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ON NEWTON-LIKE METHODS TO ENCLOSE SOLUTIONS OF NONLINEAR EQUATIONS*)

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Summary. We present a class of Newton-like methods to enclose solutions of systems of nonlinear equations. Theorems are derived concerning the feasibility of the method, its global convergence, its speed and the quality of enclosure.

Keywords: Newton-like methods, nonlinear equations, enclosure of solutions, interval analysis.
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1. INTRODUCTION

The discretization of integral equations or boundary value problems yields systems of nonlinear equations

$$F(x) = 0 \tag{1}$$

where F maps some region $D \subseteq \mathbb{R}^n$ into the set $\mathbb{R}^n$ of vectors with n real components. We present an algorithm to enclose solutions of such real systems by an interval vector. This algorithm is based on the representation

$$F(x) = F(\tilde{x}) + J(x, \tilde{x})(x - \tilde{x}), \quad x, \tilde{x} \in D, \tag{2}$$

of F obtained by applying the mean value theorem to each component F_i, $i = 1, \ldots, n$ of F. (Here we assume that F_i has continuous partial derivatives.) Therefore

$$J(x, \tilde{x}) = \left(\frac{\partial}{\partial x_j} F_i(\tilde{x} + \theta_i(x - \tilde{x}))\right) \in \mathbb{R}^{n \times n}, \quad i, j = 1, \ldots, n,$$

where $\mathbb{R}^{n \times n}$ denotes the set of real $n \times n$ matrices and θ_i, $i = 1, \ldots, n$, are some real numbers lying between 0 and 1. In the sequel let $x = z$ be a solution of (1). Split J into $M - N$ with a nonsingular $n \times n$ matrix M, and $N \in \mathbb{R}^{n \times n}$. Then (2) implies

$$z = \tilde{x} - M^{-1}\{N(\tilde{x} - z) + F(\tilde{x})\}. \tag{3}$$

*) This is an abbreviated version of the second part of the author's Habilitationsschrift [12].

Let $[x]^0 \in I\mathbb{R}^n$ (set of interval vectors with n components) and let $[A] \in I\mathbb{R}^{n \times n}$ (set of $n \times n$ interval matrices). Consider the splitting $[A] = [M] - [N]$ where $[M], [N] \in I\mathbb{R}^{n \times n}$ and where the interval Gaussian algorithm (cf [2, § 15]) is feasible for $[M]$. If $z \in [x]^0 \subseteq D$ and if $F'(x) \in [A]$ for all Jacobians

$$F'(x) = \left(\frac{\partial}{\partial x_j} F_i(x)\right) \in \mathbb{R}^{n \times n},\ x \in [x]^0\,,$$

then by (3) by the inclusion monotonicity of interval arithmetic one gets

$$z \in \tilde{x} - \mathrm{IGA}([M], [N]\,(\tilde{x} - [x]^0) + F(\tilde{x}))$$

where $\mathrm{IGA}([B], [c])$ denotes the interval vector resulting from the interval Gaussian algorithm applied to $[B] \in I\mathbb{R}^{n \times n}$ and $[c] \in I\mathbb{R}^n$. This suggests the iterative process

$$(4) \qquad [x]^{k+1} = \{\tilde{x}^k - \mathrm{IGA}\,([M]^k, [N]^k\,(\tilde{x}^k - [x]^k) + F(\tilde{x}^k))\} \cap [x]^k\,, \quad k = 0, 1, \ldots$$

with $\tilde{x}^k \in [x]^k$ and $[A]^k = [M]^k - [N]^k$ such that $F'(x) \in [A]^k$ for all $x \in [x]^k$. A modification of (4) was first considered in [19, p. 78].

To realize (4) one has to compute F' and the complete interval Gaussian algorithm for each iterate. This can become very cumbersome. Therefore we adopt the idea of some Newton-like methods described in [2]: We iterate several steps with $[M]^k$, $[N]^k$ being fixed. A more precise formulation of this method (I) can be found in Section 3. By the intersection in (4) the iterates of (I) are inclusion monotone. Under appropriate hypotheses we are able to prove that $[x]^0$ contains exactly one solution z of (1) if and only if (I) does not break down by empty intersection – see Section 4. We formulate criteria for the sequence of iterates to converge towards z and we show that these sequences may converge superlinearly. In Section 5 we illustrate our results by two examples growing out from applications.

2. NOTATION

We write matrices by capital letters, vectors and scalars by small letters. Interval quantities are written with brackets using notation like $[A] = [{}_-A, {}^-A] = ([a]_{ij}) = ([{}_-a_{ij}, {}^-a_{ij}])$ simultaneously without further reference. Real numbers are sometimes identified with point intervals by omitting the brackets. Real vectors and real matrices are often interpreted analogously. Thus 0 may be used for a real null matrix or for an interval null matrix. For $[D]$ being a diagonal matrix $[D] = \mathrm{diag}\,([d]_1, \ldots, [d]_n) \in I\mathbb{R}^{n \times n}$ we set $[D]^{-1} := \mathrm{diag}\,(1/[d]_1, \ldots, 1/[d]_n)$ if no diagonal entry $[d]_i$ of $[D]$ contains zero. We equip $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ with the usual entrywise defined partial ordering "$\leqq$", and denote by $\varrho(A)$ the spectral radius of a real $n \times n$ matrix A.

According to [16, 2.4.7] we call $A \in \mathbb{R}^{n\times n}$ an M matrix if it has only nonpositive offdiagonal entries and if it has a nonnegative inverse. By definition $[A] \in I\mathbb{R}^{n\times n}$ is an interval M matrix if all elements of $[A]$ are M matrices (cf. [4]).

We call the representation $[A] = [M] - [N]$ a triangular splitting of $[A] \in I\mathbb{R}^{n\times n}$ if $[M]$ is a lower triangular matrix (cf. [13]). For $[A] \in I\mathbb{R}^{n\times n}$ we define the absolute value by the nonnegative real matrix

$$|[A]| := (\max\{|{}_{-}a_{ij}|, |{}^{-}a_{ij}|\}) \in \mathbb{R}^{n\times n},$$

and the width by

$$d([A]) := ({}^{-}a_{ij} - {}_{-}a_{ij}) \in \mathbb{R}^{n\times n}.$$

Absolute value and width of an interval vector are defined analogously. We assume that the reader is familiar with the elementary facts of interval analysis which can be found e.g. in [2]. Here we only recall the formulae

$$\begin{aligned} &d([x] + [y]) = d([x]) + d([y]), \\ &d([A][x]) \leqq d([A])\,|[x]| + |[A]|\,d([x]), \\ &d([A][x]) = |[A]|\,d([x]) \quad \text{if} \quad 0 \in [x] \end{aligned} \tag{5}$$

and if no entry of $[A]$ contains 0 in its interior.

