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# STABILITY AND BOUNDEDNESS OF CONTROLLABLE CONTINUOUS FLOWS

FRANTIŠEK TUMAJER

*Dedicated to the memory of my father František Tumajer*

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***Summary.*** In the paper the concept of a controllable continuous flow in a metric space is introduced as a generalization of a controllable system of differential equations in a Banach space, and various kinds of stability and of boundedness of this flow are defined. Theorems stating necessary and sufficient conditions for particular kinds of stability and boundedness are formulated in terms of Ljapunov functions.

*Keywords:* controllable continuous flow, uniform stability of a set with respect to a flow, uniform boundedness of a flow with respect to a set, uniform boundedness of a flow.

*AMS Classification:* 34D20

Notation. In the paper we introduce the concept of a controllable continuous flow which is a generalization of a controllable system of differential equations in a Banach space. Before defining a controllable flow, we introduce the necessary notation. By the symbol $P$ we denote a metric space with a metric $\varrho$, by $R$ the set of all real numbers with the Euclidean metric, by $R^+$ the set of all positive numbers and by $U$ a non-empty set. By the symbol $t$ we denote a continuous flow on $P$ over $R$, i.e. a mapping $t\colon D = \{(\sigma, x, \alpha) \in R \times P \times R\colon \sigma \geqq \alpha\} \to P$ which has the following properties (we define ${}_\sigma t_\alpha x = t(\sigma, x, \alpha)$):

(i) $(x, \alpha) \in P \times R \Rightarrow {}_\alpha t_\alpha x = x$;
(ii) ${}_\gamma t_\beta \circ {}_\beta t_\alpha x = {}_\gamma t_\alpha x$ for all $\gamma \geqq \beta \geqq \alpha$ in $R$ and for any $x \in P$;
(iii) for any $(x, \alpha) \in P \times R$ the mapping $t(\cdot, x, \alpha)\colon \langle \alpha, +\infty) \to P$ is continuous.

**Definition.** We say that $\{t^u\colon u \in U\}$ is a *controllable continuous flow* on $P$ over $R \times U$ if and only if

(iv) for any $u \in U$ the mapping $t^u$ is a continuous flow on $P$ over $R$;
(v) $U$ is such a non-empty set that for any $\alpha < \beta < \gamma \in R$, $(x, u) \in P \times U$, $y = {}_\beta t^u_\alpha x$, $(z, v) \in P \times U$ for which $z = {}_\gamma t^v_\beta y$ there exists at least one element $w \in U$ such that $z = {}_\gamma t^w_\alpha x$.

The basic interpretation of the above definition is described in the following example:

Let $U$ be the set of piecewise continuous mappings of the set $R$ into a Banach space $B_1$, let $f: B \times R \times B_1 \to B$ be such a mapping into a Banach space $B$ that for each $u \in U$ the mapping $f^u: B \times R \to B$ defined by the rule $f^u(z, \sigma) = f(z, \sigma, u(\sigma))$ is locally Lipschitzian and for any piecewise continuous mapping $z: R \to B$ the mapping $f^u_z: R \to B$ determined by $f^u_z(\sigma) = f(z(\sigma), \sigma, u(\sigma))$ is piecewise continuous. The solutions of the differential equation $z' = f^u(z, \sigma)$ are continuous mapping $z^u: R \to B$, i.e., in Carathéodory's sense we have $\mathrm{d}z^u(\sigma)/\mathrm{d}\sigma = f(z^u(\sigma), \sigma, u(\sigma))$ for almost all $\sigma \in R$. The set of the differential equations of the form

$$z' = f^u(z, \sigma), \quad u \in U,$$

is called a controllable system. To this controllable system we assign a controllable continuous flow $\{t^u: u \in U\}$ on $B$ over $R \times U$ in such a way that we define $y = {}_\beta t^u_\alpha x$ if and only if $x \in B$, $y \in B$, $\alpha \leqq \beta \in R$, $u \in U$ and there exists a solution $z^u$ of the controllable system of differential equations for which $z^u(\alpha) = x$ and $z^u(\beta) = y$.

In the sequel we suppose that a continuous mapping $s: R \to P$ and a set $m = \{(s(\alpha), \alpha) \in P \times R: \alpha \in R\}$ are given.

**Definition.** The set $m$ is *uniformly stable with respect to the controllable continuous flow* $\{t^u: u \in U\}$ if and only if there exists a mapping $z_1: R^+ \to R^+$ such that

$$(1) \qquad \varrho(x, s(\alpha)) \leqq z_1(r), \quad u \in U \Rightarrow \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq r.$$

The controllable continuous flow $\{t^u: u \in U\}$ is *uniformly bounded with respect to the set* $m$, if and only if there exists a mapping $z_1: R^+ \to R^+$ such that

$$(2) \qquad \varrho(x, s(\alpha)) \leqq r, \quad u \in U \Rightarrow \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq z_1(r).$$

The controllable continuous flow $\{t^u: u \in U\}$ is *uniformly bounded*, if and only if there exists a mapping $z_1: R^+ \to R^+$ such that

$$(3) \qquad (\sigma, x_1, \alpha) \in D, \quad (\sigma, x_2, \alpha) \in D, \quad \varrho(x_1, x_2) \leqq r,$$
$$(u_1, u_2) \in U \times U \Rightarrow \varrho({}_\sigma t^{u_1}_\alpha x_1, {}_\sigma t^{u_2}_\alpha x_2) \leqq z_1(r).$$

