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THE DARBOUX THEOREM ON PLANE TRAJECTORIES OF TWO-PARAMETRIC SPACE MOTIONS

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Summary. The paper contains the proof of the classification theorem for two-parametric space motions with at least 5 points with plane trajectories. The proof is based on [1] and on the canonical form of a certain tensor of order 3. The second part of the paper deals with the problem of plane trajectories from the differential-geometrical point of view. Some applications are given.

Keywords: Kinematic geometry, differential geometry.

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1. INTRODUCTION

The Lie group $\mathscr{E}_3$ of all congruences of the Euclidean space E_3 has dimension 6. This means that for a 2-parametric space motion we can (in general) prescribe 4 surfaces as trajectories of 4 points. So we expect that there exist 2-parametric space motions which have 4 planes as trajectories of 4 points, and such a motion should be determined uniquely by those trajectories. This suggests a natural question about the existence of 2-parametric motions with more then 4 plane trajectories. This question was answered by G. Darboux in [1], where he gave an example of a 2-parametric motion with exactly 10 planes as trajectories of points. He also proved that "in general" a 2-parametric space motion can have at most 10 plane trajectories. To this end he introduced an ingenious method using quaternionic realization of $SO(3)$, and proceeded with rigorousness sufficient at his time. In the present paper we follow Darboux's ideas to give a complete proof of the classification theorem for 2-parametric space motions with at least 5 plane trajectories.

In the second part of the paper we discuss the same problem from the point of view of differential geometry and we also give some applications. Throughout the whole paper we use notation from [3] and suppose that all the 2-parametric space motions considered are regular in the sense that their spherical image is a 2-parametric motion as well.

2. QUADRATIC DARBOUX MOTIONS

Let D denote the ring of dual numbers. It is a 2-dimensional associative and commutative algebra over R with unit element 1 and with a base $\{1, e\}$, where $e^2 = 0$. Let V_n be an n-dimensional vector space over R. Then $D \otimes V_n$ is a $2n$-dimensional vector space over R which obtains a natural D-module structure by the requirement $\alpha(\beta \otimes u) = (\alpha\beta) \otimes u$ for $\alpha, \beta \in D, u \in V_n$. For the sake of simplicity we will leave out the sign of the tensor product in what follows.

Let $GL(D \otimes V_n)$ denote the group of all D-linear 1-1 maps of $D \otimes V_n$. If we choose a basis in V_n, then $GL(D \otimes V_n)$ is represented by the matrix group $GL(n, D)$, which is the group of all $n \times n$ matrices with dual entries and nonzero real part of the determinant. This is easy to see if we realize that any $g = a + eb \in GL(n, D)$ is given by the matrix $\tilde{g} = \begin{pmatrix} a, & b \\ 0, & a \end{pmatrix}$, where $D \otimes V_n$ is considered over R.

Any bilinear map (u, v) from $V_n \times V_n$ into R can be uniquely extended to a D-bilinear map from $(D \otimes V_n) \times (D \otimes V_n)$ into D by the requirement $(eu, v) = (u, ev) = e(u, v)$, $(eu, ev) = 0$ for all $u, v \in V_n$. Let us further denote by $O(n, D)$ the subgroup of $GL(n, D)$ given by the equation $g \,.\, g^{\mathrm{T}} = \mathbf{E}$, where $\mathbf{E}$ is the unit matrix, $g \in GL(n, D)$ and g^{T} is the transpose of g. $O(n, D)$ is a Lie group of dimension $n(n - 1)$ and it is given by matrices of the form $g = (1 + ex)\, a$, where $a \in O(n, R)$ and $x \in \mathfrak{O}(n, R)$, where $\mathfrak{O}(n, R)$ denotes the Lie algebra of $O(n, R)$. This is easy to see because $g \,.\, g^{\mathrm{T}} = (1 + ex)\, a \,.\, a^{\mathrm{T}}(1 + ex^{\mathrm{T}}) = a \,.\, a^{\mathrm{T}} + e(xaa^{\mathrm{T}} + aa^{\mathrm{T}}x)$ and so $aa^{\mathrm{T}} = \mathbf{E}, x + x^{\mathrm{T}} = 0$.

Theorem 1. *Let V_n be a vector space of dimension n over R with the Euclidean scalar product (u, v). Then the group of all R-linear maps of $D \otimes V_n$ which preserve the extended scalar product in $D \otimes V_n$ is isomorphic with the group $O(n, D)$.*

Proof. Let us choose an orthonormal basis in V_n and let T be an R-linear map of $D \otimes V_n$ which preserves the extended scalar product. Then $T(u_1 + eu_2) = mu_1 + nu_2 + e(pu_1 + ru_2)$, where $u_1, u_2 \in V_n$, $m, n, p, r \in GL(n, R)$. For $u = u_1 + eu_2, v = v_1 + ev_2$ we have $g(Tu, Tv) = g(u, v)$, where g is the scalar product in $D \otimes V_n$. Substitution yields $g(u_1, v_1) = g(mu_1 + nu_2, mv_1 + nv_2)$, $g(u_1, v_2) + g(u_2, v_1) = g(pu_1 + ru_2, mv_1 + nv_2) + g(mu_1 + nu_2, pv_1 + rv_2)$.

a) Put $u_1 = v_1 = 0$, $u_2 = v_2$. Then $0 = g(nu_2, nu_2)$, so $nu_2 = 0$ and $n = 0$.

b) The first equation gives $g(u_1, v_1) = g(mu_1, mv_1)$, so $m \in O(n, R)$.

c) Let us write $p = m \,.\, s$ and put $u_2 = v_2 = 0$. Then the second equation yields $0 = g(msu_1, mv_1) + g(mu_1, msv_1) = g(su_1, v_1) + g(u_1, sv_1)$, and so $s \in \mathfrak{O}(n, R)$.

d) Similarly as above let us write $r = m \,.\, w$ and use the second equation again, but with $v_1 = 0$. Then we get $g(u_1, v_2) = g(mu_1, mwv_2) = g(u_1, wv_2)$, so $g(u_1, v_2 - wv_2) = 0$ and $w = \mathbf{E}$. This proves that $T \in O(n, D)$. The converse is obvious.

Let $\mathscr{E}_3$ be the Lie group of all orientation preserving congruences of the Euclidean space E_3, and let $SO(n, D)$ be the subgroup of $O(n, D)$ of elements with determinant equal to 1.

Lemma 1. *$SO(3, D)$ and $\mathscr{E}_3$ are isomorphic Lie groups.*

Proof. Let us write

$$x^v = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} 0, & -x^3, & x^2 \\ x^3, & 0, & -x^1 \\ -x^2, & x^1, & 0 \end{pmatrix}.$$

Then the mapping $\varphi\colon \mathscr{E}_3 \to SO(3, D)$ given by

$$\varphi \begin{pmatrix} 1, & 0 \\ x^v, & a \end{pmatrix} = (1 + ex)\, a$$

is an isomorphism, because $(1 + ex)\, a(1 + ey)\, b = [1 + e(x + aya^{\mathrm{T}})]\, ab$,

$$\begin{pmatrix} 1, & 0 \\ x^v, & a \end{pmatrix} \begin{pmatrix} 1, & 0 \\ y^v, & b \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ x^v + ay^v, & ab \end{pmatrix},$$

and $ay^v = (aya^{\mathrm{T}})^v$.

Now let Q be the algebra of quaternions. Then $D \otimes Q$ obtains a D-algebra structure by defining $(u_1 + eu_2) \,.\, (u_2 + ev_2) = u_1u_2 + e(v_1u_2 + u_1v_2)$, where $u_1, u_2, v_1, v_2 \in \in Q$. This D-algebra is called the algebra of dual quaternions.

For any $\alpha = u_1 + eu_2 \in D \otimes Q$ we define $\bar{\alpha} = \bar{u}_1 + e\bar{u}_2$, $\alpha_e = u_1 - eu_2$, where the bar denotes the conjugate quaternion. A dual quaternion α is called a unit dual quaternion if $\alpha\bar{\alpha} = 1$. The set of all dual unit quaternions is a 6-dimensional Lie group U and its Lie algebra consists of all pure imaginary dual quaternions. This is easy to see, as $\alpha\bar{\alpha} = (u_1 + eu_2)(\bar{u}_1 + e\bar{u}_2) = u_1\bar{u}_1 + e(u_2\bar{u}_1 + u_1\bar{u}_2)$, so $u_1\bar{u}_1 = 1$, $u_2\bar{u}_1 + u_1\bar{u}_2 = 0$ (and for the tangent vectors at 1 we have $u_1 + \bar{u}_1 = 0$, $u_2 + \bar{u}_2 = 0$).

Lemma 2. *There is a Lie homomorphism of U onto $\mathscr{E}_3$ with the kernel $\{1, -1\}$.*

Proof. Let us consider the transformation $\varphi_\alpha(A) = \alpha A \bar{\alpha}_e$ of the Euclidean space E_3 of all quaternions A of the form $A = 1 + ex$, where x is a pure imaginary quaternion, $\alpha \in U$ (see [2]). Then we get

$$\varphi_\alpha(A) = (u_1 + eu_2)(1 + ex)(\bar{u}_1 - e\bar{u}_2) = 1 + e(u_1x\bar{u}_1 + u_2\bar{u}_1 - u_1\bar{u}_2),$$

where $u_2\bar{u}_1 - u_1\bar{u}_2 = u_1\bar{u}_2 - u_2\bar{u}_1$, so $u_2\bar{u}_1 - u_1\bar{u}_2$ is a pure imaginary quaternion. If $u_1 = x$, then $u_1x\bar{u}_1 = u_1u_1\bar{u}_1 = u_1 = x$. This shows that φ_α is a composition of a rotation with a translation, and so it belongs to $\mathscr{E}_3$.

Conversely, each translation and rotation can be obtained as φ_α for a suitable α, φ_α is a homomorphism of Lie groups by definition, and the kernel is obvious.

Lemma 3. *There is a Lie homomorphism of $U \times U$ on $SO(4, D)$ with kernel* $\{(1, 1), (-1, -1)\}$.

Proof. Let us consider the mapping $\varphi\colon U \times U \to GL(D \otimes Q)$, where $D \otimes Q$ is considered as a real vector space of dimension 8 and $\varphi(\alpha, \beta)\, u = \alpha u \bar{\beta}_e$, $\alpha, \beta \in U$, $u \in D \otimes Q$. Then φ is a homomorphism of Lie groups. Let us first consider the map $\varphi(\alpha, 1)$. Any unit quaternion α can be written in the form $\alpha = (1 + ex)\, a$, where a is a real unit quaternion and x is a pure imaginary quaternion. Further, $a(u_1 + eu_2) = au_1 + eau_2$. This means that multiplication by a preserves unit real quaternions, so $\varphi(a, 1) \in O(4, R)$. As det $\varphi(a, 1) = 1$, we have even $\varphi(a, 1) \in SO(4, R)$.

Further, $(1 + ex)(u_1 + eu_2) = u_1 + e(xu_1 + u_2)$, where multiplication by x is represented by a skew-symmetric matrix, as for instance $\mathbf{i}(1, \mathbf{i}, \mathbf{j}, \mathbf{k}) = (\mathbf{i}, -1, \mathbf{k}, -\mathbf{j})$, and similarly for $\mathbf{j}$ and $\mathbf{k}$. This shows that $x \in \mathfrak{SO}(4, R)$ and this yields $\varphi(\alpha, 1) \in SO(4, D)$. A similar result is true for the second variable β.