3. THE ALGORITHM

According to (4) and the remarks in Section 1 we consider the following iterative process (6) for $k = 0, 1, \ldots$:

(6a) $[x]^{k,0} := [x]^k$,

(6b) $[y]^{k,m} := \tilde{x}^k - \mathrm{IGA}\,([M]^k, [N]^k(\tilde{x}^k - [x]^{k,m-1}) + F(\tilde{x}^k))$,

(6c) $[x]^{k,m} := [y]^{k,m} \cap [x]^{k,m-1}, \quad m\ 1, 2, \ldots, r_k$,

(6d) $[x]^{k+1} := [x]^{k,r_k}$.

F, D and $[x]^0 \in I\mathbb{R}^n$ are defined as in Section 1; $\tilde{x}^k \in [x]^k$ can be chosen arbitrarily; $[A]^k \in I\mathbb{R}^{n\times n}$ has to enclose all Jacobians $F'(x) \in \mathbb{R}^{n\times n}$ for $x \in [x]^k$; $[M]^k - [N]^k$ is a splitting of $[A]^k$ with the interval Gaussian algorithm being feasible for $[M]^k$($[A]^k$, $[M]^k$, $[N]^k$ may depend on $[x]^k$); $\{r_k\}$ is a sequence of positive integers.

Choosing the selectable quantities in (6) in a definite way and introducing some stopping criterion results in the algorithm (I) mentioned in Section 1.

Method (6) is a generalization of the iterative processes described in [2, p. 278]. For $\tilde{x}^k := ({}_{-}x^k + {}^{-}x^k)/2$ it was presented in [11]. For $r_k = 1$ it is a modification of [19, p. 78]. To list some more well-known iterative processes contained in (6) we split $[A]^k$ in the usual way into

$$[A]^k = [D]^k - [E]^k - [F]^k \tag{7}$$

where $[D]^k$ is a diagonal matrix, $[E]^k$ is strictly lower triangular and $[F]^k$ is strictly upper triangular. Furthermore we denote by $F'([x])^k) \in I\mathbb{R}^{n\times n}$ an interval arithmetic evaluation for the Jacobian F' at $[x]^k \in I\mathbb{R}^n$ (cf. [2, p. 21]). Specifying some quantities in (6) we get the following methods:

– Newton-like total step method and its modification in [2, p. 276 and p. 278].

$$([A]^k = F'([x]^k), \quad [M]^k = [D]^k, \quad [N]^k = [E]^k + [F]^k,$$
$$r_k = 1 \text{ and } r_k \text{ arbitrary, respectively.})$$

– Newton-like single step method and its modification in [2, p. 276 and p. 278].

$$([A]^k = F'([x]^k), \quad [M]^k = [D]^k - [E]^k, \quad [N]^k = [F]^k,$$
$$r_k = 1 \text{ and } r_k \text{ arbitrary, respectively.})$$

– Newton's method in [1].

$$([A]^k = [M]^k = F'([x]^k), \quad [N]^k = 0, \quad r_k = 1)$$

– simplified Newton's method in [1].

$$([A]^k = [M]^k = F'([x]^0), [N]^k = 0, \ r_k = 1)$$

– methods in [14].

If $[M]^k - [N]^k$ is a triangular splitting of $[A]^k$ then one can modify (6b) and (6c) getting only one formula

$$[x]_i^{k,m} := \{\tilde{x}_i^k - (1/[m]_{ii}^k)(\sum_{j=1}^{i-1} - [m]_{ij}^k(\tilde{x}_j^k - [x]_j^{k,m}) +$$
$$+ ([N]^k(\tilde{x}^k - [x]^{k,m-1}))_i + F_i(\tilde{x}^k))\} \cap [x]_i^{k,m-1}, \quad 1 \leqq i \leqq n.$$

4. RESULTS

Normally interval iterative methods are considered under the following aspects:

– feasibility

For arbitrary starting vectors all iterates must be defined.

– global convergence

The methods should converge for any starting vector $[x]^0$.

The limit $[x]^*$ should be independent of $[x]^0$.

– speed of convergence

The method should converge fast in some sense to be specified. As a measure for the speed one can use the R order defined in [2, p. 286].

– inclusion monotonicity

This means that for sequences $\{[\hat{x}]^k\}$, $\{[x]^k\}$ of iterates satisfying $[\hat{x}]^0 \subseteq [x]^0$ one has $[\hat{x}]^k \subseteq [x]^k$, $k = 0, 1, \ldots$

In particular, $[x]^1 \subseteq [x]^0$ always implies $[x]^{k+1} \subseteq [x]^k$ in this case.

– enclosure

The set $S := \{z \mid z \in [x]^0 \wedge F(z) = 0\}$ of solutions should be contained in the limit $[x]^*$.

– quality of enclosure

The limit $[x]^*$ should be a good enclosure of the interval hull of S (cf. e.g. [13]). For a good method to enclose a solution z of (1) one expects the iterates $[x]^k$ to contract to z.

These aspects are basic for the theorems in this section. We formulate them under the following assumptions

$$(8) \qquad \begin{aligned} &[A] = [M] - [N],\\ &[A], [M] \quad \text{interval} \quad M \quad \text{matrices}\\ &\underline{N} \geqq 0 \end{aligned}$$

where $[A]$, $[M]$ and $[N]$ will be appropriately replaced in the sequel. In our first theorem we consider the feasibility, the global convergence and the inclusion monotonicity of (6).

Theorem 1. *Let $D \subseteq \mathbb{R}^n$ be a region and let F: $D \to \mathbb{R}^n$ be a mapping with continuous partial derivatives. Let* (8) *hold for the matrices $[A]^k$, $[M]^k$ and $[N]^k$ of* (6) *and denote by $\{[x]^k\}$ a sequence resulting from* (6). *Using the notation of* (6) *we get*

a) $[x]^{k,m+1} \subseteq [x]^{k,m}$, $1 \leqq m \leqq r_k$; $[x]^{k+1} \subseteq [x]^k$.

b) $[x]^0$ *contains at most one solution z of* (1).

c) *If $[x]^0$ contains a solution z of* (1) *then the method* (6) *does not break down by empty intersection, and we have*

$$z \in [x]^k, \quad k = 1, 2, \ldots.$$

$\{[x]^k\}$ *converges towards a limit $[x]^*$ containing z.*

d) *If the method* (6) *breaks down by empty intersection in a finite number of steps then $[x]^0$ contains no solution of* (1).