The set $m$ is *uniformly asymptotically stable with respect to the controllable continuous* flow $\{t^u: u \in U\}$, if and only if it is uniformly stable with respect to $\{t^u: u \in U\}$ and there exist a constant $r_1 \in R^+$ and a mapping $z_2: R^+ \to R^+$ such that

$$(4) \qquad \varrho(x, s(\alpha)) \leqq r_1, \quad u \in U, \quad \sigma \geqq \alpha + z_2(r) \Rightarrow \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq r.$$

The controllable continuous flow $\{t^u: u \in U\}$ is *uniformly asymptotically bounded with respect to the set* $m$, if and only if it is uniformly bounded with respect to $m$ and there exist a constant $r_1 \in R^+$ and a mapping $z_2: R^+ \to R^+$ such that

$$(5) \qquad \varrho(x, s(\alpha)) \leqq r, \quad u \in U, \quad \sigma \geqq \alpha + z_2(r) \Rightarrow \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq r_1.$$

The controllable continuous flow $\{t^u: u \in U\}$ is *uniformly asymptotically bounded*, if and only if it is uniformly bounded and there exist a constant $r_1 \in R^+$ and a mapping $z_2: R^+ \to R^+$ such that

(6) $$(\sigma, x_1, \alpha) \in D\,, \quad (\sigma, x_2, \alpha) \in D\,, \quad \varrho(x_1, x_2) \leqq r\,, \quad \sigma \geqq \alpha + z_2(r)\,,$$
$$(u_1, u_2) \in U \times U \Rightarrow \varrho({}_\sigma t_\alpha^{u_1} x_1, {}_\sigma t_\alpha^{u_2} x_2) \leqq r_1\,.$$

Remark. If the set $m$ is uniformly stable with respect to the controllable continuous flow $\{t^u: u \in U\}$, then for each $u \in U$ we have

$$(\sigma, s(\alpha), \alpha) \in D \Rightarrow {}_\sigma t_\alpha^u s(\alpha) = s(\sigma)\,.$$

Proof. Suppose that there exists $(\beta, s(\alpha), \alpha, u) \in D \times U$ such that ${}_\beta t_\alpha^u s(\alpha) \neq s(\beta)$, i.e. $\varrho({}_\beta t_\alpha^u s(\alpha), s(\beta)) = r_0 > 0$. For each $r \in (0, r_0)$ we have $0 = \varrho(s(\alpha), s(\alpha)) \leqq z_1(r)$, hence (1) implies the inequality $\varrho({}_\beta t_\alpha^u s(\alpha), s(\beta)) \leqq r$, which contradicts the assumption that $\varrho({}_\beta t_\alpha^u s(\alpha), s(\beta)) > r$.

**Theorem 1.** *The set $m$ is uniformly stable with respect to the controllable continuous flow $\{t^u: u \in U\}$, if and only if there exist a constant $\delta \in R^+$ and partial mappings $a: R^+ \to R^+$, $b: R^+ \to R^+$, $V: P \times R \to R^+$ with the following properties:*

(i) $$\text{domain } V = \{(x, \alpha) \in P \times R : 0 < \varrho(x, s(\alpha)) \leqq \delta\}$$

*and the implication*

$$(x, \alpha) \in \text{domain } V,\ u \in U,\ ({}_\beta t_\alpha^u x, \beta) \in \text{domain } V \Rightarrow V({}_\beta t_\alpha^u x, \beta) \leqq V(x, \alpha)$$

*holds whenever $V({}_\lambda t_\alpha^u x, \lambda)$ is defined for all $\lambda \in \langle \alpha, \beta \rangle$;*

(ii) *$a, b: (0, \delta\rangle \to R^+$ are increasing continuous mappings such that* $\lim_{r \to 0+} b(r) = 0$ *and the implication*

$$(x, \alpha) \in \text{domain } V \Rightarrow a(\varrho(x, s(\alpha))) \leqq V(x, \alpha) \leqq b(\varrho(x, s(\alpha)))$$

*holds.*

Proof. Let $m$ be uniformly stable with respect to $\{t^u: u \in U\}$. Without loss of generality we may suppose that the mapping $z_1$ from (1) is increasing and continuous with $\lim_{r \to 0+} z_1(r) = 0$. Choose $r_0 > 1$, $\delta \in z_1(R^+)$ and define a partial mapping $V: P \times R \to R^+$ by the rule

(7) $$V(x, \alpha) = \sup \{\varrho({}_\sigma t_\alpha^u x, s(\sigma)) \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} : (u, \sigma) \in U \times R, \sigma \geqq \alpha)\}$$

for $(x, \alpha) \in P \times R$ with $0 < \varrho(x, s(\alpha)) \leqq \delta$. (The factor $(1 + (\sigma - \alpha) r_0)/(1 + \sigma - \alpha)$ is used here only for the applications in the proofs of Theorems 4, 5, 6.) If $(x, \alpha) \in$ domain $V$, $u \in U$, $y = {}_\beta t_\alpha^u x$, $(y, \beta) \in$ domain $V$, then for each $v \in U$, $z = {}_\sigma t_\beta^v y$ there exists $w \in U$ such that $z = {}_\sigma t_\alpha^w x$. This and the inequality

$$\frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} \geqq \frac{1 + (\sigma - \beta) r_0}{1 + \sigma - \beta}$$

mply

$$V(y, \beta) = \sup \left\{ \varrho({}_\sigma t^v_\beta y, s(\sigma)) \frac{1 + (\sigma - \beta) r_0}{1 + \sigma - \beta} : (v, \sigma) \in U \times R, \sigma \geqq \beta \right\} =$$

$$= \sup \left\{ \varrho({}_\sigma t^v_\beta \circ {}_\beta t^u_\alpha x, s(\sigma)) \frac{1 + (\sigma - \beta) r_0}{1 + \sigma - \beta} : (v, \sigma) \in U \times R, \sigma \geqq \beta \right\} \leqq$$

$$\leqq \sup \left\{ \varrho({}_\sigma t^w_\alpha x, s(\sigma)) \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} : (w, \sigma) \in U \times R, \sigma \geqq \alpha \right\} = V(x, \alpha),$$

this shows that the mapping $V$ has the property (i).

