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NEW REGIONS OF STABILITY IN INPUT OPTIMIZATION[1])

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Summary. Using point-to-set mappings we identify two new regions of stability in input optimization. Then we extend various results from the literature on optimality conditions, continuity of Lagrange multipliers, and the marginal value formula over the new and some old regions of stability.

Keywords: Region of stability, point-to-set mapping, optimal input, input constraint qualification, Lagrange multipliers, marginal value.

1. INTRODUCTION

Consider the mathematical programming *model*

$$(P, \theta) \qquad \begin{aligned} &\operatorname*{Min}_{(x)} f^0(x, \theta) \\ &\text{s.t.} \\ &f^i(x, \theta) \leqq 0\,, \quad i \in \mathscr{P} \triangleq \{1, \ldots, m\}\,, \quad \theta \in I \end{aligned}$$

where all functions $f^i\colon R^n \times R^p \to R$ are continuous and $f^i(\cdot, \theta) : R^n \to R$ are convex, $i \in \{0\} \cup \mathscr{P}$. The set $I \subset R^p$ is assumed convex. Such model is termed *convex*. For a fixed parameter (input) $\theta \in I$, (P, θ) is a usual convex *program*.

With each $\theta \in I$ we associate the triple (output)

$$F(\theta) = \{x \in R^n : f^i(x, \theta) \leqq 0,\ i \in \mathscr{P}\}$$

the *feasible set*,

$$\tilde{F}(\theta) = \{\tilde{x}(\theta)\}$$

the set of all *optimal solutions* $\tilde{x}(\theta)$, and

$$\tilde{f}(\theta) = f^0(\tilde{x}(\theta), \theta)$$

the *optimal value.*

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We will study perturbations of the output $\{F(\theta), \tilde{F}(\theta), \tilde{f}(\theta)\}$ in a neighbourhood $N(\theta*)$ of an arbitrary but fixed $\theta* \in I$. It is assumed that $\tilde{F}(\theta*) \neq \theta$ and bounded. Objective functions with this property are termed *realistic*. In many models describing real-life situations (e.g., in multi-objective models), "continuity" of the output is not guaranteed for arbitrary perturbations in a neighbourhood of $\theta*$. (See, e.g., [2, 21].) However, continuity is preserved on "regions of stability". We recall

1.1 **Definition** [16, 24, 25]. *Consider a convex model (P, θ) at some $\theta = \theta* \in I$ with a realistic objective function f^0. We say that the model is stable in a region $S \subset I \subset R^p$ at $\theta* \in S$ if, for some neighbourhood $N(\theta*)$ of $\theta*$, both*

(i) $\tilde{F}(\theta) \neq \emptyset$ *for every* $\theta \in N(\theta*) \cap S$, *and*
(ii) $\theta \in N(\theta*) \cap S$ *and* $\theta \to \theta*$ *imply that* $\tilde{f}(\theta) \to \tilde{f}(\theta*)$. ■

In particular, we say that the model is *stable* at $\theta*$, if one can specify $S = N(\theta*)$.

In this paper we consider only those regions of stability for which the point-to-set mapping $\Gamma: \theta \to F(\theta)$ is lower semicontinuous at $\theta*$, relative to S. (See [1, 3, 6, 7, 8, 9].) On such regions not only the optimal value function, but also the set of optimal solutions, is "continuous" for every realistic objective function. (See [22, 25].)

The regions of stability can be expressed in terms of constructive objects of convex analysis such as (defined for each $\theta \in I$)

$$\mathscr{P}^{=}(\theta) = \{i \in \mathscr{P}: x \in F(\theta) \Rightarrow f^i(x, \theta) = 0\},$$

the *minimal index set of active constraints* (see [2, 17, 22]), and the corresponding set in R^n:

$$F^{=}(\theta) = \{x \in R^n: f^i(x, \theta) = 0,\ i \in \mathscr{P}^{=}(\theta)\}.$$

Also we use the notation $\mathscr{P}^{<}(\theta) = \mathscr{P} \setminus \mathscr{P}^{=}(\theta)$.

Among the "oldest" regions of stability are

$$M(\theta*) = \{\theta: F(\theta*) \subset F(\theta)\} \cap I,$$

$$V(\theta*) = \{\theta: F^{=}(\theta*) \subset F^{=}(\theta), \text{ and } f^i(x, \theta) \leqq 0,$$
$$\forall x \in F(\theta*),\ i \in \mathscr{P}^{=}(\theta*) \setminus \mathscr{P}^{=}(\theta)\} \cap I.$$

(See [22, 24, 25].) To simplify notation we denote

$$R_1(\theta*) = \{\theta: \mathscr{P}^{=}(\theta*) = \mathscr{P}^{=}(\theta)\} \cap I,$$
$$R_2(\theta*) = \{\theta: f^i(x, \theta) \leqq 0,\ \forall x \in F^{=}(\theta),\ i \in \mathscr{P}^{=}(\theta*) \setminus \mathscr{P}^{=}(\theta)\} \cap I,$$
$$R_3(\theta*) = \{\theta: f^i(x, \theta) \leqq 0,\ \forall x \in F^{=}(\theta*),\ i \in \mathscr{P}^{=}(\theta*) \setminus \mathscr{P}^{=}(\theta)\} \cap I,$$
$$R_4(\theta*) = \{\theta: f^i(x, \theta) \leqq 0,\ \forall x \in F(\theta*),\ i \in \mathscr{P}^{=}(\theta*) \setminus \mathscr{P}^{=}(\theta)\} \cap I.$$

Note that now

$$V(\theta*) = \{\theta: F^{=}(\theta*) \subset F^{=}(\theta)\} \cap R_4(\theta*).$$

If $\Gamma: \theta \to F^{=}(\theta)$ is lower semicontinuous at θ^*, then $R_1(\theta*)$ and $R_2(\theta^*)$ are regions of stability at $\theta*$. (See [13, 22].) For a complete list of presently used regions of stability, and for some of their applications, see [22]. (Also the last section of this paper.)