Proof. a) is apparently true.

b) If $z, \tilde{z} \in [x]^0$ are two different solutions of (1) then (2) implies $0 = J(z, \tilde{z})(z - \tilde{z})$ in contrast to the nonsingularity of $J(z, \tilde{z}) \in [A]^0$.

c) is proved by induction using (3).

d) is a consequence of c). □

The sequence of iterates need not contract to a solution z of (1) (cf. [1], [19, p. 85]). In our next theorem we formulate sufficient criteria to guarantee the convergence towards z. To prove them we need the following lemma:

Lemma 1. *Let $[b] \in I\mathbb{R}^n$ and let $[M] \in I\mathbb{R}^{n \times n}$ be an interval M matrix. Set $[M]^1 = ([m]^1_{ij}) := [M]$ and let $[M]^k = ([m]^k_{ij}) \in I\mathbb{R}^{n \times n}$, $k = 2, 3, \ldots, n$, be the interval matrix obtained from $[M]$ after $(k - 1)$ steps of the interval Gaussian algorithm.*

Then the following assertions are true:

a) IGA$([M], [b])$ *exists.*

b) IGA$([M], [b]) = [D]^1 ([U]^1 ([D]^2 ([U]^2 (\ldots ([D]^{n-1} ([U]^{n-1} \cdot$
$\cdot ([D]^n ([L]^{n-1} ([L]^{n-2} (\ldots ([L]^1 [b])) \ldots)$
with $[L]^k = ([l]^k_{ij}), [U]^k = ([u]^k_{ij}), [D]^k = ([d]^k_{ij})$ *being defined by*

$$[l]^k_{ij} := \begin{cases} 1 & \text{if } i = j, \\ -[m]^k_{ik}/[m]^k_{kk} & \text{if } j = k < i, \\ 0 & \textit{otherwise}; \end{cases}$$

$$[u]^k_{ij} := \begin{cases} 1 & \text{if } i = j, \\ -[m]^k_{kj} & \text{if } i = k < j, \\ 0 & \textit{otherwise}; \end{cases}$$

$$[d]^k_{ij} := \begin{cases} 1 & \text{if } i = j \neq k, \\ 1/[m]^k_{kk} & \text{if } i = j = k, \\ 0 & \textit{otherwise}. \end{cases}$$

c) *The matrices in* b) *can be represented as*

$$[L]^k = [{}^-L^k, {}_-L^k], \quad [U]^k = [{}^-U^k, {}_-U^k], \quad [D]^k = [{}^-D^k, {}_-D^k]$$

where ${}_-L^k$, ${}_-U^k$, ${}_-D^k$ *and* ${}^-L^k$, ${}^-U^k$, ${}^-D^k$ *are constructed as in* b) *when replacing* $[M]$ *by* ${}_-M$ *and by* ${}^-M$, *respectively. In particular* $[L]^k$, $[U]^k$ *and* $[D]^k$ *contain only nonnegative real matrices as elements.*

d) IGA$([M], [b]) \subseteq$ IGA$([M]) \cdot [b]$
with

$$\text{IGA}([M]) := [D]^1 ([U]^1 ([D]^2 ([U]^2 (\ldots ([D]^{n-1} ([U]^{n-1} ([D]^n \cdot$$
$$\cdot ([L]^{n-1} ([L]^{n-2} (\ldots ([L]^2 [L]^1) \ldots) = [{}^-M^{-1}, {}_-M^{-1}].$$

Proof. a) is proved in [4, Satz 1].

b) is the well-known representation of the interval Gaussian algorithm due to Schwandt (cf. [1] or [19, p. 32]).

c) follows at once by b) and the proof of Satz 1 in [4].

d) follows from b), c) and the inclusion

(9) $[A]([B][b]) \subseteq ([A][B])[b]$

which holds for $n \times n$ interval matrices $[A]$, $[B]$ with ${}_-A, {}_-B \geqq 0$. □

Theorem 2. *Let* $D \subseteq \mathbb{R}^n$ *be a region and let* $F\colon D \to \mathbb{R}^n$ *be a mapping with continuous partial derivatives and with a solution* $z \in [x]^0 \subseteq D$ *of* (1).

(10) *Let* $[\hat{A}]$, $[\hat{M}]$ *and* $[\hat{N}]$ *fulfil* (8) *and let the matrices* $[M]^k$, $[N]^k$ *of* (6) *be contained in* $[\hat{M}]$ *and* $[\hat{N}]$, *respectively.*

Then each of the following conditions guarantees the convergence of the iterates

$[x]^k$ *of* (6) *towards the unique solution of* (1):

α) $[\hat{M}] - [\hat{N}]$ *is a triangular splitting of* $[\hat{A}]$.

β) $\varrho({}_{-}\hat{M}^{-1}\,{}^{-}\hat{N} + ({}_{-}\hat{M}^{-1} - {}^{-}\hat{M}^{-1})\,|[\hat{A}]|) < 1$.

γ) $\varrho({}_{-}\hat{M}^{-1}\,{}^{-}\hat{N} + ({}_{-}\hat{M}^{-1} - {}^{-}\hat{M}^{-1})\,|[\hat{A}]|/2) < 1$ *and* $\tilde{x}^k := ({}_{-}x^k + {}^{-}x^k)/2$, $k = 0, 1, \ldots$.

δ) ${}_{-}\hat{M} = {}^{-}\hat{M}$.

Proof. Assumption (10) guarantees the existence of a subsequence $\{[x]^{k_l}\}$ of $\{[x]^k\}$ such that $[M]^{k_l}, [N]^{k_l}$ and $\{\tilde{x}^{k_l}\}$ converge to limits $[M] \subseteq [\hat{M}]$, $[N] \subseteq [\hat{N}]$ and $\tilde{x}$, respectively. Again by (10) the matrices $[M]$, $[N]$ and $[A] := [M] - [N]$ fulfil (8). In particular $[M] - [N]$ is a triangular splitting if the same holds true for $[\hat{M}] - [\hat{N}]$.