If $\delta \geqq \varrho(x, s(\alpha)) > 0$, there exists $r \in R^+$ such that $\varrho(x, s(\alpha)) = z_1(r)$. Then (1) implies that for any $(u, \sigma) \in U \times R, \sigma \geqq \alpha$, the inequality $\varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq r = z_1^{-1}(\varrho(x, s(\alpha)))$ holds, where $z_1^{-1}$ is the inverse mapping to $z_1$. As $(1 + (\sigma - \alpha) r_0)/(1 + \sigma - \alpha) < r_0$, this implies

$$\varrho(x, s(\alpha)) \leqq V(x, \alpha) \leqq r_0 z_1^{-1}(\varrho(x, s(\alpha))),$$

which shows that the mappings $a, b$: $(0, \delta\rangle \to R^+$ defined by $a(r) = r$, $b(r) = r_0 z_1^{-1}(r)$ have the property (ii).

Let there exist partial mappings $a$, $b$, $V$ and a constant $\delta \in R^+$ with the properties (i) and (ii). Choose $\delta_0 \in R^+$, $\delta_0 < \delta$, define a mapping $z_1$ in such a way that $0 < b(z_1(r)) \leqq a(r)$ for $0 < r \leqq \delta_0$ and $z_1(r) = z_1(\delta_0)$ for $\delta_0 < r$ holds. Let us show that (1) holds. Suppose that there exist $0 < r \leqq \delta_0$, $(x, \alpha, u) \in P \times R \times U$, $\varrho(x, s(\alpha)) \leqq z_1(r)$ such that for some $\gamma \geqq \alpha$ we have $r < \varrho({}_\gamma t^u_\alpha x, s(\gamma))$. Denote $\beta_0 = \inf\{\beta \in R: \varrho({}_\beta t^u_\alpha x, s(\beta)) = r\}$. As the mapping $\varrho(\cdot\, t^u_\alpha x, s(\cdot))$: $\langle \alpha, +\infty) \to \langle 0, +\infty)$ is continuous, we may suppose that $\varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq \delta$ for all $\sigma \in \langle \alpha, \gamma \rangle$, where $\gamma > \beta_0$. Then

$$a(\varrho({}_\gamma t^u_\alpha x, s(\gamma))) \leqq V({}_\gamma t^u_\alpha x, \gamma) \leqq V(x, \alpha) \leqq b(\varrho(x, s(\alpha))) \leqq b(z_1(r)) \leqq a(r).$$

This implies $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) \leqq r$, which contradicts the inequality $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) > r$. The theorem is proved.

**Theorem 2.** *The controllable continuous flow $\{t^u: u \in U\}$ is uniformly bounded with respect to the set $m$, if and only if there exist $\delta \in R^+$ and partial mappings $a: R^+ \to R^+$, $b: R^+ \to R^+$, $V: P \times R \to R^+$ with the following properties:*

(i) domain $V = \{(x, \alpha) \in P \times R: \delta \leqq \varrho(x, s(\alpha))\}$

*and the implication*

$$(x, \alpha) \in \text{domain } V, \quad u \in U, \quad ({}_\beta t^u_\alpha, \beta) \in \text{domain } V \Rightarrow V({}_\beta t^u_\alpha x, \beta) \leqq V(x, \alpha)$$

*holds whenever $V({}_\lambda t^u_\alpha x, \lambda)$ is defined for all $\lambda \in \langle \alpha, \beta \rangle$;*

(ii) *$a, b$: $\langle \delta, +\infty) \to R^+$ are increasing continuous mappings such that* $\lim_{r \to +\infty} a(r) = +\infty$ *and the implication*

$$(x, \alpha) \in \text{domain } V \Rightarrow a(\varrho(x, s(\alpha))) \leqq V(x, \alpha) \leqq b(\varrho(x, s(\alpha)))$$

*holds.*

Proof. Let $\{t^u : u \in U\}$ be uniformly bounded with respect to $m$. Without loss of generality we may suppose that the mapping $z_1$ from (2) is increasing and continuous with $\lim_{r \to +\infty} z_1(r) = +\infty$. Choose $r_0 > 1$, $\delta \in R^+$ and define the partial mapping $V: P \times R \to R^+$ by the rule (7) for $(x, \alpha) \in P \times R$ with $\varrho(x, s(\alpha)) \geqq \delta$. According to the first part of the proof of Theorem 1 the mapping $V$ has the property (i). If $(x, \alpha) \in$ domain $V$ and $\varrho(x, s(\alpha)) \leqq r$, then from (2), (7) and the inequality

$$\frac{1 + (\sigma - \alpha)\, r_0}{1 + \sigma - \alpha} < r_0$$

we obtain

$$\varrho(x, s(\alpha)) \leqq \sup \left\{ \varrho({}_\sigma t_\alpha^u x, s(\sigma)) \frac{1 + (\sigma - \alpha)\, r_0}{1 + \sigma - \alpha} : (u, \sigma) \in U \times R, \sigma \geqq \alpha \right\} =$$
$$= V(x, \alpha) \leqq r_0\, z_1(r)\,,$$

which shows that the mappings $a, b$: $\langle \delta, +\infty) \to R^+$ defined by $a(r) = r$, $b(r) = r_0\, z_1(r)$ have the property (ii).