Now we have to show that φ is onto. Let $\psi \in SO(4, D)$ be arbitrary. Then ψ acts on unit dual quaternions by matrix multiplication (we choose the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$). Denote $\psi(1) = \gamma$. Then γ is a unit dual quaternion and the map $\varphi(\bar{\gamma}, 1)\, \psi$ preserves 1. If we restrict this map to Q only, we get an element of $SO(4, R)$ which preserves 1, so this element is from $SO(3, R)$. It is a rotation in the space of all imaginary quaternions, which is the Lie algebra of the Lie group Q_1 of all unit real quaternions. Each rotation in the Lie algebra of Q_1 can be realized by the adjoint mapping, so there is an element β in U such that $\varphi(\beta, \beta) = \varphi(\bar{\gamma}, 1)\, \psi$ on Q. This shows that we may suppose that ψ is identical on Q. Such a map is given by a skew-symmetric matrix, $u_1 + eu_2 \to u_1 + e(u_2 + xu_1)$, with x skew-symmetric. Such a mapping can be represented by $\varphi(1 + ea, 1 + eb)$ for suitable pure imaginary quaternions a and b, which is easily verified by computation. This proves that φ is onto.

In the end we shall find the kernel of φ. Let us suppose that $\alpha u \bar{\beta}_e = u$ for all $u \in D \otimes Q$. This yields that $\alpha \bar{\beta}_e = 1$, so $\beta = \alpha_e$. Further, $(1 + ex)\, a(u_1 + eu_2) \cdot \bar{a}(1 - ex)$ has the real part equal to $au_1\bar{a}$, so $a = \pm 1$. Now $(1 + ex)\, u_1(1 - ex) = u_1 + e(xu_1 - u_1 x)$ and this yields $x = 0$.

Definition 1. *Let G_1/H_1 and G_2/H_2 be two homogeneous spaces. We say that G_1/H_1 and G_2/H_2 are locally equivalent if there exists an isomorphism $\vartheta\colon \mathfrak{G}_1$ onto $\mathfrak{G}_2$ such that $\vartheta(\mathfrak{H}_1) = \mathfrak{H}_2$. Local equivalence of homogeneous spaces will be denoted* by $\cong$.

Theorem 2. $\mathscr{E}_3 \times \mathscr{E}_3/\mathscr{E}_3 \cong SO(4, D)/\, SO(3, D) \cong U \times U/U$, *where we identify $\mathscr{E}_3$ with Diag $(\mathscr{E}_3 \times \mathscr{E}_3)$ and similarly for U.*

Proof follows from Lemmas 1–3.

Remark. From the point of view of local differential geometry we may consider locally equivalent homogeneous spaces as equal. The homogeneous space $\mathscr{E}_3 \times \mathscr{E}_3/\mathscr{E}_3$ is the so called "kinematical space" of the space kinematics.

Theorem 3. *The kinematical space of space kinematics can be realized as a regular quadratic surface K of signature $(4, 4, 0)$ in the projective space P_7 with the transformation group preserving the degenerate scalar product of signature $(4, 0, 4)$ and the quadratic form defining K.*

Remark. The isotropy space P_3 of the scalar product is considered as removed.

Proof. The scalar product preserved by $SO(4, D)$ is given by the quadratic form $\alpha\bar{\alpha}$, which for $\alpha = u_1 + eu_2, u_1 = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $u_2 = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is given by $\sum_{i=0}^{3} a_i^2 + 2e\sum_{i=0}^{3} a_i b_i$. The statement now follows from Theorem 1.

Remark. Theorem 3 characterizes the space kinematics as a geometry of Möbius type with a metric group of transformations.

Theorem 4. *The 3-dimensional projective subspaces of K without isotropic points are the left and right translates of the unit sphere Q_1 given by the equations $u_1\bar{u}_1 = 1$, $u_2 = 0$, with the opposite points identified. If such a subspace contains 1, it is of the form $\alpha Q_1\bar{\alpha}$, $\alpha \in U$, and it corresponds to all rotations around the point $u_2\bar{u}_1 - u_1\bar{u}_2$, where $\alpha = u_1 + eu_2$.*

Proof. Let S be a 4-dimensional real vector subspace of $D \otimes Q$ which determines a 3-dimensional projective subspace of K, and let $1 \in S$. Further let $\alpha = u_0 + u + e(v_0 + v) \in S$ be linearly independent with 1, $u_0, v_0 \in R$, u, v being pure imaginary. Then $\lambda . 1 + \mu\alpha \in K$ for all $\lambda, \mu \in R$. This yields $v_0 = 0$. Adding a multiple of 1, we get $u_0 = 0$ as well. Thus we have $\sum_{i=1}^{3} a_i b_i = 0$, where $u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Now S can be changed by using the group Q_1 in such a way that $u = \mathbf{i}$, because $(1 + ex)(u + ev)(1 - ex) = u + e(v + xu - ux)$ and we may choose x such that $v + xu - ux = 0$, as $\sum_{i=1}^{3} a_i b_i = 0$. This shows that we may suppose $\alpha = \mathbf{i}$.

Now let $\beta = \lambda\mathbf{j} + \mu\mathbf{k} + ev \in S$. Then $\lambda^2 + \mu^2 \neq 0$ and β can be changed to $\beta = \mathbf{j} + ev$ by using Q_1. Then $v = 2\lambda\mathbf{k}$ for $\lambda \in R$ and we have $(1 - \lambda e\mathbf{i})(\mathbf{j} + 2e\lambda\mathbf{k}) . (1 + \lambda e\mathbf{i}) = \mathbf{j}$, $(1 - \lambda e\mathbf{i})\,\mathbf{i}(1 + \lambda e\mathbf{i}) = \mathbf{i}$. This shows that we may suppose $\lambda = 0$. If S contains 1, $\mathbf{i}$, $\mathbf{j}$, it contains $\mathbf{k}$ as well and it is unique.

Let $1 \in \alpha Q_1 \bar{\beta}_e$. Then $\alpha\bar{\beta}_e = 1$, $\bar{\beta}_e = \bar{\alpha}$ and for any $v \in Q_1$ we have

$$(\alpha v \bar{\alpha})\left[1 + e(u_2\bar{u}_1 - u_1\bar{u}_2)\right](\alpha_e \bar{v} \bar{\alpha}_e) = 1 + e(u_2\bar{u}_1 - u_1\bar{u}_2),$$

which is verified by direct computation.

Using Lemma 2 we can compute explicitly the matrix of the transformation of E_3, corresponding to the unit dual quaternion $\alpha = u_1 + eu_2$, $u_1 = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $u_2 = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. As the result we obtain the matrix

(1)

$$\varphi_\alpha = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 2(b_0a_1-b_1a_0+b_3a_2-b_2a_3), & a_0^2+a_1^2-a_2^2-a_3^2, & 2(a_1a_2+a_0a_3), & 2(a_1a_3-a_0a_2) \\ 2(b_0a_2-b_2a_0+b_1a_3-b_3a_1), & 2(a_1a_2-a_0a_3), & a_0^2-a_1^2+a_2^2-a_3^2, & 2(a_2a_3+a_0a_1) \\ 2(b_0a_3-b_3a_0+b_2a_1-b_1a_2), & 2(a_0a_2+a_1a_3), & 2(a_2a_3-a_0a_1), & a_0^2-a_1^2-a_2^2-a_3^2 \end{pmatrix}$$

Two-parametric space motions can now be studied with regard to the just mentioned representation of motion as a submanifold in the kinematical space. From this point of view it would be interesting to investigate the properties of motions which lie in linear subspaces of P_7. Theorem 4 may serve as an example of this approach.

For instance, we may investigate properties of motions which lie in a linear subspace P_6 of P_7. The simplest case occurs if we suppose that such P_6 is the tangent space of K at some isotropic point. As $U \times U$ acts transitively on the isotropic space, we may suppose that the point of contact of P_6 is the point e. The equation of the tangent space of P_6 at e is $a_0 = 0$. A special class of such motions was investigated by G. Darboux in [1]. In the following part of this section we will follow his ideas and present some of this results in a more up-to-date way.

Definition 2. *A two-parametric space motion given by the matrix* $g = \begin{pmatrix} 1, & 0 \\ t, & \gamma \end{pmatrix}$, *where*

(2)
$$\gamma = \begin{pmatrix} a_1^2 - a_2^2 - a_3^2, & 2a_1a_2, & 2a_1a_3 \\ 2a_1a_2, & -a_1^2 + a_2^2 - a_3^2, & 2a_2a_3 \\ 2a_1a_3, & 2a_2a_3, & -a_1^2 - a_2^2 + a_3^2 \end{pmatrix},$$

$$\sum_{\alpha=1}^{3} a_\alpha^2 = 1\,, \quad t = (t_1, t_2, t_3)^{\mathrm{T}}\,, \quad t_\alpha = \sum_{\beta,\gamma=1}^{3} m_\alpha^{\beta\gamma} a_\beta a_\gamma\,, \quad m_\alpha^{\beta\gamma} = m_\alpha^{\gamma\beta}\,,$$

will be called a *2-parametric Darboux quadratic motion.*

Remark. If we drop the condition

$$\sum_{\alpha=1}^{3} a_\alpha^2 = 1\,,$$

we can consider a_α as homogeneous parameters of the motion in P_7.

Remark. In what follows we will use the summation convention for indices $\alpha, \beta, \gamma, \lambda, \mu, \nu = 1, 2, 3$.

Theorem 5. *Let* $M = (m_\alpha^{\beta\gamma})$ *be as in Definition* 2. *Then* M *is a tensor with respect to* $SO(3, R)$.

Proof. The spherical part of a quadratic Darboux motion is given by the condition $a_0 = 0$. For the change of coordinates we therefore must have $\alpha\,.\,1\,.\,\bar{\beta} = 1$, so $\alpha = \beta$. Each dual quaternion β with $a_0 = 0$ can be written in the form $\beta =$

$= u_1(1 - 1/2et)$, where $u_1 \in Q_1$, $t = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$ are the components of the translation. For the change of the orthonormal basis by $\alpha \in Q_1$ we have $\tilde{\beta} = \alpha\beta\bar{\alpha}$, where

$$\tilde{\beta} = \tilde{u}_1(1 - 1/2e\tilde{t}) = \alpha u_1(1 - 1/2et)\,\bar{\alpha} = \alpha u_1\bar{\alpha} - 1/2e\alpha t\bar{\alpha} = \alpha u_1\bar{\alpha}(1 - 1/2e\alpha t\bar{\alpha})\,,$$

and so $\tilde{u}_1 = \alpha u_1\bar{\alpha}$, $\tilde{t} = \alpha t\bar{\alpha}$. Let γ^α_β be the corresponding matrix from $SO(3)$. Then

$$\tilde{a}_\alpha = \gamma^\beta_\alpha a_\beta\,,\quad \tilde{t}_\alpha = \gamma^\beta_\alpha t_\beta\,,\quad \tilde{t}_\alpha = \tilde{m}^{\beta\gamma}_\alpha \tilde{a}_\beta \tilde{a}_\gamma = \gamma^\lambda_\alpha t_\lambda = \tilde{m}^{\beta\gamma}_\alpha \gamma^\mu_\beta a_\mu \gamma^\nu_\gamma a_\nu = \gamma^\lambda_\alpha m^{\mu\nu}_\lambda a_\mu a_\nu\,,$$

so

$$\gamma^\lambda_\alpha m^{\mu\nu}_\lambda = \tilde{m}^{\beta\gamma}_\alpha \gamma^\mu_\beta \gamma^\nu_\gamma\,,\quad \tilde{m}^{\beta\gamma}_\alpha = \gamma^\alpha_\lambda m^{\mu\nu}_\lambda \gamma^\beta_\mu \gamma^\gamma_\nu$$

which was to be proved.