Now

(11) $$\tilde{x} \in [x]^* := \lim_{k\to\infty} [x]^k \subseteq [y]^{k_l,1} \cap [x]^{k_l} \subseteq [x]^{k_l}$$

by Theorem 1. With

(12) $$[y] := \mathrm{IGA}\,([M], [N]\,(\tilde{x} - [x]^*) + F(\tilde{x}))$$

and with $k_l \to \infty$ in (11) one easily gets

(13) $$[x]^* = (\tilde{x} - [y]) \cap [x]^* \subseteq \tilde{x} - [y]\,.$$

Hence

(14) $$\tilde{x} \in \tilde{x} - [y] \quad \text{and} \quad 0 \in [y]\,.$$

α) Split the triangular matrix $[M]$ into

$$[M] = [D] - [E]$$

with $[D]$ and $[E]$ being defined analogously to the matrices in (7). Since $[M]$ is an interval M matrix no diagonal entry contains zero [16, 2.4.8]. Therefore $[D]^{-1}$ exists, and by (12) one easily obtains

$$[y] = [D]^{-1}\,\{[E]\,[y] + [N]\,(\tilde{x} - [x]^*) + F(\tilde{x})\}\,.$$

By virtue of (14) this implies

(15) $$0 \in [E]\,[y] + [N]\,(\tilde{x} - [x]^*) + F(\tilde{x})\,.$$

Thus the inequality

$$d([x]^*) \leqq d([y]) = |[D]^{-1}|\,\{|[E]|\,d([y]) + |[N]|\,d([x]^*)\} \leqq$$
$$\leqq |[D]^{-1}|\,\{|[E]| + |[N]|\}\,d([y])$$

follows by (13) and (5). Hence

(16) $$({}_{-}M - {}^{-}N)\,d([y]) \leqq 0\,.$$

Since ${}_-A = {}_-M - {}^-N$ is an M matrix its inverse is nonnegative, and (13) and (16) finally yield $d([x]^*) = d([y]) = 0$. The assertion is now proved by Theorem 1c).

β) γ) By (13), (12), Lemma 1d), (9) and (5) one gets

$$(17) \qquad d([x]^*) \leqq d([y]) \leqq d(\mathrm{IGA}([M])\{[N](\tilde{x} - [x]^*) + F(\tilde{x})\}) \leqq \\ \leqq d(\{\mathrm{IGA}([M])[N]\}(\tilde{x} - [x]^*)) + d(\mathrm{IGA}([M])F(\tilde{x})) = \\ = |\mathrm{IGA}([M])[N]|\, d([x]^*) + d(\mathrm{IGA}([M]))\,|F(\tilde{x})|\,.$$

Now (2) and (10) imply

$$F(\tilde{x}) \in F(z) + J(\tilde{x}, z)(\tilde{x} - z) = J(\tilde{x}, z)(\tilde{x} - z) \in [\hat{A}](\tilde{x} - z)\,.$$

Again by Lemma 1d) one thus obtains

$$(18) \qquad d([x]^*) \leqq {}_-M^{-1}\,{}^-N\, d([x]^*) + \{{}_-M^{-1} - {}^-M^{-1}\}\,|[\hat{A}]|\,.\,|\tilde{x} - z| \leqq \\ \leqq C\, d([x]^*)$$

with $0 \leqq C := {}_-\hat{M}^{-1}\,{}^-\hat{N} + ({}_-\hat{M}^{-1} - {}^-\hat{M}^{-1})\,|[\hat{A}]|$ for case β) and $0 \leqq C := := {}_-\hat{M}^{-1}\,{}^-\hat{N} + ({}_-\hat{M}^{-1} - {}^-\hat{M}^{-1})\,|[\hat{A}]|/2$ for case γ). Applying (18) iteratively we arrive at $d([x]^*) \leqq C^m\, d([x]^*)$, $m = 0, 1, \ldots$. Since $\varrho(C) < 1$ by assumption, $\lim_{m \to \infty} C^m = 0$, hence $d([x]^*) = 0$, and the proof is completed as in α).

δ) follows by β) due to the fact that ${}_-\hat{M} - {}^-\hat{N}$ is a regular splitting [16, 2.4.15] of the M matrix ${}_-\hat{A}$; thus $\varrho({}_-\hat{M}^{-1}\,{}^-\hat{N}) < 1$ by [16, 2.4.17]. □

We remark that according to Theorems 1 and 2a) the Newton-like total step method and the Newton-like single step method converge towards the zero of F if $[\hat{A}] = [A]^0$ is an M matrix.

In the cases of Newton's method and the simplified Newton's method the conditions β) and γ) of Theorem 2 are just the criteria $\varrho(B) < 1$ and $\varrho(B) < 2$, respectively, of [1, p. 368], where $B := \{({}_-\hat{M}^0)^{-1} - ({}^-\hat{M}^0)^{-1}\}\,|[\hat{M}]^0|$ and $[\hat{M}]^0 = = [{}_-\hat{M}^0, {}^-\hat{M}^0] := F'([x]^0)$.

Conditions β) and γ) guarantee the contraction of $\{[x]^k\}$ towards z at least if $d([\hat{M}])$ is sufficiently small; in this case ${}_-\hat{M}^{-1} - {}^-\hat{M}^{-1}$ will be small and β and γ) will hold by the same arguments as in the proof of Theorem 2δ) and by the continuity of the spectral radius.

In Theorem 2 we have assumed a zero of F to exist. In Theorem 1 we stated that $[x]^0$ contains certainly no zero of F is the method (6) breaks down by empty intersection. In the following theorem we show that the converse is true, too.

Theorem 3. *Let the assumptions of Theorem* 1 *be true and let* (10) *hold with* $[\hat{M}]$ *being lower triangular. Then the method* (6) *breaks down by empty intersection iff the starting vector* $[x]^0$ *contains no zero of* F.

Proof. If (6) breaks down then $[x]^0$ contains no zero by Theorem 1d). If (6) does not break down then the sequence $\{[x]^k\}$ of iterates converges to some interval vector $[x]^*$. In the same way as in the proof of Theorem 2α) one gets $d([x]^*) = = d([y]) = 0$ and $[y] = 0$ ($[y]$ as in (12)). Hence $F(\tilde{x}) = 0$ by (15) with $\tilde{x} \in [x]^* \subseteq \subseteq [x]^0$. □

Our final theorem concerns the speed of convergence of the method (6).