Let there exist partial mappings $a$, $b$, $V$ and a constant $\delta \in R^+$ with the properties (i) and (ii). Define the mapping $z_1$ in such a way that $a(z_1(r)) > b(r)$ for $r \in \langle \delta, +\infty)$ and $z_1(r) = z_1(\delta)$ for $r \in (0, \delta)$ hold, and show that (2) holds. Let $r \geqq \delta$ be given. Suppose that there exist $(x, \alpha, u) \in P \times R \times U$, $\varrho(x, s(\alpha)) \leqq r$, $\gamma > \alpha$ such that $\varrho({}_\gamma t_\alpha^u x, s(\gamma)) > z_1(r)$, If we denote $\gamma_2 = \inf \{\lambda \in \langle \alpha, \gamma \rangle : \varrho({}_\lambda t_\alpha^u x, s(\lambda)) = z_1(r)\}$ and $\gamma_1 = \sup \{\lambda \in \langle \alpha, \gamma_2 \rangle : \varrho({}_\lambda t_\alpha^u x, s(\lambda)) = r\}$, then $r \leqq \varrho({}_\lambda t_\alpha^u x, s(\lambda)) \leqq z_1(r)$ for all $\lambda \in \langle \gamma_1, \gamma_2 \rangle$. Then

$$a(\varrho({}_{\gamma_2} t_\alpha^u x, s(\gamma_2))) = a(z_1(r)) \leqq V({}_{\gamma_2} t_\alpha^u x, \gamma_2) \leqq V({}_{\gamma_1} t_\alpha^u x, \gamma_1) \leqq$$
$$\leqq b(\varrho({}_{\gamma_1} t_\alpha^u x, s(\gamma_1))) = b(r)\,,$$

which contradicts the inequality $a(z_1(r)) > b(r)$. The theorem is proved.

**Theorem 3.** *The controllable continuous flow $\{t^u : u \in U\}$ is uniformly bounded, if and only if there exist a constant $\delta \in R^+$ and partial mappings $a$: $R^+ \to R^+$, $b$: $R^+ \to R^+$, $V$: $P \times P \times R \to R^+$ with the following properties:*

(i) $\quad$ domain $V = \{(x_1, x_2, \alpha) \in P \times P \times R : \delta \leqq \varrho(x_1, x_2)\}$

*and the implication*

$$(x_1, x_2, \alpha) \in \text{domain } V, \quad (u_1, u_2) \in U \times U\,, \quad ({}_\beta t_\alpha^{u_1} x_1, {}_\beta t_\alpha^{u_2} x_2, \beta) \in \text{domain } V \Rightarrow$$
$$\Rightarrow V({}_\beta t_\alpha^{u_1} x_1, {}_\beta t_\alpha^{u_2} x_2, \beta) \leqq V(x_1, x_2, \alpha)$$

*holds whenever $V({}_\lambda t_\alpha^{u_1} x_1, {}_\lambda t_\alpha^{u_2} x_2, \lambda)$ is defined for all $\lambda \in \langle \alpha, \beta \}$;*

(ii) *$a, b$: $\langle \delta, +\infty) \to R^+$ are increasing continuous mappings such that $\lim_{r \to +\infty} a(r) = +\infty$ and the implication*

$$(x_1, x_2, \alpha) \in \text{domain } V \Rightarrow a(\varrho(x_1, x_2)) \leqq V(x_1, x_2, \alpha) \leqq b(\varrho(x_1, x_2))$$

*holds.*

Proof. The proof of this theorem follows from the proof of Theorem 2 if in (7) we replace $s(\sigma) \in P$ by the element ${}_\sigma t_\alpha^{u_2} x_2 \in P$ and define the partial mapping $V\colon P \times P \times R \to R^+$ by the rule

$$V(x_1, x_2, \alpha) = \tag{8}$$

$$= \sup \left\{ \varrho({}_\sigma t_\alpha^{u_1} x_1, {}_\sigma t_\alpha^{u_2} x_2) \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} : (u_1, u_2, \sigma) \in U \times U \times R, \sigma \geqq \alpha \right\}$$

for $(x_1, x_2, \alpha) \in P \times P \times R$ with $\varrho(x_1, x_2) \geqq \delta$. If $(x_1, x_2, \alpha) \in \text{domain } V$, $(u_1, u_2) \in U \times U$, $y_j = {}_\beta t_\alpha^{u_j} x_j$, $j = 1, 2$, $(y_1, y_2, \beta) \in \text{domain } V$, then for each $z_j = {}_\sigma t_\beta^{v_j} y_j$, $j = 1, 2$, there exists $(w_1, w_2) \in U \times U$ such that $z_j = {}_\sigma t_\alpha^{w_j} x_j$, $j = 1, 2$. From this and from the inequality

$$\frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} \geqq \frac{1 + (\sigma - \beta) r_0}{1 + \sigma - \beta}$$

we obtain

$$V(y_1, y_2, \beta) =$$

$$= \sup \left\{ \varrho({}_\sigma t_\beta^{v_1} y_1, {}_\sigma t_\beta^{v_2} y_2) \frac{1 + (\sigma - \beta) r_0}{1 + \sigma - \beta} : (v_1, v_2, \sigma) \in U \times U \times R, \sigma \geqq \beta \right\} =$$

$$= \sup \left\{ \varrho({}_\sigma t_\beta^{v_1} \circ {}_\beta t_\alpha^{u_1} x_1, {}_\sigma t_\beta^{v_2} \circ {}_\beta t_\alpha^{u_2} x_2) \frac{1 + (\sigma - \beta) r_0}{1 + \sigma - \beta} : (v_1, v_2, \sigma) \in U \times U \times R, \right.$$

$$\left. \sigma \geqq \beta \right\} \leqq \sup \left\{ \varrho({}_\sigma t_\alpha^{w_1} x_1, {}_\sigma t_\alpha^{w_2} x_2) \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} : (w_1, w_2, \sigma) \in U \times U \times R, \sigma \geqq \right.$$

$$\left. \geqq \alpha \right\} = V(x_1, x_2, \alpha),$$

which shows that the mapping $V$ has the property (i). The rest of the proof is an easy modification of the proof of Theorem 2 based on the above mentioned replacement.