Regions of stability are important in input optimization (see [18, 22]), but they are also of an independent interest (see, e.g. [4, 15]). Unlike the "usual" optimization, input optimization uses regions of stability to determine an optimal input $\theta*$ starting from an initial θ^0. The path $\sigma(\theta)$, connecting the "present" input θ^0 to the "future" input $\theta*$, passes only through regions of stability; this guarantees continuity of the economic process. The optimal input $\theta*$ (and hence the "optimal realization" $(P, \theta*)$ of the mathematical model (P, θ)) depends on the initial state of the system determined by θ^0. (Different θ^0's generally lead to different optimal inputs $\theta*$.)

In this paper, first we identify two new regions of stability in Section 2. In Section 3 we show that the optimality condition for an optimal input from [17, 20] extends to one of the new regions. Continuity of the restricted Lagrangian multiplier functions is established on any region of stability where $F(\theta*) = F^{=}(\theta*)$, in Section 4. In Sections 5 and 6 we show that the necessary condition for optimality and the marginal value formula, for differentiable bi-convex models, hold on more regions of stability than known from the literature [16, 19, 22]. Finally, in Section 7, we give a schematic comparison of all presently known regions of stability. The two regions introduced in this paper turn out to be among the largest ones.

2. TWO NEW REGIONS OF STABILITY

Take a $\theta* \in I$ and denote

$$Z(\theta*) = \{\theta : F(\theta*) \subset F^{=}(\theta)\} \cap R_4(\theta*) .$$

We claim that this set is a region of stability.

2.1. **Theorem.** *Consider the convex model (P, θ) at some $\theta = \theta*$, where $F(\theta*) \neq \emptyset$. Then $Z(\theta*)$ is a region of stability at $\theta*$ for every realistic objective function.*

Proof. It is enough to show that the point-to-set mapping $\Gamma: \theta \to F(\theta)$ is lower semicontinuous at $\theta*$, relative to the set $Z(\theta*)$. If this were not true, then there would exist an open set $\mathscr{A} \subset R^n$ such that

$$\mathscr{A} \cap F(\theta*) \neq \emptyset$$

but

$$\mathscr{A} \cap F(\theta^k) = \emptyset \tag{2.1}$$

for a sequence $\theta^k \in Z(\theta*)$, $\theta^k \to \theta*$. Now, choose an arbitrary

$$\hat{x} \in \text{rel int} \{\mathscr{A} \cap F(\theta*)\} .$$

Clearly

$$f^i(\hat{x}, \theta*) < 0\,, \quad i \in \mathscr{P}^<(\theta*)$$

and hence

(2.2) $$f^i(\hat{x}, \theta^k) < 0\,, \quad i \in \mathscr{P}^<(\theta*)$$

for all sufficiently large k's. In particular, since $\theta^k \in Z(\theta*)$,

(2.3) $$f^i(\hat{x}, \theta^k) \leqq 0\,, \quad i \in \mathscr{P}^=(\theta*) \setminus \mathscr{P}^=(\theta^k)\,.$$

Also $\hat{x} \in F(\theta*) \subset F^=(\theta^k)$ implies that

(2.4) $$f^i(\hat{x}, \theta^k) = 0\,, \quad i \in \mathscr{P}^=(\theta^k)\,.$$

The relations (2.2), (2.3) and (2.4) imply

$$\hat{x} \in F(\theta^k)$$

contradicting (2.1). ■

Since

$$F(\theta*) \subset F^=(\theta*) \subset F^=(\theta)$$

for $\theta \in V(\theta*)$ and

$$F(\theta*) \subset F(\theta) \subset F^=(\theta)$$

for $\theta \in M(\theta*)$, we note that $Z(\theta*)$ is a bigger region of stability than $M(\theta*)$ and $V(\theta*)$, i.e.

$$\{M(\theta*) \cup V(\theta*)\} \subset Z(\theta*)\,.$$

The following example shows that the inclusion may be strict.

2.2. Example. Consider a convex model with the two constraints

$$f^1 = -x + \theta + 1 \leqq 0\,,$$
$$f^2 = \max\{0, x - \theta\} - x + \theta \leqq 0$$

around $\theta* = 0$.

Since $F(\theta) = [\theta + 1, \infty)$, we have

$$M(\theta*) = \{\theta\colon [1, \infty) \subset [\theta + 1, \infty)\}$$
$$= (-\infty, 0]\,.$$

Further,

$$\mathscr{P}^=(\theta) = \{2\}$$

for every θ, while

$$F^=(\theta) = [\theta, \infty)\,.$$

Hence

$$V(\theta*) = \{\theta\colon [0, \infty) \subset [\theta, \infty)\} \cap R = (-\infty, 0]\,.$$

However, the new region of stability is bigger:

$$Z(\theta*) = \{\theta: [1, \infty) \subset [\theta, \infty)\} = (-\infty, 1].$$

Using $Z(\theta*)$ we conclude that the model is, in fact, stable at $\theta*$ for every realistic objective function. ■

Many results in input optimization require lower semicontinuity of the point-to-set mapping $F^=: \theta \to F^=(\theta)$. In particular, the mapping $F^=$ is necessarily lower semicontinuous at $\theta*$ over the region of stability $V(\theta*)$. However, the mapping $F^=$ may not be lower semicontinuous neither over $M(\theta*)$ nor over $Z(\theta*)$. ■

We will now identify another region of stability. The construction of $V(\theta*)$ suggests that, under some reasonable assumptions, such as lower semicontinuity of $F^=$, the set $R_4(\theta*)$ itself may be a region of stability. As the following example (communicated to the authors by Semple) shows, this is not enough.

2.3. Example (Semple). Consider the contraints

$$f^1 = |x_1 - \theta| \leqq 0,$$
$$f^2 = x_1 - \theta x_2 \leqq 0,$$
$$f^3 = x_2 - 2 \leqq 0,$$
$$f^4 = -x_2 \leqq 0$$

around $\theta* = 0$. Suppose that θ is taken from $\theta \in I = [0, 1]$.