Theorem 4. *Let the assumptions of Theorem 2 be true and let one of the conditions* α) – δ) *of this theorem hold. Furthermore let the inequality*

$$d([A]^k) \leqq \|d([x]^k)\|_0 \cdot B\,, \quad k = 0, 1, \ldots \tag{19}$$

be satisfied for some vector norm $\|\cdot\|_0$ *and some* $n \times n$ *matrix* B. *We consider here only sequences* $\{[x]^k\}$ *of iterates of which the starting vector* $[x]^0 \subseteq D$ *contains the zero* z *of* F.

(20) *We assume that* B *and the interval matrices* $[\hat{M}]$ *and* $[\hat{N}]$ *of* (10) *can be chosen independently of these sequences.*

Then there is a monotone vector norm $\|\cdot\|$ (cf. [16, 2.4.2]) *which is independent of* $[x]^0$ *such that the following assertions hold.*

a) $\|d([x]^{k+1})\| \leqq \alpha_k \|d([x]^k)\|$, $k = 0, 1, \ldots$, *with* $\alpha_k \leqq \alpha < 1$ *for some constant* α *and for* k *sufficiently large.*

b) *If* $\lim_{k\to\infty} r_k = \infty$ *then*

$$\|d([x]^{k+1})\| \leqq \alpha_k \|d([x]^k)\|\,, \quad k = 0, 1, \ldots,$$

with $\lim_{k\to\infty} \alpha_k = 0$.

c) *If* $[N] = 0$ *then*

$$\|d([x]^{k+1})\| \leqq \beta \|d([x]^k)\|^2\,, \quad k = 0, 1, \ldots,$$

with some constant β *which is independent of* $[x]^0$.

Proof. a) We adopt the notation of (6) and Theorem 2, set $\tilde{x}^{k,0} := \tilde{x}^k$ and choose arbitrary vectors $\tilde{x}^{k,m} \in [x]^{k,m}$, $m = 1, \ldots, r_k - 1$. Analogously to (17) and (18) one gets

$$\begin{aligned} d([x]^{k,m}) &\leqq d([y]^{k,m}) \leqq \\ &\leqq d(\mathrm{IGA}\,([M]^k)\{[N]^k(\tilde{x}^k - \tilde{x}^{k,m-1} + \tilde{x}^{k,m-1} - [x]^{k,m-1}) + F(\tilde{x}^k)\}) \leqq \\ &\leqq d(\mathrm{IGA}\,([M]^k)\,[N]^k)\,|\tilde{x}^k - \tilde{x}^{k,m-1}| + |\mathrm{IGA}\,([M]^k)|\,|[N]^k|\,d([x]^{k,m-1}) + \\ &\quad + d(\mathrm{IGA}\,([M]^k))\,|[A]^k|\,d([x]^k) \leqq \\ &\leqq \{d(\mathrm{IGA}\,([M]^k))\,|[N]^k| + |\mathrm{IGA}\,([M]^k)|\,d([N]^k)\}\,d([x]^k) + \\ &\quad + ({}_{-}M^k)^{-1}\,{}^{-}N^k\,d([x]^{k,m-1}) + d(\mathrm{IGA}\,([M]^k))\,|[A]^k|\,d([x]^k)\,. \end{aligned} \tag{21}$$

Applying (21) iteratively r_k times, one obtains

(22) $$d([x]^{k+1}) = d([x]^{k,r_k}) \leqq \{((\_M^k)^{-1}\,{}^-N^k)^{r_k} + \sum_{j=0}^{r_k-1}((\_M^k)^{-1}\,{}^-N^k)^j\,.$$
$$.\,\{d(\mathrm{IGA}([M]^k))\,.\,(|[N]^k| + |[A]^k|) + (\_M^k)^{-1}\,d([N]^k)\}\}\,d([x]^k) \leqq$$
$$\leqq \{((\_M^k)^{-1}\,{}^-N^k)^{r_k} + (I - (\_M^k)^{-1}\,{}^-N^k)^{-1}\,\{d(\mathrm{IGA}([M]^k))\,.$$
$$.\,(|[N]^k| + |[A]^k|) + (\_M^k)^{-1}\,d([N]^k)\}\}\,d([x]^k)\,.$$

Since $d([A]^k) = d([M]^k) + d([N]^k)$, the inequality (19) also holds with $[M]^k$ and $[N^k]$, respectively, instead of $[A]^k$. Especially, Lemma 1d) yields

$$d(\mathrm{IGA}([M]^k)) = (\_M^k)^{-1} - ({}^-M^k)^{-1} = (\_M^k)^{-1}\,({}^-M^k - \_M^k)\,({}^-M^k)^{-1} \leqq$$
$$\leqq \|d([x]^k)\|_0\,(\_M^k)^{-1}\,B({}^-M^k)^{-1} \leqq \|d([x]^k)\|_0\,\_\hat{M}^{-1}B\,\_\hat{M}^{-1}\,.$$

Thus by (22) and (20) one obtains for any monotone vector norm $\|\cdot\|$ with associated matrix norm

(23) $$\|d([x]^{k+1})\| \leqq \{\|\_\hat{M}^{-1}\,{}^-\hat{N}\|^{r_k} + \hat{\beta}\|d([x]^k)\|_0\}\,\|d([x]^k)\|$$

with

$$\hat{\beta} := \|(I - \_\hat{M}^{-1}\,{}^-\hat{N})^{-1}\|\,\{\|\_\hat{M}^{-1}B\,\_\hat{M}^{-1}\|\,.\,\|{}^-\hat{N} + |[\hat{A}]|\| + \|\_\hat{M}^{-1}B\|\}\,.$$

As in the proof of Theorem 2δ) one has $\varrho := \varrho(\_\hat{M}^{-1}\,{}^-\hat{N}) < 1$. Choose ε such that $\alpha := \varrho + 2\varepsilon < 1$. According to [2, p. 154] there exists a monotone vector norm $\|\cdot\|$ satisfying $\|\_\hat{M}^{-1}\,{}^-\hat{N}\| < \varrho + \varepsilon$. From now on we use only this special monotone norm. Since $\lim_{k\to\infty} d([x]^k) = 0$ there exists an index $k_0 = k_0(\varepsilon, \{[x]^k\})$ such that

(24) $$\alpha_k := \|\_\hat{M}^{-1}\,{}^-\hat{N}\|^{r_k} + \hat{\beta}\|d([x]^k)\|_0 < (\varrho + \varepsilon)^{r_k} + \varepsilon \leqq \alpha < 1$$

for $k \geqq k_0$. Now (23) yields the assertion.

b) follows by (23), (24) and $r_k \to \infty$.

c) is a direct consequence of (23) if $[\hat{N}] = 0$ and the equivalence of norms in $\mathbb{R}^m$ are is taken into account. □

Under the assumptions of Theorem 4 the method (6) converges at least linearly, in the case of $\lim_{k\to\infty} r_k = \infty$ it converges superlinearly (cf. [16, 9.2]). If $[N] = 0$ which is true for Newton's method one has quadratic convergence in the sense of [15, p. 149]. Theorem 4 contains Theorem 2 of [2, p. 278] if one assumes there $[\hat{A}]$ to be an interval M matrix. Part c) coincides essentially with Satz 3.1.10 in [19, p. 99].