Remark. We write the symmetric indices as upper ones only for convenience.

Now we shall find the canonical form of the tensor $\boldsymbol{M}$ with respect to the change of coordinates.

Lemma 4. *The origin in the fixed and moving spaces can be changed in such a way that* $m^{\alpha\alpha}_\beta = m^{\alpha\beta}_\alpha = 0$.

Proof. Let r and s be two pure imaginary quaternions. Then $(1 + er)\,.\,1(1 + es) = 1$ and $(1 + er)\,u_1(1 - 1/2et)\,(1 + es) = u_1[1 + e(-1/2t + \bar{u}_1 r u_1 + s)]$ is the result of an arbitrary translation in the fixed and moving spaces.

Let $r = r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}$, $s = s_1\mathbf{i} + s_2 j + s_3\mathbf{k}$. Then

$$\begin{pmatrix}\tilde{t}_1\\ \tilde{t}_2\\ \tilde{t}_3\end{pmatrix} = \begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}(a_1^2 + a_2^2 + a_3^2) + \begin{pmatrix} a_1^2 - a_2^2 - a_3^2, & 2a_1a_2, & 2a_1a_3\\ 2a_1a_2, & -a_1^2 + a_2^2 - a_3^2, & 2a_2a_3\\ 2a_1a_3, & 2a_2a_3, & -a_1^2 - a_2^2 + a_3^2\end{pmatrix} .$$

$$\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix} + \begin{pmatrix}t_1\\ t_2\\ t_3\end{pmatrix}.$$

Computation yields

$$\tilde{m}^{\alpha\alpha}_\beta = m^{\alpha\alpha}_\beta + 3r_\beta - s_\beta\,,\quad \tilde{m}^{\alpha\beta}_\alpha = m^{\alpha\beta}_\alpha + r_\beta + 3s_\beta\,.$$

This completes the proof.

From now on we may suppose that $\boldsymbol{M}$ satisfies the condition from Lemma 4, and it remains to find the canonical form of $\boldsymbol{M}$ with respect to $SO(3)$. As the procedure is not quite straightforward, we shall present it in more detail.

It is well known that the quadratic form $(\boldsymbol{M}, \boldsymbol{M}) = m^{\beta\gamma}_\alpha m^{\beta\gamma}_\alpha$ is an $SO(3)$ invariant scalar product in the space V of all tensors $\boldsymbol{M}$. Tensors which are symmetrical in the upper indices and have both contractions equal to zero form an invariant subspace V_{12} of the space of all tensors of degree 3; V_{12} has dimension 12. In this space we have a completely reducible representation of $SO(3)$, given by the natural action of $SO(3)$.

We shall try to decompose V_{12} into invariant subspaces, in which we may describe the orbits of the action of $SO(3)$ more easily. The space of all completely symmetric tensors (without asuming zero contractions) is a vector subspace V_{10} of dimension 10 in V and it is given by conditions $a_i^{ij} = a_j^{ii}$, $a_k^{ij} = a_i^{jk}$, where (i, j, k) is an even permutation of $(1, 2, 3)$ and no summation is used. Similar convention will be used in the following. V_{10} is generated by the following independent components: three components m_i^{ii}, six components m_j^{ii} and one component $m_k^{ij} = m_3^{12}$. The condition for zero contractions is $m_j^{\alpha\alpha} = 0$ and it yields $m_i^{ii} = -m_i^{jj} - m_i^{kk}$. This shows that we have only 7 components left.

Lemma 5. *The space V_7 of all completely symmetric tensors with zero contraction has dimension* 7. *Its orthogonal complement V_5 in V_{12} is given by the following conditions*:

$$m_i^{ii} = 0\,, \quad m_j^{ji} = -1/2 m_i^{jj}\,, \quad m_i^{jj} + m_i^{kk} = 0\,, \quad m_3^{12} + m_1^{23} + m_2^{31} = 0\,.$$

Proof. Let $A = (a_\alpha^{\beta\gamma})$ be a tensor orthogonal to all symmetric tensors with zero contraction. Then $m_\alpha^{\beta\gamma} a_\alpha^{\beta\gamma} = 0$ for all $m_\alpha^{\beta\gamma}$ in V_7. This gives the following equations:

$$a_i^{ii} = 2a_j^{ji} + a_i^{jj}\,, \quad a_3^{12} + a_1^{23} + a_2^{31} = 0\,.$$

The contraction conditions in V_{12} are $a_i^{\alpha\alpha} = 0$, $a_\alpha^{\alpha i} = 0$ and so we get the conditions $a_i^{ii} = 2a_j^{ji} + a_i^{jj} = 2a_i^{ki} + a_i^{kk}$, $-a_i^{ii} = a_i^{jj} + a_i^{kk} = a_j^{ji} + a_k^{ki}$, which yield $2a_i^{ii} = 2a_j^{ji} + 2a_k^{ki} + a_i^{jj} + a_i^{kk} = -3a_i^{ii}$. This shows that $a_i^{ii} = 0$ and the statement follows.

Lemma 6. *The representation of $SO(3)$ in V_5 is equivalent to the natural representation of $SO(3)$ in the space of all symmetric matrices with trace zero and of order* 3.

Proof. After performing necessary transformations of the formula we get, for the change of $\boldsymbol{M} \in V_5$:

$$\begin{aligned} \tilde{m}_1^{22} = \gamma_\alpha^1 m_\alpha^{\beta\gamma} \gamma_\beta^2 \gamma_\gamma^2 = m_1^{22}[\gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_2^3] + m_2^{33}[\gamma_1^2\gamma_3^3 + \gamma_3^2\gamma_1^3] +& \\ + m_3^{11}[\gamma_1^2\gamma_2^3 + \gamma_2^2\gamma_1^3] + \tfrac{2}{3}[(m_2^{13} - m_3^{12})\,\gamma_1^2\gamma_1^3 + (m_3^{12} - m_1^{23})\,\gamma_2^2\gamma_2^3 +& \\ + (m_1^{23} - m_2^{13})\,\gamma_3^2\gamma_3^3]\,,& \end{aligned}$$

$$\begin{aligned} \tfrac{2}{3}(\tilde{m}_3^{12} - \tilde{m}_1^{23}) = 2m_1^{22}\gamma_2^2\gamma_3^2 + 2m_3^{11}\gamma_1^2\gamma_2^2 + 2m_2^{33}\gamma_1^2\gamma_3^2 +& \\ + \tfrac{2}{3}(m_2^{13} - m_3^{12})\,(\gamma_1^2)^2 + \tfrac{2}{3}(m_3^{12} - m_1^{23})\,(\gamma_2^2)^2 + \tfrac{2}{3}(m_1^{23} - m_2^{13})\,(\gamma_3^2)^2\,.& \end{aligned}$$

The remaining formulas are obtained by cyclic permutations of indices.

On the other hand, if a_j^i is a symmetric matrix of degree 3, then for its transformation under $SO(3)$ we have $(\tilde{a}_\beta^\alpha) = \gamma_\mu^\alpha a_\mu^\nu \gamma_\nu^\beta$ and therefore

$$\begin{aligned} \tilde{a}_3^2 = a_3^2(\gamma_3^2\gamma_2^3 + \gamma_2^2\gamma_3^3) + a_1^2(\gamma_1^2\gamma_2^3 + \gamma_2^2\gamma_1^3) + a_1^3(\gamma_1^2\gamma_3^3 + \gamma_3^2\gamma_1^3) +& \\ + a_1^1\gamma_1^2\gamma_1^3 + a_2^2\gamma_2^2\gamma_2^3 + a_3^3\gamma_3^2\gamma_3^3\,,& \end{aligned}$$

$$\tilde{a}_2^2 = 2\gamma_2^2\gamma_3^2 a_3^2 + 2\gamma_1^2\gamma_2^2 a_2^1 + 2\gamma_1^2\gamma_3^2 a_3^1 + a_1^1(\gamma_1^2)^2 + a_2^2(\gamma_2^2)^2 + a_3^3(\gamma_3^2)^2 .$$

If we now define the correspondence χ so that $\chi(m_j^{ii}) = a_k^i$, $\chi(\frac{2}{3}(m_j^{ik} - m_k^{ij})) = a_i^i$, we immediately see that χ determines an equivalence of representations.

The orbits of the natural representation of $SO(3)$ in the space of symmetric matrices are well known; such matrices can be diagonalized. This shows that we may always choose frames in the fixed and moving spaces in such a way that $m_j^{ii} = 0$ for the projection of $\boldsymbol{M}$ into V_5.

Theorem 6. *Let a quadratic Darboux motion be given by its tensor $\boldsymbol{M} = (m_\gamma^{\alpha\beta})$. Then there exist frames in the fixed and moving spaces such that $\boldsymbol{M}$ satisfies the following conditions*:

$$(3) \qquad m_i^{ij} = m_j^{ii}, \quad m_j^{\alpha\alpha} = 0 .$$

Such frames are uniquely determined iff $m_j^{ik} - m_k^{ij} \neq m_k^{ji} - m_i^{jk}$.

Proof. For the projection A of $\boldsymbol{M}$ into V_5 from Lemma 5 we have $a_j^{ii} = 1/3(m_i^{ij} - m_j^{ii})$. From Lemma 5 we know that we may choose frames such that $a_j^{ii} = 0$. These frames are uniquely determined provided the diagonal elements of the matrix $\chi(A)$ from Lemma 5 are mutually different. This completes the proof.

Remark. From now on we will suppose that $\boldsymbol{M}$ satisfies (3), and call it the reduced form of $\boldsymbol{M}$.

G. Darboux in [1] found all plane trajectories of a quadratic Darboux motion and gave an explicit example of a motion which attains the maximal number of plane trajectories. We will repeat his reasoning as follows:

Let a two-parametric quadratic Darboux motion be given by (2). Then the trajectory X, Y, Z of the point x, y, z is given by the following equations:

$$X(a_1^2 + a_2^2 + a_3^2) = t_1(a_1, a_2, a_3) + (a_1^2 - a_2^2 - a_3^2)\, x + 2a_1a_2y + 2a_1a_3z ,$$

$$Y(a_1^2 + a_2^2 + a_3^2) = t_2(a_1, a_2, a_3) + 2a_1a_2x + (-a_1^2 + a_2^2 - a_3^2)\, y + 2a_2a_3z ,$$

$$Z(a_1^2 + a_2^2 + a_2^3) = t_3(a_1, a_2, a_3) + 2a_1a_3x + 2a_2a_3y + (-a_1^2 - a_2^2 + a_3^2)\, z ,$$

where $t(a_1, a_2, a_3) = m_\beta^{\alpha\gamma} a_\beta a_\gamma$. We ask whether there exist numbers

$$l_s,\ s = 0, \ldots, 3, \sum_{\alpha=1}^{3} l_\alpha^2 \neq 0$$

such that $l_0 + l_1X + l_2Y + l_3Z = 0$ for all a_1, a_2, a_3 and given x, y, z.