Here

$$F(\theta) = \left\{\begin{bmatrix}\theta \\ x_2\end{bmatrix} : 1 \leqq x_2 \leqq 2\right\}$$

for $\theta > 0$, while

$$F(\theta*) = \left\{\begin{bmatrix}0 \\ x_2\end{bmatrix} : 0 \leqq x_2 \leqq 2\right\}.$$

So the mapping F is not lower semicontinuous at $\theta*$. But

$$F^=(\theta) = \left\{\begin{bmatrix}\theta \\ x_2\end{bmatrix} : x_2 \in R\right\}$$

for every $\theta \geqq 0$, clearly lower semicontinuous, and

$$R_4(\theta*) = [0, 1].$$ ■

However, the conjecture is valid with the additional assumption that the feasible set $F(\theta*)$ has interior points. The latter is typically satisfied in the so-called "lexicographic optimization", without Slater's condition being satisfied. (See [2, 21, 22].)

2.4. **Theorem.** *Consider the convex model (P, θ) at some $\theta = \theta *$. If $F(\theta *)$ has non-empty interior and if the point-to-set mapping $F^=$ is lower semicontinuous at $\theta *$, then $R_4(\theta *)$ is a region of stability at $\theta *$ for every realistic objective function.*

Proof. The result is proved by contradiction. Suppose that the mapping F is not lower semicontinuous at $\theta *$. Then there exists an open set $\mathscr{A}$, such that

$$\mathscr{A} \cap F(\theta *) \neq \emptyset$$

but

$$\mathscr{A} \cap F(\theta^k) = \emptyset \tag{2.5}$$

for a sequence $\theta^k \in R_4(\theta *)$, $\theta^k \to \theta *$. Now choose an arbitrary

$$\hat{x} \in \text{int}\,\{\mathscr{A} \cap F(\theta^*)\}\,.$$

Clearly

$$f^i(\hat{x}, \theta *) < 0\,, \quad i \in \mathscr{P}^<(\theta *)$$

and further, by continuity

$$f^i(\hat{x}, \theta^k) < 0\,, \quad i \in \mathscr{P}^<(\theta *) \tag{2.6}$$

for all sufficiently large k's. Since $\hat{x} \in F(\theta *)$ and $\theta^k \in R_4(\theta *)$, it follows that

$$f^i(\hat{x}, \theta^k) \leqq 0\,, \quad i \in \mathscr{P}^=(\theta *) \setminus \mathscr{P}^=(\theta^k)$$

which, together with (2.6), yields

$$f^i(\hat{x}, \theta^k) \leqq 0\,, \quad i \in \mathscr{P}^<(\theta^k)\,. \tag{2.7}$$

This means that $\hat{x} \notin F^=(\theta^k)$. (Otherwise, $\hat{x} \in F^=(\theta^k)$ and (2.7) would imply $\hat{x} \in F(\theta^k)$, contradicting (2.5).) Since $F(\theta *)$ has an interior, we can place a small open ball $B(\hat{x})$, centered at $\hat{x}$, inside $\mathscr{A} \cap F(\theta *)$, such that

$$B(\hat{x}) \cap F^=(\theta^k) = \emptyset\,. \tag{2.8}$$

But

$$\hat{x} \in F(\theta *) \subset F^=(\theta *)$$

and hence

$$B(\hat{x}) \cap F^=(\theta *) \neq \emptyset\,. \tag{2.9}$$

So, now we have an open set $B(\hat{x})$ such that (2.9) holds but, for a sequence $\theta^k \to \theta *$, also (2.8). This violates the assumption on lower semicontinuity of the mapping $F^=$. ∎

3. A NECESSARY CONDITION FOR OPTIMALITY

Optimality conditions for a convex model are stated in the literature only for the regions of stability $M(\theta *)$ and $V(\theta *)$, e.g. [17, 20, 22]. We now extend these conditions to the larger region of stability $Z(\theta *)$.

First we recall the notions of an "optimal input" and an "input constraint qualification" (abbreviation: ICQ).

3.1. **Definition** [17]. *Consider the convex model (P, θ) at some $\theta* \in I$ with a realistic objective function. We say that $\theta*$ is a locally optimal input for the model, with respect to a region of stability $S(\theta*)$, if*

$$\bar{f}(\theta*) \leqq \bar{f}(\theta)$$

for every $\theta \in N(\theta*) \cap S(\theta*)$, where $N(\theta*)$ is some neighbourhood of $\theta*$. The corresponding program $(P, \theta*)$ is a locally optimal realization and $\bar{f}(\theta*)$ is a locally optimal value of the model (P, θ). ■

Recall that the optimal input $\theta*$ depends on the choice of the initial input θ^0. In order to formulate a necessary condition for optimality, we generally need an ICQ.

3.2. **Definition** [20]. *An input constraint qualification for the convex model (P, θ) at $\theta* \in I$, with respect to a region of stability $S(\theta*)$, is a condition on the constraints of the model with the property that for every $\theta \in N(\theta*) \cap S(\theta*)$, where $N(\theta*)$ is a neighbourhood of $\theta*$, the system*

$$(c, \theta) \qquad \begin{aligned} f^0(x, \theta) &< \bar{f}(\theta*) \\ f^i(x, \theta) &< 0\,, \quad i \in \mathscr{P}^<(\theta*) \\ x &\in F^=(\theta*) \end{aligned}$$

is inconsistent. ■

An ICQ guarantees the existence of a "saddle point" for the "restricted Lagrangian"

$$L_*^<(x, u; \theta) = f^0(x, \theta) + \sum_{i \in \mathscr{P}^<(\theta*)} u_i f^i(x, \theta)\,.$$

A particular ICQ for the region of stability $S(\theta*) = M(\theta*) \cup V(\theta*)$, at an optimal input $\theta*$, is the following condition:

> "For every $\theta \in N(\theta^*) \cap S(\theta*)$, where $N(\theta*)$ is a neighbourhood of $\theta*$, and for every $x \in F^=(\theta*)$ such that
>
> $$f^i(x, \theta) < 0\,, \quad i \in \mathscr{P}^<(\theta*)$$
>
> it follows that
>
> $$f^i(x, \theta*) \leqq 0\,, \quad i \in \mathscr{P}^<(\theta*)\,."$$

This condition is referred to as ICQ1 in [20, 22]. We will now show that ICQ1 is ICQ also for the region of stability $Z(\theta*)$.