5. EXAMPLES

We illustrate the results of Section 4 by some examples. They have all been computed on a KWS SAM 68 K computer using the programming language PASCAL SC, an extension of standard PASCAL ([10], [18]). This extension affects the computing according to the machine interval arithmetic described, e.g., in [2, § 4] or [9, § 4].

Example 1. Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected region, and let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $g_u(s, t, u) \geqq 0$ for $(s, t, u) \in \Omega \times \mathbb{R}$.

We start with the following Dirichlet problem

$$\Delta u := \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = g(s, t, u), (s, t) \in \Omega, \tag{25}$$
$$u(s, t) = r(s, t), \quad (s, t) \in \partial\Omega$$

where $r: \partial\Omega \to \mathbb{R}$ is defined on the boundary $\partial\Omega$ of Ω. For the sake of simplicity we assume $\Omega = (0, 1) \times (0, 1)$. Under mild conditions on r problem (25) has a unique solution (cf. [16, 1.2], [6, p. 784]). In order to determine numerical approximations for this solution we discretize (25) using an equidistant grid with mesh size $h = 1/(n + 1)$ in each direction. Replacing u by the usual five point formula leads to the system of n^2 nonlinear equations

$$F(x) := Ax + d(x) - b = 0.$$

Here the components x_i of $x \in \mathbb{R}^{n^2}$ are the approximate values of the exact solution u of (25) at the grid points (ph, qh) where

$$i = (p - 1)\, n + q, \quad p, q = 1, \ldots, n; \tag{26}$$

A is the well-known $n^2 \times n^2$ matrix due to the discretization of the Laplacian operator [16, 1.2 (7), (8)], and $d(x) = (d_i(x)) \in \mathbb{R}^{n^2}$ is defined by

$$d_i(x) = h^2 g(ph, qh, x_i)$$

where again p and q are related to i by (26); $b \in \mathbb{R}^{n^2}$ is the constant vector containing the boundary values. Let D be a diagonal matrix with diagonal entries $h^2 g_u(ph, qh, x_i) \geqq 0$, and let us assume (26). Then the Jacobian F' of F can be represented by $F' = A + D$ and is thus an M matrix [16, 2.4.14]. In particular, $F'([x]^0) := [\inf\{F'(x) \mid x \in [x]^0\}, \sup\{F'(x) \mid x \in [x]^0\}]$ is an interval M matrix for all interval vectors $[x]^0 \in I\mathbb{R}^{n^2}$. Here infimum and supermum are taken entrywise.

To get an initial enclosure $[x]^0 \in I\mathbb{R}^n$ for the zero z of F it is important to note that F is an M function in the sense of [16, 13.5.3, 13.5.6, 13.5.7]. Thus F is inverse isotone and if one knows two vectors ${}_{-}x^0$, ${}^{-}x^0 \in \mathbb{R}^{n^2}$ satisfying $F({}_{-}x^0) \leqq 0 = F(z) \leqq F({}^{-}x^0)$ then $z \in [{}_{-}x^0, {}^{-}x^0]$. Such vectors ${}_{-}x^0$, ${}^{-}x^0$ can be found constructively (cf. [16, 13.4.6 (c)]). The numerical results to follow have been derived for the special case of the radiation equation, i.e., $g(s, t, u) := e^u$, $(s, t) \in \Omega = (0, 1) \times (0, 1)$ (cf. [8, p. 107]). Furthermore, set $r(s, t) = 0$ for $(s, t) \in \partial\Omega$. Denoting by $v_e \in \mathbb{R}^{n^2}$ the real vector of which all components are equal to one we chose $n = 5$, $r_1 = 1$, $r_{k+1} = r_k + 1$, $[x]^0 = [-1, 0] \cdot v_e \in I\mathbb{R}^{25}$, $\tilde{x}^k = ({}_{-}x^k + {}^{-}x^k)/2$, $[A]^k = F'([x]^k) := [F'({}_{-}x^k), F'({}^{-}x^k)]$. We stopped the iteration whenever the inequality $d([x]^k) < 10^{-10} \cdot v_e$ was fulfilled. The following tables show the enclosures $[x]^k_{13}$ of the approximation of $u(0{\cdot}5, 0{\cdot}5)$ for different splittings.

Table 1a: Method (6) with interval Gaussian algorithm. ($[M^k] = [A]^k$)

k	$[x]^k_{13}$
0	[−1·000000000000E+00, 0·000000000000E+00]
1	[−7·554343472580E−02, −5·585029604050E−02]
2	[−6·837287950662E−02, −6·837063076031E−02]
3	[−6·837191347055E−02, −6·837191347050E−02]

Table 1b: Method (6) with $[M]^k$ being a lower Hessenberg matrix. ($[m]^k_{ij} = 0$ if $j > i + 1$; $[m]^k_{ij} = [a]^k_{ij}$ otherwise)

k	$[x]^k_{13}$
0	[−1·000000000000E+00, 0·000000000000E+00]
1	[−8·183494718785E−01, −1·758101719710E−02]
2	[−4·123657187960E−01, −4·448757007530E−02]
3	[−1·337551605389E−01, −6·250496861940E−02]
4	[−7·529605905361E−02, −6·769620743821E−02]
5	[−6·878854112493E−02, −6·833069692459E−02]
6	[−6·838619416489E−02, −6·837049893331E−02]
7	[−6·837219244189E−02, −6·837188583615E−02]
8	[−6·837191657653E−02, −6·837191316283E−02]
9	[−6·837191349026E−02, −6·837191346856E−02]