Substitution yields

$$(4) \qquad \begin{aligned} &l_0 + l_1x - l_2y - l_3z + P_1 = 0 , \qquad && l_2z + l_3y + P_1' = 0 , \\ &l_0 - l_1x + l_2y - l_3z + P_2 = 0 , && l_1z + l_3x + P_2' = 0 , \\ &l_0 - l_1x - l_2y + l_3z + P_3 = 0 , && l_1y + l_2x + P_3' = 0 , \end{aligned}$$

where $P_i = l_\alpha m_\alpha^{ii}$, $P'_i = l_\alpha m_\alpha^{jk}$.

Consequently,

$$l_1 P'_1 + l_2 P'_2 - l_3 P'_3 + 2l_1 l_2 z = 0\,, \quad P_1 + P_2 + 2l_0 - 2l_3 z = 0\,.$$

Let us first suppose that $l_1 l_2 l_3 \neq 0$. Then

$$(P_1 + P_2)\, l_1 l_2 + l_3(l_1 P'_1 + l_2 P'_2 - l_3 P'_3) + 2l_0 l_1 l_2 = 0$$

and by cyclic permutations

$$\begin{aligned} -2l_0 l_1 l_2 l_3 &= (P_1 + P_2)\, l_1 l_2 l_3 + l_3^2(l_1 P'_1 + l_2 P'_2 - l_3 P'_3) = \\ &= (P_2 + P_3)\, l_1 l_2 l_3 + l_1^2(-l_1 P'_1 + l_2 P'_2 + l_3 P'_3) = \\ &= (P_1 + P_3)\, l_1 l_2 l_3 + l_2^2(l_1 P'_1 - l_2 P'_2 + l_3 P'_3)\,. \end{aligned}$$

This yields

$$(5) \qquad (l_i^2 + l_j^2)(l_j P'_j - l_i P'_i) + l_k P'_k(l_i^2 - l_j^2) - l_i l_j l_k (P_i - P_j) = 0\,,$$

where (i, j, k) is again a cyclic permutation of $(1, 2, 3)$.

Proposition 1. *A quadratic Darboux motion is spherical iff the reduced form of M is equal to zero.*

Proof. Let a quadratic Darboux motion be spherical. Then (5) must be satisfied for any l_1, l_2, l_3. If we write out explicitly one of (5), we obtain

$$(6) \qquad \begin{aligned} & l_1^2[m_1^{23}(l_2^2 - l_3^2) + l_2 l_3(m_1^{33} - m_1^{22})] + \\ & + l_1[l_2^2 l_3(3m_2^{33} - 2m_2^{11}) - l_2 l_3^2(3m_3^{22} + 2m_3^{11}) + l_2^3(m_3^{22} - m_3^{11}) + l_3^3(m_2^{11} - m_2^{33})] + \\ & + (l_2^2 + l_3^2)[l_3^2 m_3^{12} + l_2 l_3(m_1^{22} - m_1^{33}) - l_2^2 m_2^{13}] = 0\,. \end{aligned}$$

This yields $m_k^{ij} = 0$ and $m_j^{ii} = m_j^{kk}$, $2m_j^{ii} = 3m_j^{kk}$, so $m_j^{ii} = 0$. This completes the proof.

Theorem 7. (*Darboux*). *If a quadratic Darboux motion has a finite number of plane trajectories, then they are at most* 10.

Proof. For the normal vector of the plane trajectory we have equations (5), which determine two curves of degree 4 in the projective plane with homogeneous coordinates l_1, l_2, l_3. Then l_0 is uniquely determined by l_α. This means that we have at most 16 solutions for l_α. Six of them are of the form

$$(7) \qquad l_i = 0\,, \quad l_j P'_j - l_k P'_k = 0\,.$$

Further, we see that any plane trajectory with $l_i = 0$ must satisfy (5) and (4). We shall prove that any plane trajectory with $l_i \doteq 0$ corresponds to a double solution of (5) and so it was already counted. So let $l_1 = 0$, l_{20}, l_{30} be a solution of (5). Let us denote by P_{i0}, P'_{i0} the corresponding expressions for P_i, P'_i with $l_1 = 0$, $l_2 = l_{20}$, $l_3 = l_{30}$ substituted.

Then we must have

$$l_2 z + l_3 y + P'_{10} = l_3 x + P'_{20} = l_{2x} + P'_{30} = 0\,,$$

$$l_0 - l_{20} y - l_{30} z + P_{10} = l_0 + l_{20} y - l_{30} z + P_{20} = l_0 - l_{20} y + l_{30} z + P_{30} = 0\,.$$

This yields

$$l_{20} y = 1/2(P_{10} - P_{20})\,, \quad l_{30} z = 1/2(P_{10} - P_{30})$$

and we obtain

$$\begin{aligned} &l_{20} P'_{20} - l_{30} P'_{30} = 0\,, \\ &l_{20}^2 (P_{10} - P_{30}) + l_{30}^2 (P_{10} - P_{20}) + 2 l_{20} l_{30} P'_{10} = 0\,. \end{aligned} \tag{8}$$

Let us compute the tangent lines of (5) at the common point $l_1 = 0$, $l_2 = l_{20}$, $l_3 = l_{30}$, where $l_{20} P'_{20} - l_{30} P'_{30} = 0$. Let $l_{20} l_{30} \neq 0$ (otherwise it is easy). Two equations of (5) are

$$l_1^2 (l_1 P'_1 - l_3 P'_3 - l_2 P'_2) + l_3^2 (l_1 P'_1 - l_3 P'_3 + l_2 P'_2) + l_1 l_2 l_3 (P_1 - P_3) = 0\,,$$

$$l_1^2 (l_2 P'_2 - l_1 P'_1 + l_3 P'_3) + l_2^2 (l_2 P'_2 - l_1 P'_1 - l_3 P'_3) + l_1 l_2 l_3 (P_2 - P_1) = 0\,.$$

Equations of the tangent lines are

$$l_1 \left\{ l_{30}^2 \left[P'_{10} - l_{30} \left(\frac{\partial P'_3}{\partial l_1} \right)_0 + l_{20} \left(\frac{\partial P'_2}{\partial l_1} \right)_0 \right] + (P_{10} - P_{30})\, l_{20} l_{30} \right\} +$$

$$+ l_2 l_{30}^2 \frac{\partial}{\partial l_2} (l_2 P'_2 - l_3 P'_3)_0 + l_3 l_{30}^2 \frac{\partial}{\partial l_3} (l_2 P'_2 - l_3 P'_3)_0 = 0,$$

$$l_1 \left\{ l_{20}^2 \left[-P'_{10} - l_{30} \left(\frac{\partial P'_3}{\partial l_1} \right)_0 + l_{20} \left(\frac{\partial P'_2}{\partial l_1} \right)_0 \right] + (P_{20} - P_{10})\, l_{20} l_{30} \right\} +$$

$$+ l_2 l_{20}^2 \frac{\partial}{\partial l_2} (l_2 P'_2 - l_3 P'_3)_0 + l_3 l_{20}^2 \frac{\partial}{\partial l_3} (l_2 P'_2 - l_3 P'_3)_0 = 0\,.$$

These two lines coincide iff

$$2 l_{20}^2 l_{30}^2 P'_{10} + l_{20}^3 l_{30} (P_{10} - P_{30}) + l_{20} l_{30}^3 (P_{10} - P_{20}) = 0$$

and this is what we need to complete the proof.

Example 1 (Darboux). Let us consider the case $m_k^{ij} = 1$, otherwise $m_\gamma^{\alpha\beta} = 0$. Then $P_i = 0$, $P'_i = l_i$, and (5) reads

$$(l_i^2 - l_j^2)(l_i^2 + l_j^2 - l_k^2) = 0.$$

We have to consider the following cases:

1) $l_i^2 = l_j^2 = l_k^2$, so $l_2 = \varepsilon_2 l_1$, $l_3 = \varepsilon_3 l_1$, $\varepsilon_2, \varepsilon_3 = \pm 1$. The solution is: the point A with a plane trajectory is

$$A = [-1/2\varepsilon_2\varepsilon_3, -1/2\varepsilon_3, -1/2\varepsilon_2],$$

the plane trajectory is given by the equation

$$-1/2\varepsilon_2\varepsilon_3 + X + \varepsilon_2 Y + \varepsilon_3 Z = 0.$$

We have 4 solutions corresponding to the choice of $\varepsilon_2, \varepsilon_3$.

2) $l_i = 0$, $l_k = \varepsilon_k l_j$, $\varepsilon_k = \pm 1$. The point with a plane trajectory is $A = [\varepsilon, 0, 0]$, the plane trajectory is given by the equation $Y + \varepsilon Z = 0$, $\varepsilon = \pm 1$, together with the cyclic permutations. We have 6 solutions in this case.

Remark. Example 1 proves the existence of a 2-parametric space motion with exactly 10 plane trajectories.

At the end of this chapter we will discuss the case when the quadratic Darboux motion has infinitely many plane trajectories. This can occur only if (5) has infinitely many solutions. It is easy to see that the curves given by (5) cannot coincide, because in such a case each variable would be at most in the second power and similarly as in the proof of Proposition 1 we should get that the reduced form of M is zero.

This means that (5) can have infinitely many solutions only if the curves given by (5) split into curves of lower orders. Such cases must be discussed separately for each possibility:

1) Let (5) have a common conic section $Al_1^2 + Bl_2^2 + Cl_3^2 + Dl_1l_2 + El_1l_3 + Fl_2l_3 = 0$.

a) Let $A \neq 0$. Then we may suppose $A = 1$ and (6) can be written as

$$(l_1^2 + Bl_2^2 + Cl_3^2 + Dl_1l_2 + El_1l_3 + Fl_2l_3)(B_1l_2^2 + C_1l_3^2 + F_1l_2l_3) = 0.$$

Comparison with (6) yields $B_1 = -C_1 = m_1^{23}$, $F_1 = m_1^{33} - m_1^{22}$, where $(m_1^{23})^2 + (m_1^{33} - m_1^{22})^2 \neq 0$. From terms not containing l_1 we obtain

$$2Fm_1^{23} + (B - C)(m_1^{33} - m_1^{22}) = 0,$$

$$F(m_1^{33} - m_1^{22}) - 2(B - C)m_1^{23} = 0.$$

This yields $B = C$, $F = 0$.

Now we use the second equation from (5). $B \neq 0$ implies $B = C = A = 1$. $E = M = 0$, the conic section is $\sum_{\alpha=1}^{3} l_\alpha^2 = 0$ and we have no real solution. If $B = 0$, then the common conic section is given by $l_1(l_1 + Dl_2 + El_3) = 0$. $l_1 = 0$ is a solution for all l_2 and l_3 iff $l_2^2(P_1 - P_3) + l_3^2(P_1 - P_2) + 2l_2l_3P_1' = 0$ for all l_2, l_3. This yields $D = E = 0$ and the common conic section is $l_1^2 = 0$, which is a special case of a common straight line; this case will be treated later on.

b) Let $A = B = C = 0$, $DEF \neq 0$. The common conic section is $Dl_1l_2 + El_1l_3 + Fl_2l_3 = 0$. (6) can be written as

$$(Dl_1l_2 + El_1l_3 + Fl_2l_3)(A_1l_1^2 + B_1l_2^2 + C_1l_3^2 + D_1l_1l_2 + E_1l_1l_3 + F_1l_2l_3) = 0.$$

We have no terms with l_2^4 and l_3^4, so $m_3^{12} = m_1^{23} = m_2^{13} = 0$, $A_1 = 0$. The term with l_1^2 gives $D_1 = E_1 = 0$, $m_1^{33} = m_1^{22}$ and $C_1 = F_1 = B_1 = 0$, which is impossible. A similar result is obtained for $D = 0$.