3.3. **Lemma.** *Consider the convex model (P, θ) at an optimal input $\theta* \in I$ with respect to the region of stability $Z(\theta*)$. Then ICQ1 is ICQ.*

Proof. Suppose that the condition ICQ1 holds, but not ICQ. Then there exist sequences $\theta^k \in Z(\theta*)$, $\theta^k \to \theta*$ and $x^k = x^k(\theta^k) \in F^=(\theta*)$ such that

$$f^0(x^k, \theta^k) < \tilde{f}(\theta*), \tag{3.1}$$

$$f^i(x^k, \theta^k) < 0, \quad i \in \mathscr{P}^<(\theta*). \tag{3.2}$$

Since ICQ1 holds, (3.2) implies

$$f^i(x^k, \theta*) \leqq 0, \quad i \in \mathscr{P}^<(\theta*)$$

and hence $x^k \in F(\theta*)$. On the other hand, $\theta^k \in Z(\theta*)$ implies

$$f^i(x^k, \theta^k) \leqq 0, \quad i \in \mathscr{P}^=(\theta*) \setminus \mathscr{P}^=(\theta^k)$$

which together with (3.2) gives

$$f^i(x^k, \theta^k) \leqq 0, \quad i \in \mathscr{P}^<(\theta^k) \tag{3.3}$$

and also

$$x^k \in F^=(\theta^k). \tag{3.4}$$

The two relations (3.3) and (3.4) imply $x^k \in F(\theta^k)$. Since $\theta*$ is locally optimal, we have a contradiction to (3.1). ■

3.4. **Theorem.** *Consider the convex model (P, θ) with a realistic objective function at some $\theta* \in I$. Suppose that $\theta*$ is a locally optimal input with respect to the region of stability $Z(\theta*)$ and that the condition ICQ1 is satisfied at $\theta*$ with respect to $Z(\theta*)$. Let $\tilde{x}(\theta*)$ be a corresponding optimal solution. Then there exists a neighbourhood $N(\theta*)$ and a non-negative vector function*

$$\mathscr{U}: N(\theta*) \cap Z(\theta*) \to R_+^{q(\theta*)}$$

such that, whenever $\theta \in N(\theta*) \cap Z(\theta*)$,

$$L_*^<(\tilde{x}(\theta*), u; \theta*) \leqq L_*^<(\tilde{x}(\theta*), \mathscr{U}(\theta*); \theta*) \leqq L_*^<(x, \mathscr{U}(\theta); \theta)$$

for every $u \in R_+^{q(\theta*)}$ (the non-negative orthant in $R^{q(\theta*)}$, where $q(\theta*)$ is cardinality of $\mathscr{P}^<(\theta*)$) and every $x \in F^=(\theta*)$.

Proof. Since ICQ1 is indeed ICQ for $Z(\theta*)$, the result is an immediate consequence of, say, [22, Theorem 7.4]. ■

The importance of Theorem 3.4 is that a necessary condition for an optimal input is now stated over a larger region of stability than $M(\theta*) \cup V(\theta*)$. Of course, the result also holds under some more restrictive ICQ's such as ICQ2 or Slater's condition. (See [20].)

4. CONTINUITY OF A LAGRANGE MULTIPLIER FUNCTION

Consider the convex model (P, θ) at some $\theta \in I$. It is well known that an $x* \in F(\theta)$ is optimal if, and only if, there exists $u* \in R_+^{q(\theta)}$, such that

$$L^<(x^*, u; \theta) \leqq L^<(x^*, u^*; \theta) \leqq L^<(x, u^*; \theta)$$

for every $x \in F^=(\theta)$ and every $u \in R_+^{q(\theta)}$; see [17, 22]. Here

$$L^<(x, u; \theta) = f^0(x, \theta) + \sum_{i \in \mathscr{P}^<(\theta)} u_i f^i(x, \theta) .$$

(This claim also follows from Theorem 3.4 for a fixed θ.) The corresponding Lagrange multipliers

$$U^<(\theta) = \{u_i^*(\theta): i \in \mathscr{P}^<(\theta)\}$$

are generally discontinuous on regions of stability; see [12, 14, 19]. However, on the regions of stability we have $\mathscr{P}^<(\theta*) \subset \mathscr{P}^<(\theta)$ for every θ in a neighbourhood $N(\theta*)$ of $\theta*$; see [13]. Hence it follows that the Lagrangian $L_*^<$ and the subset

$$U_*^<(\theta) = \{u_i(\theta): i \in \mathscr{P}^<(\theta*)\}$$

of $U^<(\theta)$ are well defined. Unfortunately, $U_*^<(\theta)$ can still be discontinuous on some regions of stability; see [12]. If $F^=$ is lower semicontinuous at $\theta*$, then a region of stability, where $U_*^<(\theta)$ is continuous, is $R_2(\theta*)$. (See [14].) Continuity is also proved on

$$V_1(\theta*) = \{\theta: F^=(\theta*) \subset F^=(\theta)\} \cap R_3(\theta*) .$$

(See [12].)

In this section we will establish continuity of $U_*^<(\theta)$ on every subset of an *arbitrary* region of stability, provided that

$$F(\theta*) = F^=(\theta*) . \tag{4.1}$$

The condition (4.1) looks somewhat restrictive. In particular, if Slater's condition holds at $\theta*$, then $F^=(\theta*) = R^n$ and the condition holds only for unconstrained models. However, another extreme case is when the feasible set is determined only by linear equations, in which case the condition is trivially satisfied. A general situation, where (4.1) holds, is described by the example below.