**Theorem 4.** *The set $m$ is uniformly asymptotically stable with respect to the controllable continuous flow $\{t^u : u \in U\}$, if and only if there exist $\delta \in R^+$ and partial mappings $a\colon R^+ \to R^+$, $b\colon R^+ \to R^+$, $c\colon R^+ \to R^+$, $V\colon P \times R \to R^+$ with the following properties:*

(i) $$\text{domain } V = \{(x, \alpha) \in P \times R : 0 < \varrho(x, s(\alpha)) \leqq \delta\},$$

*$c\colon (0, \delta\rangle \to R^+$ is a continuous increasing mapping, $\lim_{r \to 0+} c(r) = 0$, such that the implication*

$$(x, \alpha) \in \text{domain } V, \quad u \in U \Rightarrow V({}_\beta t_\alpha^u x, \beta) - V(x, \alpha) \leqq$$
$$\leqq - \int_\alpha^\beta c(\varrho({}_\lambda t_\alpha^u x, s(\lambda)))\, d\lambda$$

*holds whenever* $V({}_\lambda t_\alpha^u x, \lambda)$ *is defined for all* $\lambda \in \langle \alpha, \beta \rangle$;

(ii) $a, b\colon (0, \delta\rangle \to R^+$ *are increasing continuous mappings such that* $\lim_{r \to 0+} b(r) = 0$,

$$(x, \alpha) \in \text{domain } V \Rightarrow a(\varrho(x, s(\alpha))) \leqq V(x, \alpha) \leqq b(\varrho(x, s(\alpha))) .$$

Proof. Let $m$ be uniformly asymptotically stable with respect to $\{t^u\colon u \in U\}$. The mapping $z_2$ from (4) may be supposed to be continuous and decreasing and $\lim_{r \to 0+} z_2(r) = +\infty$. Choose $r_0 > 1$, $\delta \in z_1(R^+)$, $\delta \leqq r_1$ and define the partial mapping $V\colon P \times R \to R^+$ by the rule (7) for $(x, \alpha) \in P \times R$ with $0 < \varrho(x, s(\alpha)) \leqq \delta$. According to the first part of the proof of Theorem 1 the mapping $V$ has the property (ii). We shall show that it has the property (i). If $(x, \alpha) \in \text{domain } V$, $u \in U$, then (4) implies that for all

$$\sigma \geqq \alpha + z_2(\varrho(x, s(\alpha))/r_0^2) \quad \text{we have} \quad \varrho({}_\sigma t_\alpha^u x, s(\sigma)) \leqq \varrho(x, s(\alpha))/r_0^2 .$$

Therefore

$$\varrho({}_\sigma t_\alpha^u x, s(\sigma)) \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} \leqq \frac{\varrho(x, s(\alpha))}{r_0^2} \cdot \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} <$$
$$< \varrho(x, s(\alpha))/r_0 < \varrho(x, s(\alpha)) \leqq V(x, \alpha) ,$$

which implies

$$\sup \left\{ \varrho({}_\sigma t_\alpha^u x, s(\sigma)) \frac{1 + (\sigma - \alpha) r_0}{1 + \sigma - \alpha} : (u, \sigma) \in U \times R, \sigma \geqq \alpha + z_2(\varrho(x, s(\alpha))/r_0^2) \right\} <$$
$$< \varrho(x, s(\alpha)) \leqq V(x, \alpha) .$$

This implies that for each $\varepsilon_1 \in R^+$ there exists $(u_0, \sigma_0) \in U \times R$, $\sigma_0 \in \langle \alpha, \alpha + z_2(\varrho(x, s(\alpha))/r_0^2) \rangle$ such that

$$V(x, \alpha) < \varrho({}_{\sigma_0} t_\alpha^{u_0} x, s(\sigma_0)) \frac{1 + (\sigma_0 - \alpha) r_0}{1 + \sigma_0 - \alpha} + \varepsilon_1 . \tag{9}$$

If we denote $y = {}_\beta t_\alpha^u x$, $\varepsilon_1 = \varepsilon(\beta - \alpha)$, $\varepsilon \in R^+$, then for $(y, \beta) \in \text{domain } V$, we have

$$V(y, \beta) < \varrho({}_{\beta_0} t_\beta^{v_0} y, s(\beta_0)) \frac{1 + (\beta_0 - \beta) r_0}{1 + \beta_0 - \beta} + (\beta - \alpha) \varepsilon =$$
$$= \varrho({}_{\beta_0} t_\beta^{v_0} \circ {}_\beta t_\alpha^u x, s(\beta_0)) \frac{1 + (\beta_0 - \beta) r_0}{1 + \beta_0 - \beta} + (\beta - \alpha) \varepsilon =$$
$$= \varrho({}_{\beta_0} t_\alpha^v x, s(\beta_0)) \frac{1 + (\beta_0 - \beta) r_0}{1 + \beta_0 - \beta} + (\beta - \alpha) \varepsilon =$$
$$= \varrho({}_{\beta_0} t_\alpha^v x, s(\beta_0)) \frac{1 + (\beta_0 - \alpha) r_0}{1 + \beta_0 - \alpha} \left[ 1 - \frac{(r_0 - 1)(\beta - \alpha)}{(1 - \beta_0 - \beta)(1 + (\beta_0 - \alpha) r_0)} \right] +$$