2) Let (5) have a common cubic curve. Easy computation shows that this cubic curve must be singular.

3) Let (5) have a common straight line $Al_1 + Bl_2 + Cl_3 = 0$ with $ABC \neq 0$. Let us denote $m_i^{kk} - m_i^{jj} = \alpha_i$, $3m_2^{33} + 2m_2^{11} = \beta$ $3m_3^{22} + 2m_3^{11} = \gamma$. Then (6) changes to

$$(9) \qquad l_1^2[m_1^{23}(l_2^2 - l_3^2) + l_2l_3\alpha_1] + l_1(l_2^3\alpha_3 + l_3^3\alpha_2 + l_2^2l_3\beta - l_2l_3^2\gamma) + \\ + (l_2^2 + l_3^2)(l_3^2m_3^{12} - l_2^2m_2^{13} - l_2l_3\alpha_1) = 0\,.$$

Let us write (6) in the form

$$(10) \qquad (Al_1 + Bl_2 + Cl_3)(A_1l_1l_2^2 + B_1l_1l_3^2 + C_1l_1l_2l_3 + D_1l_2^3 + \\ + E_1l_2^2l_3 + F_1l_2l_3^2 + H_1l_3^3) = 0\,,$$

Comparison of (9) and (10) gives the following equations:

$$(11) \qquad AA_1 = -AB_1 = m_1^{23}\,, \quad AC_1 = \alpha_1\,, \quad AD_1 + A_1B = \alpha_3\,,$$
$$AH_1 + B_1C = \alpha_2\,, \quad AE_1 + A_1C + BC_1 = \beta\,, \quad AF_1 + B_1B + C_1C = -\gamma\,,$$
$$BE_1 + CD_1 = -\alpha_1\,, \quad BH_1 + CF_1 = -\alpha_1\,, \quad BD_1 = -m_2^{13}\,,$$
$$CH_1 = m_3^{12}\,, \quad BF_1 + CE_1 = m_3^{12} - m_2^{13}\,.$$

For A, B, C we obtain the following equations:

$$\alpha_3AB = B^2m_1^{23} - A^2m_2^{13}\,,$$
$$\alpha_1(A^2 - B^2) - 2BCm_1^{23} + \alpha_3AC + \beta AB = 0\,,$$
$$\alpha_1(A^2 - C^2) + 2BCm_1^{23} - \gamma AC + \alpha_2AB = 0\,,$$
$$A^2(m_3^{12} - m_2^{13}) = m_1^{23}(B^2 - C^2) - 2\alpha_1BC - \gamma AB + \beta AC\,,$$

and their cyclic permutations.

By forming suitable combinations, we arrive at two equations

$$\alpha_1BCm_1^{23} + \alpha_2ACm_2^{13} + \alpha_3ABm_3^{12} = 0\,, \quad \alpha_1BC + \alpha_2AC + \alpha_3AB = 0\,.$$

Let us denote $m_i^{jk} - k_j^{ki} = \lambda_k$, $m_i^{kk} - m_i^{jj} = \alpha_i$, $m_i^{kk} + m_i^{jj} = \beta_i$. Then we obtain (up to a constant factor)

$$(12) \qquad A = \alpha_1\lambda_2\lambda_3\,, \quad B = \alpha_2\lambda_1\lambda_3\,, \quad C = \alpha_3\lambda_1\lambda_2\,.$$

The remaining equations are

$$(13) \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0\,,$$

$$(14) \qquad \alpha_2\lambda_1(\alpha_1\lambda_2\alpha_3 - \alpha_2\lambda_1\lambda_3) = (\alpha_2^2\lambda_1^2 - \alpha_1^2\lambda_2^2)\, m_2^{13}\,,$$

$$(15) \qquad \alpha_1\lambda_3(\alpha_1^2\lambda_2^2 - \alpha_2^2\lambda_1^2) - 2\alpha_2\alpha_3\lambda_1^2\lambda_2 m_1^{23} + \alpha_3^2\alpha_1\lambda_1\lambda_2^2 + $$
$$+ 1/2\alpha_1\alpha_2\lambda_1\lambda_2\lambda_3(5\beta_2 - \alpha_2) = 0\,,$$

and their cyclic permutations. This means that it is enough to choose a solution of (13) and (14) and we get a solution of our problem. Equations (13) and (14) have sufficiently many solutions (for instance $\alpha_i = \lambda_i$ is such a solution).

For the point (x, y, z) which has a plane trajectory we have $l_1P_1' + l_2P_2' - l_3P_3' = -2l_1l_2z$.

Substitution gives

$$l_1^2m_1^{23} + l_2^2m_2^{13} - l_3^2m_3^{12} + l_1l_2\alpha_3 - l_1l_3\alpha_2 + l_2l_3\alpha_1 = 2l_1l_2z\,.$$

If we now substitute for l_3 from $Al_1 + Bl_2 + Cl_3 = 0$, for A, B, C from (12) and then from (13) and (14), we see that $z =$ const. A similar result is true for x and y. This shows that infinitely many plane trajectories correspond to a straight trajectory of a point. We obtain a similar result in the case $ABC = 0$.

Theorem 8. *A quadratic Darboux motion has infinitely many plane trajectories iff it has a point with a straight trajectory. In addition to this point it can have six plane trajectories. In the case* $\alpha_1\alpha_2\alpha_3\lambda_1\lambda_2\lambda_3 \neq 0$ *such motions are determined by the solutions of* (13)–(15).

Proof. If we remove the common line $Al_1 + Bl_2 + Cl_3 = 0$ from (5), we get two cubic curves with at most 9 common points. Inspection of equation (6) shows that 3 of those points are points with $l_i = 0$, which are not counted.

Remark. For the sake of simplicity we shall call planes p_1, p_2, p_3 independent if their normal vectors are linearly independent.

Theorem 9. (*Darboux*). *Let a 2-parametric space motion have* 4 *plane trajectories, each* 3 *of them independent. Then, if such a motion has one more plane trajectory, it is either a quadratic Darboux motion, or all* 5 *points with plane trajectories lie on a straight line and each point of this line has a plane curve as its trajectory.*

Proof. Let a matrix $\gamma = (\gamma_{\alpha\beta})$ of a spherical motion be given by a unit quaternion $\alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Then there exists a 1-1 linear correspondence between the homogeneous quadratic polynomials in a_α and the linear equations in $\gamma_{\alpha\beta}$, given by (1). Explicitly, we have

$$1 + \gamma_{ii} - 2a_0^2 = 2a_i^2\,, \quad i = 1, 2, 3\,, \quad \gamma_{\alpha\alpha} + 1 = 4a_0^2\,,$$
$$\gamma_{12} = 2(a_1a_2 + a_0a_3)\,, \quad \gamma_{21} = 2(a_1a_2 - a_0a_3)$$

and similarly for other $\gamma_{\alpha\beta}$. This shows that from any quadratic homogeneous polynomial in a_α we get a linear equation in $\gamma_{\alpha\beta}$ and the correspondence is linear. The converse is obvious from (1), only the constant term of the linear equation must be multiplied by $1 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Now let a motion be given by the matrix $g = \begin{pmatrix} 1, & 0 \\ t_\alpha, & \gamma_{\alpha\beta} \end{pmatrix}$.

The trajectory X^i of the point $x^i = (1, x_1^i, x_2^i, x_3^i)^T$ is given by

$$X^i = \begin{pmatrix} 1, & 0 \\ t, & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix} = \begin{pmatrix} 1 \\ t + \gamma x^i \end{pmatrix}.$$

The trajectory X^i of x^i lies in a plane iff there exist $l_j^i, j = 1, \ldots, 4$, such that

$$l_4^i + \sum_{j=1}^{3} l_j^i (t_j + \sum_{\alpha=1}^{3} \gamma_{\alpha j} x^i) = 0 .$$

Let points x^1, x^2, x^3 have plane trajectories given by l^1, l^2, l^3 such that $|l_j^i| \neq 0$, $i, j = 1, 2, 3$, where vertical lines denote the determinant. Then we have

$$\sum_{j=1}^{3} l_j^i t_j = -l_4^i - \sum_{\alpha, j=1}^{3} l_j^i \gamma_{j\alpha} x_\alpha^i , \quad i = 1, 2, 3 .$$

The solution for t_j is

(16) $$t_1 = -|l_1, l_2, l_3|^{-1} [|l_4, l_2, l_3| + \sum_{j,\alpha=1}^{3} \gamma_{j\alpha} |l_j x_\alpha, l_2, l_3|] .$$

$$t_2 = -|l_1, l_2, l_3|^{-1} [|l_1, l_4, l_3| + \sum_{j,\alpha=1}^{3} \gamma_{j\alpha} |l_1, l_j x_\alpha, l_3|] ,$$

$$t_3 = -|l_1, l_2, l_3|^{-1} [|l_1, l_2, l_4| + \sum_{j,\alpha=1}^{3} \gamma_{j\alpha} |l_1, l_2, l_j x_\alpha|] ,$$

where l_j denotes l_j^i while $l_j x_\alpha$ denotes $l_j^i x_x^i$, $\alpha, i, j = 1, 2, 3$, written in columns.

The substitution into the 4-th plane trajectory gives the equation

(17) $$- |l_1, l_2, l_3, l_4| + \sum_{\alpha, j=1}^{3} \gamma_{j\alpha} |l_j x_\alpha, l_1, l_2, l_3| = 0 ,$$

where l_j now denotes l_j^i, $l_j x_\alpha$ denotes $l_j^i x_\alpha^i$, $i, j = 1, 2, 3, 4$, $\alpha = 1, 2, 3$. (17) is a linear equation in $\gamma_{\alpha\beta}$, so it gives a homogeneous quadratic equation in $a_0, \ldots, a_3$.

First we shall show that (17) is a nontrivial equation. So, let $|l_j x_\alpha, l_1, l_2, l_3| = 0$ for all $\alpha, j = 1, 2, 3$. Then $|l_1, l_2, l_3, l_4| = 0$ and

$$l_j^4 = \sum_{i=1}^{3} \Theta^i l_j^i , \quad \text{with} \quad \sum_{i=1}^{3} (\Theta^i)^2 \neq 0 .$$

This yields

$$l_j^4 x_\alpha^4 = \sum_{i=1}^{3} \Theta^i l_j^i x_\alpha^i .$$

Substitution for l_j^4 yields

$$(\sum_{i=1}^{3} \Theta^i l_j^i)\, x^4 = \sum_{i=1}^{3} \Theta^i l_j^i x_\alpha^i \quad \text{and} \quad \sum_{i=1}^{3} \Theta^i l_j^i (x_\alpha^4 - x_\alpha^i) = 0\,.$$

As $x^4 \neq x^i$, $i = 1, 2, 3$, the last system of equations must have a nontrivial solution, and this means that the determinant of this system must be equal to zero. The determinant of the system is equal to $\Theta^1 \Theta^2 \Theta^3 |l_1, l_2, l_3|$, where l_j denotes l_j^i, $i, j = 1, 2, 3$.