4.1. Example. Consider a convex model with only one constraint

$$f^1 = |x| - (1 + \theta^2)\, x \leqq 0$$

around $\theta = \theta* = 0$.

Here $F(\theta) = [0, \infty)$ for every θ, but

$$\mathscr{P}^=(\theta) = \begin{cases} \{1\} & \text{if } \theta = 0 \\ \emptyset & \text{if } \theta \neq 0 \end{cases}$$

and hence

$$F^{=}(\theta) = \begin{cases} [0, \infty) & \text{if} \quad \theta = 0 \\ R & \text{if} \quad \theta \neq 0 . \end{cases}$$

The point-to-set mapping $F^{=}$ is lower semicontinuous at $\theta*$ and

$$R_2(\theta*) = \{\theta*\} .$$

Hence the result from the literature on continuity of the Lagrange multiplier function $U_*^{<}(\theta)$ is not useful here. However, since the requirement (4.1) is satisfied, and int $F(\theta*) \neq \emptyset$, we will be able to establish continuity (using the result given below) on the region of stability, say, $R_4(\theta*) = R$.

4.2. **Theorem.** *Consider the convex model (P, θ) at some $\theta* \in I$ with $\tilde{F}(\theta*) \neq \emptyset$ and bounded and let $F(\theta*) = F^{=}(\theta*)$. If $S(\theta*)$ is an arbitrary region of stability at $\theta*$ then, for an arbitrary sequence $\theta^k \in S(\theta*)$, $\theta^k \to \theta*$:*

(i) *The sequence $U_*^{<}(\theta^k)$ is bounded for all sufficiently large k's and*

(ii) *the set of all limit points of $U_*^{<}(\theta^k)$, as $\theta^k \to \theta*$, is nonempty, and it is contained in $U_*^{<}(\theta*)$.*

Proof. (i) This statement holds for any region of stability, as one can see from the proofs of [12, Theorem 3.1] or [13, Theorem 2.1].

(ii) The existence of limit points of $U_*^{<}(\theta^k)$, as $\theta^k \to \theta*$, follows from the boundedness of each component $u_i(\theta^k)$, $\theta^k \to \theta^*$, $i \in \mathscr{P}^{<}(\theta^*)$. It is left to prove the "inclusion statement". The proof follows exactly the arguments of the proof given in [13, Theorem 2.1], up to the inequality

$$\tilde{f}(\theta^k) \leqq f^0(x, \theta^k) + \sum_{i \in \mathscr{P}^{<}(\theta^k)} u_i(\theta^k) f^i(x, \theta^k) \tag{4.2}$$

for every $x \in F^{=}(\theta^k)$. Since for every $x \in F(\theta^k)$ we have

$$f^i(x, \theta^k) \leqq 0 , \quad i \in \mathscr{P}^{<}(\theta^k) \setminus \mathscr{P}^{<}(\theta*)$$

it follows that the right-hand side in (4.2) can be modified and we have

$$\tilde{f}(\theta^k) \leqq L_*^{<}(x, u(\theta^k); \theta^k) \tag{4.3}$$

for every $x \in F(\theta^k)$. Now we pick an arbitrary $x \in F(\theta*)$. Since θ^k is chosen from a region of stability $S(\theta*)$, the mapping F is lower semicontinuous at $\theta*$ with respect to $S(\theta*)$. Therefore there exists a sequence $x^k = x^k(\theta^k) \in F(\theta^k)$ such that $x^k \to x$ as $\theta^k \to \theta*$. The inequality (4.3) gives in the limit

$$\tilde{f}(\theta*) \leqq L_*^{<}(x, u(\theta*); \theta*) \tag{4.4}$$

for every $x \in F(\theta*)$. (Here $u(\theta*)$ is an arbitrary limit point of the sequence $U_*^{<}(\theta^k)$ that generates the above $\theta^k \in S(\theta*)$, $\theta^k \to \theta*$.) Since $F(\theta*) = F^{=}(\theta*)$, the inequality

(4.4) also holds for every $x \in F^{=}(\theta*)$. This is the right-hand side of the saddle-point inequality at $\theta*$. The left-hand side was proved [13, Theorem 2.1]. So we conclude that the limit point $u(\theta*)$ of $u(\theta^k)$, as $\theta^k \in S(\theta*)$, $\theta^k \to \theta*$, is indeed in $U_*^{\leq}(\theta*)$. ■

It is easy to construct examples showing that Theorem 4.2 is applicable to different regions of stability from $V_1(\theta*)$ and $R_2(\theta*)$. (See Example 4.1.)

5. A NECESSARY CONDITION FOR BI-CONVEX MODELS

The necessary condition for an optimal input for general convex models can be strengthened for bi-convex models, i.e., for the models (P, θ) where both $f^i(\cdot, \theta)$: $R^n \to R$ and $f^i(x, \cdot)$: $R^p \to R$, $i \in \{0\} \cup \mathscr{P}$ are convex functions. The result, given below, is proved for a subset of the region of stability $Z(\theta*)$, namely for

$$Z_1(\theta*) = \{\theta: F(\theta*) \subset F^{=}(\theta)\} \cap R_2(\theta*).$$

Following the ideas from [18], we use a part of the unit ball $B = B(\theta*)$ in R^p defined by

$$B = \left\{\frac{\theta - \theta*}{\|\theta - \theta\|} : \theta \in N(\theta*) \cap Z_1(\theta*),\ \theta \neq \theta*\right\}$$

for some fixed neighbourhood $N(\theta*)$ of $\theta*$. We denote by B^0 the derived set of B, i.e., the set of all limit points as $\theta \in Z_1(\theta*)$, $\theta \to \theta*$. Also

$$(B^0)^+ = \{u: u^{\mathrm{T}} b \geqq 0 \text{ for every } b \in B^0\}$$

is the polar of B^0.