$$+ (\beta - \alpha)\,\varepsilon \leqq V(x, \alpha)\left[1 - \frac{(r_0 - 1)(\beta - \alpha)}{(1 + \beta_0 - \beta)(1 + (\beta_0 - \alpha) r_0)}\right] + (\beta - \alpha)\,\varepsilon$$

and therefore

$$\text{(10)} \qquad \frac{V(y, \beta) - V(x, \alpha)}{\beta - \alpha} < -(r_0 - 1)\frac{V(x, \alpha)}{(1 + \beta_0 - \beta)(1 + (\beta_0 - \alpha) r_0)} + \varepsilon\,.$$

From this and from the inequalities $0 \leqq \beta_0 - \beta \leqq z_2(\varrho(y, s(\beta))/r_0^2)$, $0 < \beta_0 - \alpha \leqq$ $\leqq z_2(\varrho(y, s(\beta))/r_0^2) + \beta - \alpha$ we obtain the inequality

$$\text{(11)} \qquad \frac{V(y, \beta) - V(x, \alpha)}{\beta - \alpha} \leqq$$

$$\leqq - \frac{(r_0 - 1) V(x, \alpha)}{(1 + z_2(\varrho(y, s(\beta))/r_0^2))(1 + r_0\, z_2(\varrho(y, s(\beta))/r_0^2) + r_0(\beta - \alpha))} \leqq$$

$$\leqq - \frac{(r_0 - 1)\,\varrho(x, s(\alpha))}{(1 + z_2(\varrho(y, s(\beta))/r_0^2))(1 + r_0\, z_2(\varrho(y, s(\beta))/r_0^2) + r_0(\beta - \alpha))}\,.$$

As $\lim_{\beta \to \alpha+} \varrho(y, s(\beta)) = \lim_{\beta \to \alpha+} \varrho({}_\beta t^u_\alpha x, s(\beta)) = \varrho(x, s(\alpha))$ for each $u \in U$ and the mapping $z_2$ is continuous by the assumption, (11) implies the inequality

$$\text{(12)} \qquad \limsup_{\beta \to \alpha+} \frac{V({}_\beta t^u_\alpha x, \beta) - V(x, \alpha)}{\beta - \alpha} \leqq$$

$$\leqq - \frac{(r_0 - 1)\,\varrho(x, s(\alpha))}{(1 + z_2(\varrho(x, s(\alpha))/r_0^2))(1 + r_0\, z_2(\varrho(x, s(\alpha))/r_0^2))}\,.$$

If we define the mapping $c: (0, \delta\rangle \to R^+$ by the rule

$$c(r) = \frac{(r_0 - 1)\, r}{(1 + z_2(r/r_0^2))(1 + r_0\, z_2(r/r_0^2))}\,,$$

we see that the mapping $V$ determined by (7) has the property (i).

Let there exist partial mappings $a$, $b$, $c$, $V$ and $\delta \in R^+$ with the properties (i), (ii). According to Theorem 1 the properties (i), (ii) imply that $m$ is uniformly stable with respect to $\{t^u : u \in U\}$. Choose $\delta_0 \in R^+$, $\delta_0 < \delta$ and put $r_1 = b^{-1}(a(\delta_0))$. Suppose that there exists $(x, \alpha) \in$ domain $V$, $\varrho(x, s(\alpha)) \leqq r_1, \gamma > \alpha$, $u \in U$ such that $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) > \delta_0$ holds. Similarly as in the proof of Theorem 1 it can be easily shown that it is possible to suppose $\varrho({}_\lambda t^u_\alpha x, s(\lambda)) \leqq \delta$ for all $\lambda \in \langle \alpha, \gamma \rangle$. Then $a(\varrho({}_\gamma t^u_\alpha x, s(\gamma))) \leqq V({}_\gamma t^u_\alpha x, \gamma) \leqq$ $\leqq V(x, \alpha) \leqq b(\varrho(x, s(\alpha))) \leqq b(r_1) = a(\delta_0)$, which implies $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) \leqq \delta_0$, and this contradicts the assumption that $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) > \delta_0$. Thus if $(x, \alpha) \in$ domain $V$, $\varrho(x,$ $s(\alpha)) \leqq r_1$, $u \in U$, then also $\varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq \delta$ for each $\sigma \geqq \alpha$. Define the mapping $z_2$: $R^+ \to R^+$ by the rule $z_2(r) = b(r_1)/c(z_1(r))$, where $z_1$ is the mapping from (1), and show that $r_1$ and $z_2$ fulfill (4). Suppose that there exist $(x, \alpha, u) \in P \times R \times U$, $\varrho(x, s(\alpha)) \leqq r_1$ and $r \in R^+$ such that for some $\beta \geqq \alpha + z_2(r)$ we have $\varrho({}_\beta t^u_\alpha x, s(\beta)) > r$. If $z_1(r) < \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq \delta$ for all $\sigma \in \langle \alpha, \beta \rangle$, then (i) implies the inequality

$$V({}_\beta t^u_\alpha x, \beta) \leqq V(x, \alpha) - \int_\alpha^\beta c(\varrho({}_\lambda t^u_\alpha x, s(\lambda)))\, \mathrm{d}\lambda < b(r_1) - c(z_1(r))\, z_2(r) = 0\,,$$

which contradicts the positivity of the mapping $V$. If there exists $\gamma \in \langle \alpha, \beta \rangle$ such that $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) \leqq z_1(r)$, then ${}_\beta t^u_\alpha x = {}_\beta t^u_\gamma \circ {}_\gamma t^u_\alpha x$ implies the inequality $\varrho({}_\beta t^u_\alpha x, s(\beta)) \leqq r$, which contradicts the assumption $\varrho({}_\beta t^u_\alpha x, s(\beta)) > r$. From this we obtain that for each $(\sigma, x, \alpha, u) \in R \times P \times R \times U$, $\varrho(x, s(\alpha)) \leqq r_1$, $\sigma \geqq \alpha + z_2(r)$ we have $\varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq r$. The theorem is proved.