Table 1c: Method (6) with Gauss-Seidel splitting. ($[m]^k_{ij} = 0$ if $j > i$; $[m]^k_{ij} = [a]^k_{ij}$ otherwise)

k	$[x]^k_{13}$
0	[−1·000000000000E+00, 0·000000000000E+00]
1	[−9·396739219836E−01, −1·020441822320E−02]
2	[−7·176171701259E−01, −2·862273555180E−02]
3	[−3·497480671343E−01, −4·870645472820E−02]
4	[−1·532587240125E−01, −6·173372639350E−02]
5	[−8·726681388736E−02, −6·682425850797E−02]
6	[−7·148164389479E−02, −6·811373109524E−02]
7	[−6·875062318619E−02, −6·834038232352E−02]
8	[−6·840604989363E−02, −6·836907009522E−02]
9	[−6·837419112189E−02, −6·837172374657E−02]
10	[−6·837202596188E−02, −6·837190410017E−02]
11	[−6·837191758313E−02, −6·837191312794E−02]
12	[−6·837191358185E−02, −6·837191346124E−02]
13	[−6·837191347278E−02, −6·837191347032E−02]

Table 1d: Method (6) with tridiagonal matrix $[M]^k$. ($[m]^k_{ij} = [a]^k_{ij}$ if $|i-j| \leq 1$)

k	$[x]^k_{13}$
0	[−1·000000000000E+00, 0·000000000000E+00]
1	[−9·674933774774E−01, −1·050652446660E−02]
2	[−6·324595863319E−01, −2·958257971440E−02]
3	[−3·635030182095E−01, −4·895639717660E−02]
4	[−1·653399741726E−01, −6·142217013950E−02]
5	[−8·884683714901E−02, −6·659981329261E−02]
6	[−7·216417446688E−02, −6·804020775374E−02]
7	[−6·898890097368E−02, −6·832543750548E−02]
8	[−6·843701666856E−02, −6·836700724279E−02]
9	[−6·837636907854E−02, −6·837152248472E−02]
10	[−6·837218142219E−02, −6·837188995738E−02]
11	[−6·837192763015E−02, −6·837191240340E−02]
12	[−6·837191395587E−02, −6·837191343393E−02]
13	[−6·837191348134E−02, −6·837191346956E−02]

Table 1e: Method (6) with Jacobi splitting. [$(m]^k_{ij} = 0$ if $j \neq i$; $[m]^k_{ij} = [a]^k_{ij}$ otherwise)

k	$[x]^k_{13}$
0	[−1·000000000000E+00, 0·000000000000E+00]
1	[−1·000000000000E+00, −5·475389562400E−03]
2	[−9·445425381095E−01, −1·637533865160E−02]
3	[−6·692129920699E−01, −3·128744075070E−02]
4	[−4·019454115632E−01, −4·617451627540E−02]
5	[−2·244661113557E−01, −5·724550566680E−02]
6	[−1·317794405294E−01, −6·369210813702E−02]
7	[−9·075306489820E−02, −6·670937323925E−02]
8	[−7·509331412084E−02, −6·786904930303E−02]
9	[−7·008972629320E−02, −6·824167420502E−02]
10	[−6·875368492206E−02, −6·834294763968E−02]
11	[−6·844569771728E−02, −6·836637454852E−02]
12	[−6·838405136348E−02, −6·837100225416E−02]
13	[−6·837361306062E−02, −6·837178448959E−02]
14	[−6·837212043056E−02, −6·837189776442E−02]
15	[−6·837193538700E−02, −6·837191182519E−02]
16	[−6·837191544604E−02, −6·837191332220E−02]
17	[−6·837191362212E−02, −6·837191345900E−02]
18	[−6·837191348067E−02, −6·837191346973E−02]

For the method (6) with Gauss-Seidel splitting and with Jacobi splitting, Theorem 2α) guarantees the convergence towards the zero z. For the other splittings the tables indicate that condition β) of Theorem 2 is fulfilled if k is sufficiently large. Thus convergence is guaranteed a posteriori.

The starting point of our second example is the so called H-equation, a nonlinear integral equation occurring in connection with radiative transfer (cf. [5], [16, p. 18]).

Example 2.

$$(26) \qquad u(s) = (Tu)(s) := 1 + \lambda\, u(s) \int_0^1 (u(t)\, s/(s+t))\, dt\,, \quad \lambda \in [0, 1/4]\,.$$

One sees at once that $u(0) = 1$ for each continuous solution of (26). Let C be the space of all continuous functions u being defined on $[0, 1]$ with values out of the interval $[1, 2]$. Then Banach's fixed point theorem shows that (26) has a unique solution in C. (Equip C with the maximum norm and take

$$1 \leqq Tu_1 \leqq Tu \leqq Tu_2 \leqq 2$$

for $u_1 = 1$, $u_2 = 2$, $u \in C$ into account. Cf. [17, p. 74].) Discretizing (26) using any quadrature formula of the form

$$\int_a^b f(t)\, dt \approx \sum_{j=0}^{n} w_j f(t_j)$$

with $a \leqq t_0 < t_1 < \ldots < t_n \leqq b$, $\sum_{j=0}^{n} w_j = 1$, $w_j \geqq 0$ yields the system of $(n+1)$ nonlinear equations

$$(27) \qquad x_i = g_i(x_0, \ldots, x_n) := 1 + \lambda x_i \sum_{j=0}^{n} (w_j x_j t_i/(t_i + t_j))\,, \quad i = 0, 1, \ldots, n\,,$$

where x_i is an approximation of $u(t_i)$.