We also need a condition on indices of the constraints, known as the "index condition" (see [22, 23]). First we denote by

$$\mathscr{P}(\tilde{x}(\theta*), \theta*) = \{i \in \mathscr{P}: f^i(\tilde{x}(\theta*), \theta*) = 0\}$$

the set of active constraints for $\theta*$ at the optimal solution $\tilde{x}(\theta*)$. We recall that, for differentiable functions, the index condition is said to hold at $\theta*$, with respect to a region of stability $S(\theta*)$, if

$$\text{(IND)} \qquad \{\mathscr{P}^{<}(\theta^k) \cap \mathscr{P}(\tilde{x}(\theta*), \theta*)\} \subset \mathscr{P}^{<}(\theta*)$$

for all but possibly finitely many k's, for every sequence $\theta^k \in S(\theta*)$, $\theta^k \to \theta*$.

To simplify the notation, we introduce the abbreviation

$$g(\theta) = L_*^{\leq}(\tilde{x}(\theta*), \tilde{u}(\theta*); \theta)$$

where $(\tilde{x}(\theta*), \tilde{u}(\theta*))$ is a restricted saddle point. Note that $g(\theta)$ is a convex function.

5.1. **Theorem.** *Consider the bi-convex model (P, θ) at $\theta = \theta* \in I$ with a realistic objective function. Suppose that the corresponding saddle point $\{\tilde{x}(\theta*), \tilde{u}_i(\theta*)$: $i \in$*

$\in \mathscr{P}^<(\theta*)\}$ *of the restricted Lagrangian is unique and that the index condition (IND) holds relative to* $Z_1(\theta*)$. *We also assume that the point-to-set mapping* $F^=$ *is lower semicontinuous at* $\theta*$ *and that all functions* $f^i(\tilde{x}(\theta*), \theta)$, $i \in \{0\} \cup \mathscr{P}^<(\theta*)$ *are differentiable at* $\theta*$. *If* $\theta*$ *is a locally optimal input with respect to* $Z_1(\theta*)$, *then*

$$\nabla_\theta g(\theta*) \in (B^0)^+ .$$

Proof. In the proof we use the fact that $Z_1(\theta*) \subset R_2(\theta*)$ and the result on continuity of the restricted Lagrange multipliers with respect to the set $R_2(\theta*)$. (See [18].) So, take an arbitrary $l \in B^0$. This point is generated by some $\theta^k \in Z_1(\theta*)$, $\theta^k \to \theta*$. By the continuity of Lagrange multipliers, there exist $u_i(\theta^k) \to \tilde{u}_i(\theta*)$, $i \in \mathscr{P}^<(\theta*)$. For this sequence θ^k define

$$\mathscr{E}(\theta^k) = L^<(\tilde{x}(\theta*), \tilde{u}(\theta*); \theta^k) - L^<(\tilde{x}(\theta*), \tilde{u}(\theta^k); \theta^k) \tag{5.1}$$

and, using the gradient inequality for $g(\theta)$, we find that

$$(\nabla g(\theta^k), \theta^k - \theta*) \geqq L^<(\tilde{x}(\theta*), \tilde{u}(\theta^k); \theta^k) - g(\theta*) + \mathscr{E}(\theta^k) .$$

(Here $(u, v) = u^T v$ denotes tke Euclidean inner product.) Now we invoke the estimate

$$\tilde{f}(\theta) - \tilde{f}(\theta*) \leqq L^<(z, \tilde{u}(\theta); \theta) - L^<_*(\tilde{x}(\theta*), v; \theta*) \tag{5.2}$$

holding for every $z \in F^=(\theta)$ and $v \in R_+^{q(\theta*)}$, as it is known from [16]. Using the fact that $\theta^k \in Z_1(\theta*)$, we note that

$$\tilde{x}(\theta*) \in F(\theta*) \subset F^=(\theta^k) .$$

So we can specify $z = \tilde{x}(\theta*)$ and $v = \tilde{u}(\theta*)$ in (5.2), and (5.1) now gives

$$\begin{aligned}(\nabla g(\theta^k), \theta^k - \theta*) &\geqq \tilde{f}(\theta^k) - \tilde{f}(\theta*) + \mathscr{E}(\theta^k) \\ &\geqq \mathscr{E}(\theta^k)\end{aligned} \tag{5.3}$$

for all sufficiently large k's, because $\theta*$ is a locally optimal input. Hence

$$\left(\nabla g(\theta^k), \frac{\theta^k - \theta*}{\|\theta^k - \theta*\|}\right) \geqq \frac{\mathscr{E}(\theta^k)}{\|\theta^k - \theta*\|} . \tag{5.4}$$

The index condition guarantees non-negativity of the limit when $k \to \infty$ of the right-hand side term in (5.4). (See [23] for details.) This, together with continuous differentiability property of the differentiable convex function $g(\theta)$, gives the desired result. ■

A result of the above kind was proved in the literature (see [18, 23]) but only for the region of stability $V_1(\theta*)$.

The marginal value formula (i.e., the path derivative of the optimal value function) was proved in the literature on the region of stability

$$V_3(\theta*) = \{\theta: F^=(\theta*) = F^=(\theta)\} \cap R_3(\theta*) .$$

(See [16, 19, 23].) Here we extend its validity to four different regions

$$Z_2(\theta*) = \{\theta: F(\theta) \subset F^=(\theta*)\} \cap Z_1(\theta*) ,$$

$$Z_3(\theta*) = \{\theta: F(\theta) \subset F^=(\theta*) \subset F^=(\theta)\} \cap R_3(\theta*) ,$$

$$R_5(\theta*) = \{\theta: F(\theta) \subset F(\theta*)\} \cap R_3(\theta*)$$

and

$$R_6(\theta*) = \{\theta: F(\theta) \subset F^=(\theta*)\} \cap R_1(\theta*) .$$

Note that $V_3(\theta*) \subset \{Z_2(\theta*) \cap Z_3(\theta*)\}$ but, generally, $V_3(\theta*)$ is different from $R_5(\theta*)$ and $R_6(\theta*)$. However, some extra assumptions are needed for the extension: The point-to-set mapping $F^=$ will always be assumed lower semicontinuous at $\theta*$; also, for $R_5(\theta*)$, we have to assume that $\operatorname{int} F(\theta*) \neq \emptyset$ (in which case $R_3(\theta*) \subset \subset R_4(\theta*)$ is indeed a region of stability) and also that $F(\theta*) = F^=(\theta*)$) in order to apply Theorem 4.2.).