As $|l_1, l_2, l_3| \neq 0$, we have $\Theta^1 \Theta^2 \Theta^3 = 0$. Let $\Theta^3 = 0$. Then $l^4 = \Theta^1 l^1 + \Theta^2 l^2$, which is a contradiction with $|l_i^1, l_i^2, l_i^4| \neq 0$, $i = 1, 2, 3$, as any 3 plane trajectories are independent by assumption. This completes the proof.

So far we have shown that g is a two-parametric motion uniquely determined by points x^i and planes l^i, $i = 1, 2, 3, 4$. Let us now have one more point x^5 which has a plane trajectory determined by l^5. This gives a new homogeneous quadratic equation in $a_0, \ldots, a_3$. If we consider $a_0, \ldots, a_3$ as homogeneous coordinates in the projective space P_3, we get two quadratic surfaces in this space. Two quadratic surfaces in P_3 have a 2-dimensional intersection iff they coincide or if they have a common plane. If they have a common plane, the motion is a quadratic Darboux motion – the spherical part satisfies a linear equation and translations are quadratic in $a_0, \ldots, a_3$, as we see from (17).

Consequently, we can suppose that the quadratic surfaces determined by (17) are identical. This means that the corresponding linear equations given by (17) are proportional. Hence there exists a number $\Theta^4 \neq 0$ such that

$$\begin{vmatrix} l_j^1 x_\alpha^1, & l_1^1, & l_2^1, & l_3^1 \\ l_j^2 x_\alpha^2, & l_1^2, & l_2^2, & l_3^2 \\ l_j^3 x_\alpha^3, & l_1^3, & l_2^3, & l_3^3 \\ l_j^5 x_\alpha^5, & l_1^5, & l_2^5, & l_3^5 \end{vmatrix} = \Theta^4 \begin{vmatrix} l_j^1 x_\alpha^1, & l_1^1, & l_2^1, & l_3^1 \\ l_j^2 x_\alpha^2, & l_1^2, & l_2^2, & l_3^2 \\ l_j^3 x_\alpha^3, & l_1^3, & l_2^3, & l_3^3 \\ l_j^4 x_\alpha^4, & l_1^4, & l_2^4, & l_3^4 \end{vmatrix}$$

for all $j, \alpha = 1, 2, 3$. This yields

$$l_i^5 = \sum_{j=1}^{4} \Theta^j l_i^j\,, \quad l_i^5 x_\alpha^5 = \sum_{j=1}^{4} \Theta^j l_i^j x_\alpha^j\,, \quad \text{where} \quad \Theta^1 \Theta^2 \Theta^3 \Theta^4 \neq 0\,,$$

$i, \alpha = 1, 2, 3$. (We know only that $\Theta^4 \neq 0$, but all x^i and l^i play symmetrical roles in the discussion.) Now, following Darboux, let us consider the following system of linear equations:

$$\sum_{j=1}^{5} u_j = 0\,, \quad \sum_{j=1}^{5} x_\alpha^j u_j = 0\,, \quad \alpha = 1, 2, 3\,. \tag{18}$$

(18) is a system of 4 linear equations for unknowns $u_1, \ldots, u_5$. This system has 3 linearly independent solutions

$$(\Theta^1 l_i^1, \Theta^2 l_i^2, \Theta^3 l_i^3, \Theta^4 l_i^4, -l_i^5)\,, \quad i = 1, 2, 3\,,$$

because $\Theta^1 \Theta^2 \Theta^3 \neq 0$ and $|l_1, l_2, l_3| \neq 0$. This proves that the rank of the matrix of (18) is at most 2. The matrix of (18) is

$$\begin{pmatrix} 1, & 1, & 1, & 1, & 1 \\ x_j^1, & x_j^2, & x_j^3, & x_j^4, & x_j^5 \end{pmatrix}, \quad j = 1, 2, 3$$

and this shows that the points x^i lie on a straight line, $i = 1, \ldots, 5$.

It remains to prove that any point of the straight line determined by the points x^i has a plane trajectory. We may suppose that $x_1^i = x_2^i = 0$ for $i = 1, \ldots, 5$. Let x_3^i, l^i, $i = 1, \ldots, 4$ be given.

Let us choose an arbitrary x_3^5. Then we have to solve the following equations:

$$l_i^5 = \sum_{j=1}^{4} \Theta^j l_i^j, \quad l_i^5 x_3^5 = \sum_{j=1}^{4} \Theta^j l_i^j x_3^j, \quad i = 1, 2, 3.$$

We have 6 equations for 7 unknowns $\Theta^j, j = 1, \ldots, 4, l_i^5,\ i = 1, 2, 3$. The rank of the marix of this system of equations is 6, so l_i^5, Θ^j are fixed up to a factor, Θ^j determine l_4^5, so the plane trajectory is uniquely determined. Further, the trajectory of any point on the third axis depends only on γ_{i3}, which are bound by a linear equation. This means that the trajectory of such a point depends on a single parameter and so it is a curve.

To complete the classification of motions with at least 5 plane trajectories, we have to consider the remaining cases. So far we have discussed the case of 5 plane trajectories such that there exist four of them with every three independent. Let us now consider the other possibility. So, let us have 5 planes, no two of them parallel, in such a position that every quadruple of them has 3 planes dependent. It is easy to see that in this case there exist 4 planes parallel to one straight line.

Remark. A 2-parametric motion is called singular if the corresponding spherical motion is one-parametric.

Lemma 7. *Let a 2-parametric space motion have 4 plane trajectories parallel to a given straight line and such that no two of them are parallel. Then the motion is singular.*

Proof. We proceed in a similar way as above. We suppose $l_3^i = 0, i = 1, \ldots, 4$. Then t_3 is arbitrary and we get two linear equations for $\gamma_{\alpha\beta}$. This shows that the motion is singular.

A similar result is obtained if we consider the case when two planes are parallel and the other 3 are parallel with a straight line. We also get a singular motion.

3. THE SET OF FLAT POINTS

In the third part of the paper we shall present the instantaneous version of the Darboux theorem. If $\mathscr{S}$ is a 2-dimensional surface in E_3, we call a point $A \in \mathscr{S}$ flat, if the second fundamental form of $\mathscr{S}$ vanishes at A. A plane is then characterized as a surface with only flat points.

Let g be a 2-parametric (regular) space motion. Let us denote by F the set of all points $\bar{X}$ in the moving space $\bar{E}_3$ such that the trajectory of $\bar{X}$ has a flat point at $\bar{X}$. In what follows we shall describe the set F for a general regular 2-parametric space motion and show some of its properties.

If $g = g(u_1, u_2)$ is a 2-parametric space motion, then according to [3] we have the following equalities for the canonical frames $\bar{\mathscr{F}}$ and $\mathscr{F}$ in $\bar{E}_3$ and E_3, respectively: $d\mathscr{F} = \mathscr{F}\varphi$, $d\bar{\mathscr{F}} = \bar{\mathscr{F}}\psi$, $\omega = 1/2(\varphi - \psi)$, $\eta = 1/2(\varphi + \psi)$, where

$$\omega = \begin{pmatrix} 0, & 0, & 0, & 0 \\ v\omega_1, & 0, & 0, & \omega_2 \\ w\omega_2, & 0, & 0, & -\omega_1 \\ 0, & -\omega_2, & \omega_1, & 0 \end{pmatrix}, \tag{19}$$

$$\eta = \begin{pmatrix} 0, & 0, & 0, & 0 \\ (\beta w - n)\,\omega_1 - p\omega_2, & 0, & -a_1\omega_1 - a_2\omega_2, & \alpha\omega_1 + \beta\omega_2 \\ m\omega_1 + (n - \beta v)\,\omega_2, & a_1\omega_1 + a_2\omega_2, & 0, & \beta\omega_1 + \gamma\omega_2 \\ b_1\omega_1 + b_2\omega_2, & -\alpha\omega_1 - \beta\omega_2, & -\beta\omega_1 - \gamma\omega_2, & 0 \end{pmatrix}.$$

The integrability conditions are

$$\begin{gathered} d\omega_1 = a_1\omega_1 \wedge \omega_2\,, \quad d\omega_2 = a_2\omega_1 \wedge \omega_2\,, \\ b_1 = -(v)_2 + a_1(v - w)\,, \quad b_2 = (w)_1 - a_2(v - w)\,, \\ -(a_1)_2 + (a_2)_1 + a_1^2 + a_2^2 = \beta^2 - \alpha\gamma - 1\,, \\ (\beta)_2 - (\gamma)_1 - 2a_1\beta + a_2(\alpha - \gamma) = 0\,, \quad -(\alpha)_2 + (\beta)_1 + 2a_2\beta + a_1(\alpha - \gamma) = 0\,, \\ -(b_1)_2 + (b_2)_1 + b_1a_1 + b_2a_2 + \alpha p + \gamma m - 2\beta n + (v + w)(1 + \beta^2) = 0\,, \\ (w\beta - n)_2 + (p)_1 + a_1[2n - \beta(v + w)] + a_2(p - m) + b_1\beta - b_2\alpha = 0\,, \\ -(m)_2 + (n - v\beta)_1 + a_1(m - p) + a_2[2n - \beta(v + w)] - b_1\gamma + b_2\beta = 0\,, \end{gathered} \tag{20}$$

where $df = (f)_1\,\omega_1 + (f)_2\,\omega_2$ for any function f.

For the trajectory $A(u_1, u_2)$ of the point $\bar{A} \in \bar{E}_3$ at A we have

$$\Delta A = 2\mathscr{F}\omega X_A\,, \quad \Delta^2 A = 2\mathscr{F}(\varphi\omega - \omega\psi + \Delta\omega)\,X_A\,, \tag{21}$$

where X_A are the coordinates of the point $\bar{A}$ with respect to $\bar{\mathscr{F}}$, Δ denotes the ordinary (symmetric) differential.

X_A satisfy the following system of differential equations:

$$dX_A = -\psi X_A\,. \tag{22}$$

Let $X_A = (1, x, y, z)^{\mathrm{T}}$. Then the normal vector $\boldsymbol{n}$ of the trajectory of $\bar{A}$ at A is given by

$$\boldsymbol{n} = (0, zx - yw, vx + yz, vw + z^2)^{\mathrm{T}}\,. \tag{23}$$

The set of points where $\boldsymbol{n} = 0$ consists in general of two straight lines given by the equations

$$(24) \qquad zx - yw = 0\,, \quad vx + yz = 0\,, \quad vw + z^2 = 0\,.$$

Let us denote by S the set of all points given by (24). It is the set of singular points of the trajectories of points — the tangent space of the trajectory of a point from S has dimension at most 1.

