\,, \tag{49}$$

then (9) *is satisfied and the function* $F = F(x, y, z)$ *which is defined by* (20) *is positive definite in* $\tilde{P}(K)$ *with*

$$K = 1{\cdot}98\,. \tag{50}$$

Proof. First of all, on the basis of (3), (49) implies that $x_0 = 1{\cdot}542496$, $y_0 = 0{\cdot}036401$ and hence, the left-hand side of (45) is equal to $0{\cdot}33544$. This implies that (45) is satisfied with K given by (50). Clearly, (49) implies (9).

Denote the right-hand side of the inequality (48) by ϱ_0. If $\varrho_0 < \varrho_4$ with ϱ_4 mentioned in Lemma 4, then for all $\varrho \in (\varrho_0, \varrho_4)$ Lemma 4 as well as (48) are true. Hence, by Lemma 4 F is positive definite in $M(\varrho)$, and since for such ϱ both (48) and (47) are true, $P_1(K) \subset M(\varrho)$, which implies that the function F is positive definite in $P_1(K)$. By (49) we have that (45) is satisfied with K determined in (50) and thus

$\tilde{P}(K) \subset P_1(K)$. Hence we have to show that $\varrho_0 < \varrho_1$, $\varrho_0 < \varrho_2$, $\varrho_0 < \varrho_3$. Direct calculation yields

$$\varrho_0 = 0{\cdot}785046\,, \quad \varrho_1 = 0{\cdot}792314\,, \quad \varrho_2 = 0{\cdot}787336\,, \quad \varrho_3 = 0{\cdot}792069\,.$$

This completes the proof of the lemma.

Theorem 3. *If* (49) *is satisfied, then the equilibrium point* $a_1(x_0, y_0, z_0)$ *of the system* (1) *is asymptotically stable in the large with respect to the set*

$$(51) \qquad P(1{\cdot}98) = \{(X, Y, Z) \in R^3 : 1 \leqq X \leqq 1{\cdot}98,\ 0{\cdot}020134 \leqq Y \leqq \leqq 0{\cdot}0594,\ 1 \leqq Z \leqq 1{\cdot}98\}$$

in the sense of Definition 3.

Proof. By Lemma 7, the conditions (49) imply that (9) is satisfied and that k determined by (38) is equal to 1·5625. Hence we can consider the set $P(K)$ given by (43) for $K = 1{\cdot}98 > k$ and $h = 1, f = 0{\cdot}06$. This set is defined by (51). For its image $\tilde{P}(1{\cdot}98)$ under the transformation (4) the following statements are true.

1. By virtue of Lemma 6 and the transformation (4), each solution of (5) remains in $\tilde{P}(1{\cdot}98)$ for all $t \geqq t_0$ if its initial value lies in $\tilde{P}(1{\cdot}98)$ at $t = t_0$.

2. $\tilde{P}(1{\cdot}98)$ is a compact set and $(0, 0, 0) \in \tilde{P}(1{\cdot}98)$.

3. There exists a continuously differentiable function $V(x, y, z)$ defined by

$$V(x, y, z) = (x, y, z) \,.\, W \,. \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where W is the matrix defined by (10), with the following properties:

a) By (15), $V(x, y, z)$ is positive definite in $\tilde{P}(1{\cdot}98)$.

b) By Lemma 7 the time derivative $\dot{V}(x(t), y(t), z(t))|_{(5)}$ of the function $V(x(t), y(t), z(t))$ along a solution of the system (5)

$$\dot{V}(x(t), y(t), z(t))|_{(5)} = -F(x(t), y(t), z(t))$$

is negative definite.

Then by the La Salle theorem [5, p. 76] on the stability in the large, the equilibrium point (0, 0, 0) of the system (5) is stable with respect to the set $\tilde{P}(1{\cdot}98)$ and each solution system which begins in $\tilde{P}(1{\cdot}98)$ is approaching the origin (0, 0, 0) as $t \to \infty$. Similar properties are exhibited by the solutions of (1) in $P(1{\cdot}98)$, and hence the equilibrium point $a_1(x_0, y_0, z_0)$ of the system (1) is asymptotically stable in the large with respect to the set $P(1{\cdot}98)$.

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Súhrn

STABILITA MODELI BELOUSOVEJ-ŽABOTINSKÉHO REAKCIE

VLADIMÍR HALUŠKA

V práci sa pojednáva o Fieldovom-Körösovom-Noyesovom modeli Belousovej-Žabotinského reakcie. Metódou Ljapunovovej funkcie je stanovená postačujúca podmienka na to, aby netriviálny kritický bod tohoto modelu bol asymptoticky stabilný vzhľadom na istú množinu.

Резюме

УСТОЙЧИВОСТЬ МОДЕЛИ РЕАКЦИИ БЕЛОУСОВА-ЖАБОТИНСКОГО

VLADIMÍR HALUŠKA

Настоящая работа занимается моделью Фильда-Кереша-Нойеса реакции Белоусова-Жаботинского. Методом функции Ляпунова установлено достаточное условие для того, чтобы нетривиальная критическая точка этой модели была асимптотически устойчивой относительно определенного множества.

Author's address: RNDr. *Vladimír Haluška,* Katedra matematiky, Chemicko-technologická fakulta SVŠT, Radlinského 8, 811 07 Bratislava.