**Theorem 5.** *The controllable continuous flow $\{t^u : u \in U\}$ is uniformly asymptotically bounded with respect to the set m, if and only if there exist $\delta \in R^+$ and partial mappings $a: R^+ \to R^+$, $b: R^+ \to R^+$, $c: R^+ \to R^+$, $V: P \times R \to R^+$ with the following properties:*

(i) $$\text{domain } V = \{(x, \alpha) \in P \times R : \delta \leqq \varrho(x, s(\alpha))\}\,,$$

*$c: \langle \delta, +\infty) \to R^+$ is a continuous mapping such that*

$$(x, \alpha) \in \text{domain } V,\ u \in U \Rightarrow V({}_\beta t^u_\alpha x, \beta) - V(x, \alpha) \leqq$$
$$\leqq - \int_\alpha^\beta c(\varrho({}_\lambda t^u_\alpha x, s(\lambda)))\, \mathrm{d}\lambda\,,$$

*whenever $V({}_\lambda t^u_\alpha x, \lambda)$ is defined for all $\lambda \in \langle \alpha, \beta \rangle$;*

(ii) *$a, b: \langle \delta, +\infty) \to R^+$ are increasing continuous mappings, $\lim_{r \to +\infty} a(r) = +\infty$, such that*

$$(x, \alpha) \in \text{domain } V \Rightarrow a(\varrho(x, s(\alpha))) \leqq V(x, \alpha) \leqq b(\varrho(x, s(\alpha)))\,.$$

Proof. Let $\{t^u : u \in U\}$ be uniformly asymptotically bounded with respect to $m$. Choose $r_0 > 1$, $\delta > r_0 r_1$ and define the partial mapping $V: P \times R \to R^+$ by (7) for $(x, \alpha) \in P \times R$ with $\varrho(x, s(\alpha)) \geqq \delta$. According to the first part of the proof of Theorem 2 the mapping $V$ has the property (ii). We show that it has also the property (i). The mapping $z_2$ from (5) may be supposed to be continuous. If $(x, \alpha) \in \text{domain } V$, $u \in U$, then (5) implies that for all $\sigma \geqq \alpha + z_2(\varrho(x, s(\alpha)))$ we have $\varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq r_1$. Therefore

$$\varrho({}_\sigma t^u_\alpha x, s(\sigma)) \frac{1 + (\sigma - \alpha)\, r_0}{1 + \sigma - \alpha} < r_0 r_1 < \delta \leqq \varrho(x, s(\alpha)) \leqq V(x, \alpha)\,,$$

which implies

$$\sup \left\{ \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \frac{1 + (\sigma - \alpha)\, r_0}{1 + \sigma - \alpha} : (u, \sigma) \in U \times R, \sigma \geqq \alpha + z_2(\varrho(x, s(\alpha))) \right\} <$$
$$< \varrho(x, s(\alpha)) \leqq V(x, \alpha)\,.$$

This implies that for each $\varepsilon_1 \in R^+$ there exists $(u_0, \sigma_0) \in U \times R$, $\sigma_0 \in \langle\alpha, \alpha + z_2(\varrho(x, s(\alpha)))\rangle$ such that (9) holds. If we denote $y = {}_\beta t^u_\alpha x$, $\varepsilon_1 = (\beta - \alpha)\,\varepsilon$, $\varepsilon \in R^+$, then for $(y, \beta) \in$ domain $V$ the relation (10) holds. From this and from $0 \leqq \beta_0 - \beta \leqq z_2(\varrho(y, s(\beta)))$, $0 < \beta_0 - \alpha \leqq z_2(\varrho(y, s(\beta))) + \beta - \alpha$ we obtain the inequality

(13)

$$\frac{V(y, \beta) - V(x, \alpha)}{\beta - \alpha} \leqq - \frac{(r_0 - 1)\, V(x, \alpha)}{[1 + z_2(\varrho(y, s(\beta)))]\,[1 + r_0\, z_2(\varrho(y, s(\beta))) + r_0(\beta - \alpha)]} \leqq$$

$$\leqq - \frac{(r_0 - 1)\, \varrho(x, s(\alpha))}{[1 + z_2(\varrho(y, s(\beta)))]\,[1 + r_0\, z_2(\varrho(y, s(\beta))) + r_0(\beta - \alpha)]}\,.$$

As $\lim_{\beta \to \alpha+} \varrho(y, s(\beta)) = \lim_{\beta \to \alpha+} \varrho({}_\beta t^u_\alpha x, s(\beta)) = \varrho(x, s(\alpha))$ for each $u \in U$ and the mapping $z_2$ is continuous by the assumption, (13) implies the inequality

$$(14) \qquad \limsup_{\beta \to \alpha+} \frac{V({}_\beta t^u_\alpha x, \beta) - V(x, \alpha)}{\beta - \alpha} \leqq$$

$$\leqq - \frac{(r_0 - 1)\, \varrho(x, s(\alpha))}{[1 + z_2(\varrho(x, s(\alpha)))]\,[1 + r_0\, z_2(\varrho(x, s(\alpha)))]}\,.$$

If we define the mapping $c: \langle\delta, +\infty) \to R^+$ by the rule

$$c(r) = \frac{(r_0 - 1)\, r}{[1 + z_2(r)]\,[1 + r_0\, z_2(r)]}\,,$$

we see that the mapping $V$ determined by (7) has the property (i).