Two crucial arguments used, in deriving the marginal value formula, are:

(i) continuity of the restricted Lagrange multiplier function $U_*^{<}(\theta)$, and

(ii) showing that $x = \tilde{x}(\theta^k) \in F(\theta^k)$ implies $x \in F^=(\theta*)$. The latter argument is obviously valid for the above four regions, so we verify validity only of the continuity argument: The argument is valid for $Z_2(\theta*)$ because

$$Z_2(\theta*) \subset Z_1(\theta*) \subset R_2(\theta*)$$

and $U_*^{\leq}(\theta)$ is continuous on $R_2(\theta*)$, under the lower semicontinuity assumption on $F^=$. (See [13, Theorem 2.1].) Since $R_5(\theta*) \subset R_3(\theta*)$, and the latter is a region of stability, if $F^=$ is lower semicontinuous and if $F(\theta*)$ has interior points, it follows that $R_5(\theta*)$ itself is a region of stability. The additional assumption $F(\theta*) = F^=(\theta*)$ guarantees continuity of $U_*^{\leq}(\theta)$, by our Theorem 4.2. The region $Z_3(\theta*)$ enjoys continuity because $Z_3(\theta*) \subset V_1(\theta*)$ and continuity is established on $V_1(\theta*)$ in [12, Theorem 3.1]. Note that $F^=$ is lower semicontinuous on $Z_3(\theta*)$, because of the requirement $F^=(\theta*) \subset F^=(\theta)$. Finally, $R_6(\theta*) \subset R_1(\theta*) \subset R_2(\theta*)$ and continuity, being established on $R_2(\theta*)$, guarantees continuity of $R_6(\theta*)$, provided, of course, that $F^=$ is lower semicontinuous. ■

To simplify notation, we use again the abbreviation $g(\theta)$ for the Lagrangian $L_*^{<}$ at $x = \tilde{x}(\theta*)$ and $u = \tilde{u}(\theta*)$.

6.1. **Theorem.** *Consider the bi-convex model* (P, θ) *at* $\theta = \theta* \in I$ *with a realistic objective function. Suppose that the corresponding saddle point* $\{\tilde{x}(\theta*), \tilde{u}_i(\theta*): i \in \mathscr{P}^<(\theta*)\}$ *is unique and that the index condition (IND) holds at* $\theta*$ *with respect to a region of stability* $S(\theta*) = Z_i(\theta*)$, $i = 2, 3$ *or* $S(\theta*) = R_i(\theta*)$, $i = 5, 6$, *and it is assumed that the mapping* $F^=$ *is lower semicontinuous at* $\theta*$, *and, in the case of* $R_5(\theta*)$, *that* $F(\theta*) = F^=(\theta*)$ *and that* $F(\theta*)$ *has interior points. Also, suppose that* $f^i(x, \cdot)$, $i \in \{0\} \cup \mathscr{P}^<(\theta*)$ *are differentiable functions in* $S(\theta*) \cap N(\theta*)$, *where* $N(\theta*)$ *is some neighbourhood of* $\theta*$, *and that the derivatives* $\nabla_\theta f^i(x, \theta)|_{\theta=\theta*}$, $i \in \{0\} \cup \mathscr{P}^<(\theta*)$ *are continuous functions in* x *at* $\tilde{x}(\theta*)$. *Then for every fixed path* $\theta^k \in S(\theta*)$, $\theta^k \to \theta*$ *such that the limit*

$$\lim_{\substack{\theta^k \in S(\theta*) \\ \theta^k \to \theta*}} \frac{\theta^k - \theta*}{\|\theta^k - \theta*\|} = l \tag{6.1}$$

exists, we have

$$\lim_{\substack{\theta^k \in S(\theta*) \\ \theta^k \to \theta*}} \frac{\tilde{f}(\theta^k) - \tilde{f}(\theta*)}{\|\theta^k - \theta*\|} = (\nabla g(\theta*), l)\,. \tag{6.2}$$

Proof. Take a sequence $\theta^k \in S(\theta*)$, $\theta^k \to \theta*$ such that the limit l in (6.1) exists. Without loss of generality we can assume that for this sequence

$$u_i(\theta^k) \to \tilde{u}_i(\theta*)\,, \quad i \in \mathscr{P}^<(\theta*)$$

by the continuity argument given prior to the theorem. We now follow the proof of Theorem 5.1 and arrive at the inequality

$$(\nabla g(\theta^k), \theta^k - \theta*) \geqq \tilde{f}(\theta^k) - \tilde{f}(\theta*) + \mathscr{E}(\theta^k) \tag{5.3}$$

for all sufficiently large k's. This gives an upper bound for $\tilde{f}(\theta^k) - \tilde{f}(\theta*)$. A lower bound is obtained from the estimate

$$\tilde{f}(\theta^k) - \tilde{f}(\theta*) \geqq L^<(\tilde{x}(\theta^k), u; \theta^k) - L^<_*(x, \tilde{u}(\theta*); \theta*) \tag{6.3}$$

valid for every $x \in F^=(\theta*)$ and every $u \in R_+^{q(\theta)}$, where $q(\theta) = \text{card}\, \mathscr{P}^<(\theta)$. (See [16].) Here we can specify $x = \tilde{x}(\theta^k) \in F(\theta^k) \subset F^=(\theta*)$, by the second argument given prior to the theorem, and

$$u_i = \begin{cases} \tilde{u}_i(\theta*)\,, & \text{if} \quad i \in \mathscr{P}^<(\theta*) \\ 0\,, & \text{if} \quad i \in \mathscr{P}^<(\theta^k) \setminus \mathscr{P}^<(\theta*)\,. \end{cases}$$