Lemma 8. *Let Φ_A be the second fundamental form of the trajectory of the point A. Then*

$$\Phi_A = \lambda(x, y, z)\,(C_1\omega_1^2 + C_2\omega_2^2 + 2C_3\omega_1\omega_2)\,,$$

where

(25)

$$C_1 = (zx - yw)(\alpha y + (v)_1) + (vx + yz)(B_1 - \alpha x - 2y) - (vw + z^2)(M + 2z)\,,$$

$$C_2 = (zx - yw)(B_2 - 2x + \gamma y) + (vx + yz)((w)_2 - \gamma x) - (vw + z^2)(P + 2z)\,,$$

$$C_3 = (zx - yw)((v)_2 + (1 + \beta)\,y) + (vx + yz)((w)_1 + (1 - \beta)\,x) - (vw + z^2)\,N,$$

$$B_1 = 2b_1 + (v)_2\,, \quad B_2 = -2b_2 + (w)_1\,, \quad M = \alpha v + m\,, \quad P = \gamma w + p\,,$$

$$N = n + v - w\,.$$

Lemma 9. *F is given by the equations $C_i = 0$, $i = 1, 2, 3$. We have $S \subset F$ and if a straight line belongs to F, then it is parallel to the plane $z = 0$.*

Proof. Let a straight line $\boldsymbol{p} = X + \lambda u$ be in F where X is a point, u is a vector, $X = (x, y, z)^{\mathrm{T}}$, $u = (a, b, c)^{\mathrm{T}}$. If we substitute $\boldsymbol{p}$ in $C_i = 0$ and look at the terms of the 3rd degree in λ, we get $abc = 0$, $c(b^2 + c^2) = 0$, $c(a^2 + c^2) = 0$. The only solution is $c = 0$. The formulas for C_i must be found by direct computation using (21) and (23).

Lemma 10. *If the set $F - S$ is finite, then it contains at most 16 points. If $K_0 = 1 + \alpha\gamma - \beta^2 = 0$, they are at most 15.*

Proof. a) Let $v = w = 0$. Then S is the plane $z = 0$. So let $z \neq 0$. The equations for $F - S$ are

$$(26) \qquad \begin{aligned} 2y^2 + 2z^2 - x(v)_1 - yB_1 + zM &= 0\,,\\ 2x^2 + 2z^2 - xB_2 - y(w)_2 + zP &= 0\,,\\ 2xy + x(v)_2 + y(w)_1 - zN &= 0\,. \end{aligned}$$

It is not difficult to see that (26) may have at most 8 solutions, provided their number is finite.

b) Let $v^2 + w^2 \neq 0$. Let us define the following parametrization of the set F: $x = wr + ts$, $y = tr - vs$, $z = t$ with parameters r, s, t. Then $zx - yw = s(t^2 + vw)$, $vx + yz = r(t^2 + vw)$, $vw + z^2 = t^2 + vw$. This shows that $t^2 + vw = 0$ is the equation of S. For $F - S$ we obtain from (25) the following equations:

$$(27)\qquad \begin{aligned} &2t(r^2 + 1) + \alpha(vs^2 + wr^2) - 2vrs - s(v)_1 - rB_1 + M = 0\,,\\ &2t(s^2 + 1) + \gamma(vs^2 + wr^2) + 2wrs - r(w)_2 - sB_2 + P = 0\,,\\ &2trs - vs^2(1 + \beta) + wr^2(1 - \beta) + r(w)_1 + s(v)_2 - N = 0\,. \end{aligned}$$

The solvability conditions for (27) form 2 curves of order 4 in the r, s plane, so we may have at most 16 solutions. The terms of the highest degree in r and s are $r(vs^2 + wr^2)(r(\beta - 1) + \alpha s)$ and $s(vs^2 + wr^2)(\gamma r + s(\beta + 1))$, which shows that they are two common points at infinity and one more for $K_0 = 0$. On the other hand, the plane $z = \pm\sqrt{(-vw)}$ intersects the surfaces $C_i = 0$ in 3 straight lines (apart from the singular line and the line at infinity). These 3 lines have at most 1 common point, which gives at most 2 new points.

Example 2. We will discuss the plane trajectories of 2-parametric motions with constants invariants. For such a motion we have $a_1 = a_2 = b_1 = b_2 = 0$, $\beta^2 - \alpha\gamma = 1$, $\alpha p + \gamma m - 2\beta n + (v + w)(1 + \beta^2) = 0$. We shall consider only the general case, so let $v - w \neq 0$, $w \neq 0$, $v > 0$. Let us find all motions of this type which have at least one plane trajectory, which is not a curve. This means that we exclude the points of S. Equations (25) yield the following equations for F:

$$(28)\qquad \begin{aligned} &(zx - yw)(\alpha y) + (vx + yz)(-\alpha x - 2y) + (vw + z^2)(-\alpha v - m - 2z) = 0\,,\\ &(zx - yw)(-2x + \gamma y) + (vx + yz)(-\gamma x) + (vw + z^2)(-\gamma w - p - 2z) = 0\,,\\ &2(zx - yw)(\beta + 1)\,y + 2(vx + yz)(1 - \beta)\,x - (vw + z^2)(n + v - w) = 0\,. \end{aligned}$$

Further, (22) yields

$$(29)\qquad \begin{aligned} &(x)_1 = v - \beta w + n - \alpha z\,, &&(x)_2 = p - (\beta - 1)\,z\,,\\ &(y)_1 = -m - (\beta + 1)\,z\,, &&(y)_2 = w + \beta v - n - \gamma z\,,\\ &(z)_1 = \alpha x + (\beta + 1)\,y\,, &&(z)_2 = (\beta - 1)\,x + \gamma y\,. \end{aligned}$$

As $\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0$, we may write $\omega_1 = \mathrm{du}$, $\omega_2 = \mathrm{du}_1$ and we can integrate (29) with respect to u and u_1. Integration with respect to u gives a one-parametric subgroup and its trajectories are

$$(30)\qquad \begin{aligned} x(\mathrm{u}) &= \vartheta^{-2}[\sigma\,\mathrm{u}(\beta + 1) + x_0(\alpha^2 \cos \vartheta\mathrm{u} + (\beta + 1)^2) +\\ &\quad + y_0\alpha(\beta + 1)(\cos \vartheta\mathrm{u} - 1) - (z_0 - \vartheta^{-2}\varrho)\,\alpha\vartheta \sin \vartheta\mathrm{u}]\,,\\ y(\mathrm{u}) &= \vartheta^{-2}[-\sigma\,\mathrm{u}\alpha + x_0\alpha(\beta + 1)(\cos \vartheta\mathrm{u} - 1) + y_0((\beta + 1)^2 \cos \vartheta\mathrm{u} +\\ &\quad + \alpha^2) - (\beta + 1)\,\vartheta \sin \vartheta\mathrm{u}(z_0 - \vartheta^{-2}\varrho)\,,\\ z(\mathrm{u}) &= \vartheta^{-2}[x_0 a\,\vartheta \sin \vartheta\mathrm{u} + y_0(\beta + 1)\,\vartheta \sin \vartheta\mathrm{u} + \vartheta^2 \cos \vartheta\mathrm{u}(z_0 - \vartheta^{-2}\varrho) + \varrho]\,, \end{aligned}$$

where $\sigma = \alpha m + (\beta + 1)(v - \beta w + n)$, $\varrho = -m(\beta + 1) + \alpha(v - \beta w + n)$, $\vartheta = [\alpha^2 + (\beta + 1)^2]^{1/2}$, $\alpha^2 + (\beta + 1)^2 \neq 0$. (30) gives the trajectory passing through the point (x_0, y_0, z_0).

(30) must satisfy (28) for all values of u. We substitute (30) into (28) and consider only the terms with the highest powers in $\cos \vartheta u$ and $\sin \vartheta u$. Let us denote by $\bar{x}, \bar{y}, \bar{z}$ the part of x, y, z in (30) which is linear in $\sin \vartheta u$ and $\cos \vartheta u$, without the absolute term. Then we get

$$\bar{z}(\bar{z}^2 + \bar{y}^2) \quad \text{and} \quad \bar{z}(\bar{z}^2 + \bar{x}^2),$$

which must be zero mod $\cos^2 \vartheta u + \sin^2 \vartheta u = 1$.

Let us write $\bar{z} = \lambda \cos \varphi + \mu \sin \varphi$, $\bar{y} = r \cos \varphi + s \sin \varphi$, $\varphi = \vartheta u$. Then we obtain the equations

$$\lambda(\lambda^2 - \mu^2 + r^2 - s^2) - 2\mu(\lambda\mu + rs) = 0,$$
$$\mu(\lambda^2 - \mu^2 + r^2 - s^2) + 2\lambda(\lambda\mu + r) = 0.$$

Here either $\lambda^2 + \mu^2 = 0$ and $\lambda = \mu = 0$, or $\lambda^2 - \mu^2 + r^2 - s^2 = 0$ and $\lambda\mu + rs = 0$. The second possibility yields $\lambda = \varepsilon s$, $\mu = -\varepsilon r$, $\varepsilon = \pm 1$. We use the term $\bar{z}(\bar{z}^2 + \bar{x}^2)$ to obtain $\lambda = \mu = 0$.

This shows that we always have $\bar{z} = 0$. This yields $x_0\alpha + (\beta + 1) y_0 = 0$, $z_0 = \vartheta^{-2}\varrho$.

It remains to consider the case $\beta + 1 = 0$, $\alpha = 0$. In this case we use the second part of (29) similarly as above. Since now $(\beta - 1)^2 + \gamma^2 \neq 0$, we get a similar result.

We have proved so far that if a point has a plane trajectory, it must lie on the axis **p** of the one-parametric subgroup (30). Parametric equations of **p** are

$$x = (\beta + 1)\,t, \quad y = -\alpha t, \quad z = \vartheta^{-2}\varrho, \quad \text{where} \quad (\beta + 1)^2 + \alpha^2 \neq 0.$$

a) We shall investigate under what conditions all points of **p** have plane trajectories. This means that we suppose that **p** belongs to $F - S$. Substitution into (28) shows that then we must have

$$z = -1/2(\alpha w + \gamma v)$$

and

$$(vw + z^2)(\alpha v + m + 2z) = 0,$$
$$(vw + z^2)(\gamma w + p + 2z) = 0,$$
$$(vw + z^2)(n + v - w) = 0.$$

Let $vw + z^2 = 0$. Then $\alpha^2 w^2 + 2(\alpha\gamma + 2) vw + \gamma^2 v^2 = 0$. The discriminant of this equation is $D = 16(1 + \alpha\gamma) = 16\beta^2$. We obtain $w/v = -(\beta + 1)^2/\alpha^2$ or $w/v = -(\beta - 1)^2/\alpha^2$. The first possibility gives a singular line, so we may suppose that

$$wα^2 + v(\beta - 1)^2 = 0. \tag{31}$$

Substitution into the equation $z = \vartheta^{-2}\varrho$ yields

$$(32) \qquad -m(\beta + 1) + \alpha(v - \beta w + n) = -1/2(\alpha w + \gamma v)(\alpha^2 + (\beta + 1)^2).$$

The remaining possibility is $\alpha v + m + 2z = \gamma w + p + 2z = n + v - w = 0$. Substitution into the integrability condition shows that in this case $\mathbf{p}$ belongs to S.

b) A point of $\mathbf{p}$ with a plane trajectory is an isolated solution of (28). This means that the coordinates of this point with respect to $\mathscr{F}$ are constant. From (29) we obtain $n = w + \beta v - \gamma z = \beta w - v + \alpha z$, $m = -(\beta + 1) z$, $p = (\beta - 1) z$, $\alpha x + (\beta + 1) y = 0$, $(\beta - 1) x + \gamma y = 0$.

The last two equations are compatible as $K_0 = 0$; from the first two equations we get $w(1 - \beta) + v(1 + \beta) = (\alpha + \gamma) z$.