If $t_0 = 0$ we set $x_0 = 1$ and begin (27) with $i = 1$. Without loss of generality we assume $t_0 > 0$ in the sequel. Using the vector v_e from Example 1 and taking $\lambda \in [0, 1/4]$ into account one easily sees that the right hand side of (27) maps $[x]^0 := [1, 2]\, v_e \in I\!R^{n+1}$ into itself. Brouwer's fixed point theorem [16, 6.3.2] therefore guarantees a solution z of (27) which lies in $[x]^0$. This solution is a zero of the function $F = (F_i): \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F_i(x_0, \ldots, x_n) := x_i - g_i(x_0, \ldots, x_n)$. The elements of the Jacobian F' of F can be represented by

$$\frac{\partial F_i}{\partial x_k} = \begin{cases} 1 - \lambda\{\sum_{j=0}^{n} (w_j x_j t_i/(t_i + t_j)) + x_i w_i/2\} & \text{if } k = i \\ -\lambda x_i w_k t_i/(t_i + t_k) & \text{if } k \neq i\,. \end{cases}$$

For each vector $x = (x_i) \in [x]^0$ we therefore get

$$1 - \lambda\{\sum_{j=0}^{n} (w_j x_j t_i/(t_i + t_j)) + x_i w_i/2\} - \lambda x_i \sum_{\substack{k=0 \\ k \neq i}}^{n} (w_k t_i/(t_i + t_k)) =$$

$$= 1 - \lambda\{ \sum_{j=0}^{n} (w_j x_j t_i/(t_i + t_j) + x_i \sum_{k=0}^{n} (w_k t_i/(t_i + t_k))\} >$$

$$> 1 - \{ \sum_{j=0}^{n} w_j + \sum_{k=0}^{n} w_k\}/2 = 0\,, \quad i = 0, \ldots, n\,.$$

This shows that $F'(x)$ is diagonally dominant for each $x \in [x]^0$. Thus it is an M matrix (cf. [16], 2.4.14]), which is also remarked in [7]. Hence $F'([x]^0) := [F'(2v_e), F'(v_e)]$ is an interval M matrix, and Theorem 1b) guarantees uniqueness of z in $[x]^0$.

To enclose z we applied the method (6) with several splittings — see Tables 2a–f. We used the composed trapezoidal rule with 65 support abscissas which were equally spaced. Furthermore we chose $\lambda = 0{\cdot}25$, $[x]^0 = [1, 2]\, v_e \in I\mathbb{R}^{65}$ and $[A]^k$, $[M]^k - [N]^k$, $\tilde{x}^k$, r^k and the stopping criterion as in Example 1. Tables 2a–2e show enclosures of the approximation x_{64} of $u(1)$. Table 5f lists enclosures of the approximations $x_{8.i}$ of $u(i \,.\, 0{\cdot}125)$, $i = 0, 1, \ldots, 8$, when iterating four times using (6) with the Gauss-Seidel splitting. This table confirms the results in [3, p. 697] and in [7].

Table 2a: Method (6) with interval Gaussian algorithm. ($[M]^k = [A]^k$)

k	$[x]^k_{64}$
0	[1·000000000000E+00, 2·000000000000E+00]
1	[1·100013617153E+00, 1·301760354995E+00]
2	[1·249283180501E+00, 1·251518823147E+00]
3	[1·251259395815E+00, 1·251259664593E+00]
4	[1·251259545112E+00, 1·251259545114E+00]

Table 2b: Method (6) with Gauss-Seidel splitting after renumbering the equations and the unknowns. ($[m]^k_{ij} = [a]^k_{ij}$ if $i \leq j$, $[m]^k_{ij} = 0$ otherwise)

k	$[x]^k_{64}$
0	[1·000000000000E+00, 2·000000000000E+00]
1	[1·068543325523E+00, 1·589988154754E+00]
2	[1·241930893813E+00, 1·258084754789E+00]
3	[1·251256026409E+00, 1·251262389424E+00]
4	[1·251259545090E+00, 1·251259545132E+00]

Table 2c: Method (6) with tridiagonal matrix $[M]^k$. ($[m]^k_{ij} = [a]^k_{ij}$ if $|i-j| \leqq 1$)

k	$[x]^k_{64}$
0	[1·000000000000E+00, 2·000000000000E+00]
1	[1·068958556406E+00, 1·587463429936E+00]
2	[1·236739405516E+00, 1·263264263363E+00]
3	[1·251218169305E+00, 1·251298419806E+00]
4	[1·251259527012E+00, 1·251259562010E+00]
5	[1·251259545111E+00, 1·251259545115E+00]

Table 2d: Method (6) with Jacobi splitting. ($[m]^k_{ij} = 0$ if $j \neq i$; $[m]^k_{ij} = [a]^k_{ij}$ otherwise)

k	$[x]^k_{64}$
0	[1·000000000000E+00, 2·000000000000E+00]
1	[1·068543325523E+00, 1·589988154754E+00]
2	[1·235933119494E+00, 1·264046609821E+00]
3	[1·251211568200E+00, 1·251305034077E+00]
4	[1·251259521253E+00, 1·251259567589E+00]
5	[1·251259545110E+00, 1·251259545115E+00]

Table 2e: Method (6) with Gauss-Seidel splitting. ($[m]^k_{ij} = 0$ if $j > i$; $[m]^k_{ij} = [a]^k_{ij}$ otherwise)

k	$[x]^k_{64}$
0	[1·000000000000E+00, 2·000000000000E+00]
1	[1·094692712518E+00, 1·314535563627E+00]
2	[1·249592739429E+00, 1·252832234667E+00]
3	[1·251259447758E+00, 1·251259637024E+00]
4	[1·251259545112E+00, 1·251259545114E+00]

Table 2f: Enclosure of the solution of the discretized problem applying 4 iterations of method (6) using the Gauss-Seidel splitting.

X[0] = 1
X[8] = [1·084121858890E+00, 1·084121858894E+00]
X[16] = [1·129671974950E+00, 1·129671974956E+00]
X[24] = [1·162426375007E+00, 1·162426375012E+00]
X[32] = [1·187741899436E+00, 1·187741899440E+00]
X[40] = [1·208106868712E+00, 1·208106868715E+00]
X[48] = [1·224934973221E+00, 1·224934973224E+00]
X[56] = [1·239118548540E+00, 1·239118548543E+00]
X[64] = [1·251259545112E+00, 1·251259545114E+00]

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Souhrn

NEWTONOVSKÉ METODY K ODHADU INTERVALU ŘEŠENÍ NELINEÁRNÍCH ROVNIC

GÜNTER MAYER

Je popsána třída Newtonovských metod k odhadu intervalu řešení nelineárních rovnic. Jsou dokázány věty týkající se vhodnosti metody, její globální konvergence, rychlosti a kvality odhadu.

Резюме

МЕТОДЫ НЬЮТОНА ДЛЯ ОЦЕНКИ ИНТЕРВАЛА РЕШЕНИЙ НЕЛИНЕЙНЫХ УРАВНЕНИЙ

GÜNTER MAYER

В статье описан класс методов Ньютона для оценки интервала решений нелинейных уравнений. Доказаны теоремы об их удобности, сходимости в целом, скорости и качестве оценки.

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