Now the lower bound

$$\tilde{f}(\theta^k) - \tilde{f}(\theta*) \geqq (\nabla_\theta L^<_*(\tilde{x}(\theta^k), \tilde{u}(\theta*); \theta)|_{\theta=\theta*}, \theta^k - \theta*)$$

follows from (6.3), convexity of $g(\theta)$, and the gradient inequality applied to $g(\theta)$ at $\theta*$. The next step is division by $\|\theta^k - \theta*\|$ and the limiting process $\theta^k \to \theta*$. Now the continuity properties of both the Lagrangian multiplier functions and constraints yield the marginal value formula (6.2) in the limit. (See [23] for details.) ■

The example that follows shows that the marginal value formula is now indeed applicable on larger regions of stability than $V_3(\theta*)$.

6.2. Example. Consider a bi-convex model with the following two constraints:

$$f^1(x, \theta) = 2 - x \leqq 0,$$
$$f^2(x, \theta) = \max\{0, x - \theta\} - (x - \theta) \leqq 0$$

where $\theta \in I = R$.

Here we find that $F(\theta) = [2, \infty) \cap [\theta, \infty)$, $\mathscr{P}^=(\theta) = \{2\}$ and $F^=(\theta) = [\theta, \infty)$ for every θ.

At $\theta* = 0$ we have

$$V_3(\theta*) = \{\theta: [0, \infty) = [\theta, \infty)\} \cap R = \{\theta*\}.$$

Since the point-to-set mapping $\Gamma: \theta \to F^=(\theta)$ is lower semicontinuous at $\theta*$, we conclude that

$$Z_2(\theta^*) = \{\theta: ([2, \infty) \cap [\theta, \infty)) \subset [0, \infty\} \cap (-\infty, 2]$$
$$= R \cap (-\infty, 2] = (-\infty, 2]$$

and

$$R_6(\theta*) = \{\theta: ([2, \infty) \cap [\theta, \infty)) \subset [0, \infty)\} \cap R = R$$

are regions of stability. On the other hand, since $\mathscr{P}^=(\theta)$ does not depend on θ, the index condition (IND) is satisfied over every region of stability. Hence we conclude that the marginal value formula is applicable at $\theta* = 0$ for any realistic bi-convex objective function, provided that the uniqueness and differentiability assumptions in the theorem hold.

7. COMPARISON OF REGIONS OF STABILITY

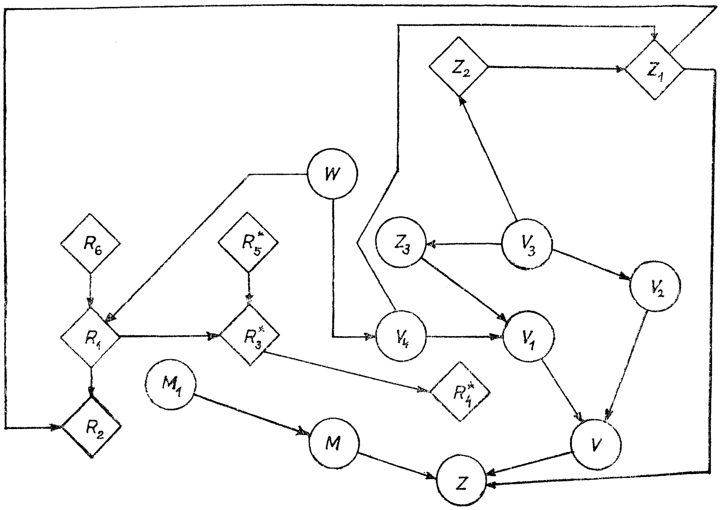
All presently used regions of stability are compared by inclusion below. For the sake of completeness we also include the regions

$$M_1(\theta*) = M(\theta*) \cap \{\theta: F(\theta) \subset F^=(\theta*)\}$$

and

$$W(\theta*) = R_1(\theta*) \cap \{\theta: F^=(\theta*) \subset F^=(\theta)\}$$

from, e.g. [22] and $V_4(\theta*) = \{\theta : F^=(\theta*) \subset F^=(\theta)\} \cap R_2(\theta*)$. The arrows mean inclusion. Thus $M_1 \to M$ means $M_1 \subset M$, etc. The sets in the "diamond shapes" are regions of stability under the additional assumption that the point-to-set mapping $\Gamma : \theta \to F^=(\theta)$ be lower semicontinuous at $\theta*$. The sets with an asterisk are regions of stability if, in addition to lower semicontinuity of $F^=$, the feasible set $F(\theta^*)$ has nonempty interior. Some regions are generally



incomparable, such as M and V. Among "smallest" regions of stability are R_6 and W, while some of the "largest" ones are R_2, R_4 and Z.

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Souhrn

NOVÉ OBLASTI STABILITY OPTIMALIZACE VSTUPNÍCH DAT

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S použitím zobrazení bod-množina se identifikují dvě nové oblasti stability optimalizace vstupních dat. Dále se rozšiřují různé výsledky z literatury týkající se podmínkek optimality, jako spojitost Lagrangeových multiplikátorů a formule pro marginální hodnoty, na nové a některé staré oblasti stability.

Резюме

НОВЫЕ ОБЛАСТИ УСТОЙЧИВОСТИ ОПТИМИЗАЦИИ ВХОДНЫХ ДАННЫХ

SHENG HUANG, S. ZLOBEC

При помощи многозначных отображений (переводящих точки в множества) устанавливаются две новые области устойчивости оптимизации входных данных и различные известные результаты, касаючиеся условий оптимальности, как напр. непрерывность множителей Лагранжа и формула для маргинальных значений, распространяются на новые и некоторые старые области устойчивости.

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