Let there exist partial mappings $a$, $b$, $c$, $V$ and $\delta \in R^+$ with the properties (i), (ii). According to Theorem 2 the properties (i), (ii) imply that $\{t^u: u \in U\}$ is uniformly bounded with respect to $m$. If $z_1$ is the mapping from (2), put $\tau(r) = \inf\{c(r): \delta \leqq r \leqq z_1(r)\}$ and define $r_1 = z_1(\delta)$, $z_2(r) = b(r)/\tau(r)$. Suppose that there exists $(x, \alpha, u) \in P \times R \times U$, $\varrho(x, s(\alpha)) \leqq r$ such that for some $\beta \geqq \alpha + z_2(r)$ we have $\varrho({}_\beta t^u_\alpha x, s(\beta)) > r_1$. If $\delta < \varrho({}_\sigma t^u_\alpha x, s(\sigma)) \leqq z_1(r)$ for all $\sigma \in \langle\alpha, \beta\rangle$, then (i) implies the inequality $V({}_\beta t^u_\alpha x, \beta) \leqq V(x, \alpha) - \int_\alpha^\beta c(\varrho({}_\lambda t^u_\alpha x, s(\lambda)))\, d\lambda \leqq b(r) - \tau(r)\, z_2(r) = 0$, which contradicts the positivity of the mapping $V$. If there exists $\gamma \in \langle\alpha, \beta\rangle$ such that $\varrho({}_\gamma t^u_\alpha x, s(\gamma)) \leqq \delta$, then ${}_\beta t^u_\alpha x = {}_\beta t^u_\gamma \circ {}_\gamma t^u_\alpha x$ implies the inequality $\varrho({}_\beta t^u_\alpha x, s(\beta)) \leqq z_1(\delta) = r_1$, which contradicts the assumption $\varrho({}_\beta t^u_\alpha x, s(\beta)) > r_1$. The theorem is proved.

**Theorem 6.** *The controllable continuous flow $\{t^u: u \in U\}$ is uniformly asymptotically bounded, if and only if there exist $\delta \in R^+$ and partial mappings $a: R^+ \to R^+$, $b: R^+ \to R^+$, $c: R^+ \to R^+$, $V: P \times P \times R \to R^+$ with the following properties:*

(i) domain $V = \{(x_1, x_2, \alpha) \in P \times P \times R: \delta \leqq \varrho(x_1, x_2)\}$,

*$c: \langle\delta, +\infty) \to R^+$ is a continuous mapping such that*

$$(x_1, x_2, \alpha) \in \text{domain } V, \quad (u_1, u_2) \in U \times U \Rightarrow$$

$$\Rightarrow V\left({}_\beta t_\alpha^{u_1} x_1, {}_\beta t_\alpha^{u_2} x_2, \beta\right) - V(x_1, x_2, \alpha) \leqq - \int_\alpha^\beta c\left(\varrho\left({}_\lambda t_\alpha^{u_1} x_1, {}_\lambda t_\alpha^{u_2} x_2\right)\right) \mathrm{d}\lambda ,$$

*whenever* $V({}_\lambda t_\alpha^{u_1} x_1, {}_\lambda t_\alpha^{u_2} x_2, \lambda)$ *is defined for all* $\lambda \in \langle \alpha, \beta \rangle$;

(ii) $a, b: \langle \delta, +\infty) \to R^+$ *are increasing continuous mappings,* $\lim_{r \to +\infty} a(r) = +\infty$, *such that*

$$(x_1, x_2, \alpha) \in \text{domain } V \Rightarrow a(\varrho(x_1 . x_2)) \leqq V(x_1, x_2, \alpha) \leqq b(\varrho(x_1, x_2)) .$$

Proof. The proof of the theorem follows from the proof of Theorem 5 if in (7) we replace $s(\sigma) \in P$ by the element ${}_\sigma t_\alpha^{u_2} x_2 \in P$ and define the partial mapping $V$: $P \times P \times R \to R^+$ by the rule (8) for $(x_1, x_2, \alpha) \in P \times P \times R$ with $\varrho(x_1, x_2) \geqq \delta$. The rest of the proof is an easy modification of the proof of Theorem 5 based on the above mentioned replacement.

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## Souhrn

### STABILITA A OHRANIČENOST REGULOVATELNÝCH SPOJITÝCH TOKŮ

FRANTIŠEK TUMAJER

V článku je zaveden pojem regulovatelného spojitého toku v metrickém prostoru jako zobecnění regulovatelného systému diferenciálních rovnic v Banachově prostoru a jsou definovány různé druhy stability a ohraničenosti tohoto toku. Pomocí ljapunovských funkcí jsou formulovány věty dávající nutné a postačující podmínky pro jednotlivé druhy stability a ohraničenosti.



## Резюме

### УСТОЙЧИВОСТЬ И ОГРАНИЧЕННОСТЬ РЕГУЛИРУЕМЫХ НЕПРЕРЫВНЫХ ПОТОКОВ

FRANTIŠEK TUMAJER

В статье введено понятие регулируемого непрерывного потока в метрическом пространстве как обобщение регулируемой системы дифференциальных уравнений в пространстве

Банаха и определенны различные виды устойчивости и ограниченности такого потока. При помощи функций Ляпунова сформулированы теоремы, дающие необходимые и достаточные условия для отдельных видов устойчивости и ограниченности.

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