(28) changes to

$$\begin{aligned} \alpha^2 x^2(2z + \alpha w + \gamma v) + (vw + z^2)(\beta + 1)^2 [(1 - \beta) z + \alpha v] &= 0, \\ x^2(2z + \alpha w + \gamma v) + (vw + z^2)[(1 + \beta) z + \gamma w] &= 0, \\ 2\alpha x^2(2z + \alpha w + \gamma v) + (vw + z^2)(\beta + 1)[\gamma z - v(\beta + 1)] &= 0, \end{aligned}$$

where we suppose $\beta + 1 \neq 0$.

i) $vw + z^2 = 0$, $2z + \alpha w + \gamma v = 0$. Here we get a special case of a).

ii) $vw + z^2 = 0$, $2z + \alpha w + \gamma v \neq 0$. Then $x = y = 0$ and the point is from S.

iii) $vw + z^2 \neq 0$. Then the determinant of (28) must be zero, which yields $\alpha z = w(1 - \beta)$, $\gamma z = v(1 + \beta)$, so $\alpha\gamma(vw + z^2) = 0$ and $\alpha\gamma = 0$. This implies $\beta = 1$, $\gamma z = 2v$, $\alpha z = 0$. As $\gamma = 0$ implies $v = 0$, we may suppose $\gamma \neq 0$. Then $\alpha = 0$, $y = 0$, $z = 2v/\gamma$ and $x^2\gamma^2(4 + \gamma^2) + (4v + \gamma^2 w)^2 = 0$, so $x = 0$, $4v + \gamma^2 w = 0$, $vw + z^2 = v\gamma^{-2}(4v + w\gamma^2) = 0$, which is a contradiction and we have no solution in the case b) for $\beta + 1 \neq 0$.

Now let $\beta = -1$. Then $x = 0$, $x = 1/2\gamma y$.

i) $\alpha \neq 0$. Then $\gamma = 0$. The equality $vw + z^2 = 0$ implies $y^2(\alpha w + 2z) = 0$. While $y = 0$ gives a singular point, $\alpha w + 2z = 0$ gives $\alpha^2 + 4 = 0$, which is impossible. Hence $vw + z^2 \neq 0$ leads to a contradiction.

ii) $\alpha = 0$. Then $vw + z^2 = 0$ gives a singular point $x = y = 0$, and $vw + z^2 \neq 0$ leads to a singular point as well.

Theorem 10. *Regular 2-parametric space motions with a 2-dimensional group of automorphisms are given as products*

$$g(\mathrm{u}_1, \mathrm{u}_2) = g_1(\alpha_1 \mathrm{u}_1 + \beta_1 \mathrm{u}_2, \gamma_1 \mathrm{u}_1 + \delta_1 \mathrm{u}_2) \,.\, g_2(\alpha_2 \mathrm{u}_1 + \beta_2 \mathrm{u}_2, \gamma_2 \mathrm{u}_1 + \delta_2 \mathrm{u}_2),$$

where $g_i(\mathrm{r}, \mathrm{s})$ is a commutative two-dimensional subgroup of $\mathscr{E}_3$ and r *denotes the angle of rotation,* s *is a translation,* rank $\begin{pmatrix} \alpha_i, & \beta_i \\ \gamma_i, & \delta_i \end{pmatrix} \geqq 1$, $i = 1, 2$. *Such a motion is a product of two one-parametric subgroups iff $m = w\alpha$, $p = \gamma v$, $n = \beta(v + w)$.*

Proof. If a motion has a 2-dimensional group of automorphisms, then it has constant invariants and it is an orbit of a 2-dimensional subgroup of $\mathscr{E}_3 \times \mathscr{E}_3$. Let G be a 2-dimensional subgroup of $\mathscr{E}_3 \times \mathscr{E}_3$. Then its canonical projections $p_i(G)$ must be subgroups of $\mathscr{E}_3$, as p_i is a homomorphism of groups, $i = 1, 2$. If $p_1(G)$ has dimension 0, then G is isomorphic with a commutative subgroup of $\mathscr{E}_3$ and the resulting motion is not regular. As any homomorphism between commutative groups is given by a linear mapping in the canonical coordinates, the first statement of the theorem follows. If such a motion is a product of two one-parametric subgroups, then φ and ψ must be functions of one parameter only. This yields $(\beta + 1)(w + \beta v - n) + \gamma m = 0$, $(\beta - 1)(-w + \beta v - n) + \gamma m = 0$, which together with the integrability condition gives the statement.

The following theorem has been already proved above.

Theorem 11. *A regular space motion with constant invariants, where $vw \neq 0$, $v - w \neq 0$, has a plane as a trajectory, which is not a curve iff (31) and (32) are satisfied. In such a case it has infinitely many such points and they are all points of the axis of g_2.*

Remark. In the case when $g(u_1, u_2)$ is a product of two one-parametric subgroups, we get that the points of the axis of g_2 have plane trajectories iff g_1 is a rotation, and the axes of g_1 and g_2 are mutually orthogonal, as we would expect.

Proof of Remark. We substitute $m = w\alpha$, $p = v\gamma$, $n = \beta(v + w)$ into (31) and (32). Then we get $\beta(\alpha^2 + \beta^2 - 1) = 0$. While $\beta = 0$ leads to singular points, $\alpha^2 + \beta^2 = 1$ is the condition of perpendicularity of axes. Further g_1 is a rotation iff $\alpha m + (\beta - 1)(-v - \beta w + n) = 0$. Substitution shows that the last equation is satisfied. The converse is obvious.

Example 3. We shall discuss plane trajectories of a motion given as a rolling of two isometric surfaces. In such a case we have $v = w = \beta = 0$, $\alpha p + \gamma m = 0$. Integrability conditions are

$$(a_2)_1 - (a_1)_2 + a_1^2 + a_2^2 + 1 = -\alpha\gamma,$$
$$(\gamma)_1 = a_2(\alpha - \gamma), \quad (\alpha)_2 = a_1(\alpha - \gamma),$$
$$-(n)_2 + (p)_1 + 2a_1 n + a_2(p - m) = 0,$$
$$-(m)_2 + (n)_1 + (m - p)\, a_1 + 2a_2 n = 0.$$

(29) will change to

$$(x)_1 = n + a_1 y - \alpha z, \quad (x)_2 = p + a_2 y + z,$$
$$(y)_1 = -m - a_1 x - z, \quad (y)_2 = -n - a_2 x - \gamma z,$$
$$(z)_1 = \alpha x + y, \quad (z)_2 = -x + \gamma y.$$

Equations of $F - S$ are

$$(33) \qquad 2(z^2 + y^2) + mz = 0\,, \quad 2(z^2 + x^2) + pz = 0\,, \quad 2xy - nz = 0\,.$$

First we shall show that in the case $n = 0$ there is no solution. So let $n = 0$. Then $xy = 0$; let $x = 0$. Then $z = -1/2p$, $y = 1/2\lambda$, where $\lambda = \sqrt{(pm - p^2)}$. Now $a_1 = -\alpha p\lambda^{-1}$, $a_2 = p\lambda^{-1}$, $(p)_1 = -\lambda$, $(p)_2 = -\lambda\gamma$, $(\lambda)_1 = -2m + p$, $(\lambda)_2 = \gamma p$, $(\alpha)_1 = p(\alpha - \gamma)\lambda^{-1}$, $(\alpha)_2 = -\alpha p(\alpha - \gamma)\lambda^{-1}$. The first integrability condition yields $(a_2)_1 - (a_1)_2 + a_1^2 + a_2^2 + 1 = \lambda^{-2}(2pm + \alpha\gamma\lambda^2 + 2\alpha^2p^2) = -\alpha\gamma$. This implies $pm(1 + \gamma^2) = 0$. As $z = 0$ is the equation of S, we must have $p \neq 0$. Then $m = 0$ and so $p = 0$, which is a contradiction. This shows that we may suppose $n \neq 0$.

The derivatives of (33) yield

$$(34) \qquad \begin{aligned}
\alpha mx + y(2\alpha n + 3m) + z(2a_1n - (m)_1) &= 0\,,\\
(3p - 2\gamma n)\,x - \gamma py + z(2a_2n + (p)_2) &= 0\,,\\
-mx + y(2n + \alpha p) + z(a_1(p - m) - (n)_1) &= 0\,,\\
(\gamma m - 2n)\,x + py + z(a_2(p - m) - (n)_2) &= 0\,.
\end{aligned}$$

Let (34) have more than one nontrivial solution. It is easy to see that any straight line passing through the origin has at most one intersection with (33) apart from the origin. This means that the matrix of (34) must have rank one. This yields

$$\begin{aligned}
4n^2 + 4n\alpha p - pm(1 + \alpha\gamma) &= 0\,,\\
4n^2\alpha\gamma - 12np\alpha - pm(9 + \alpha\gamma) &= 0\,,\\
pn(3 - \alpha\gamma) - 2\gamma(n^2 + pm) &= 0\,,\\
mn(3 - \alpha\gamma) + 2\alpha(n^2 + pm) &= 0\,.
\end{aligned}$$

These equations imply $(n^2 - pm)(3 + \alpha\gamma) = 0$.

i) Let $\alpha\gamma = -3$. Then $2n^2 + 2\alpha pn + pm = 0$, $3pn - \gamma(n^2 + pm) = 0$, $\gamma n = 3p$, $-\alpha n = 3m$ and $n = 0$, which is a contradiction.

ii) Let $n^2 = pm$. Then $(3 - \alpha\gamma)^2 + 16\alpha\gamma = 0$, so $\alpha\gamma = -1$ or $\alpha\gamma = -9$. If $\alpha\gamma = -1$, then $p = -n\alpha^{-1}$, $m = -\alpha n$, if $\alpha\gamma = -9$, then $p = -3n\alpha^{-1}$, $m = -1/3n\alpha$.

Let us write $\alpha\gamma = -k^2$, $p = -kn\alpha^{-1}$, $m = -\alpha nk^{-1}$, where $k = 1, 3$. Then $y = \alpha xk^{-1}$ or $y = -kx\alpha^{-1}$.

First, let $y = \alpha xk^{-1}$. Then $z = 2\alpha x^2n^{-1}k^{-1}$. This implies $2x^2 + pz = 0$ and from the second equation in (33) we have $z = 0$, which is a contradiction.

Now let $y = -kx\alpha^{-1}$. Then $pz = 2k^2x^2\alpha^{-2}$ and similarly as above we get a contradiction with $z \neq 0$.

Theorem 12. *The rolling of two surfaces has at most one plane trajectory which does not degenerate to a curve. Such motions exist and they have $n \neq 0$.*

Proof. It remains to prove the existence of such a motion. For this purpose let us consider the following example:

Example 4. Let us consider a rolling of two surfaces such that $\alpha = -\gamma = -1$, $n + m = -2k$, where k is a constant. Then $p = m$ and $a_1 = a_2 = 0$. For a point with a plane trajectory we get $x^2 = y^2$; let $x = y$. Then $z = k$, $x = \sqrt{(1/2kn)}$, and the remaining equations for x are $(x)_1 = -(x)_2 = n + k$. Integrability conditions reduce to one equation $(n)_1 + (n)_2 = 0$. As $d\omega_1 = d\omega_2 = 0$, we can write $\omega_1 = du_1 + du_2$, $\omega_2 = -du_1, + du_2$.

For any function $f(u_1 . u_2)$ we have

$$\frac{\partial f}{\partial u_1} = (f)_1 - (f)_2 , \quad \frac{\partial f}{\partial u_2} = (f)_1 + (f)_2 .$$

This yields

$$\frac{\partial x}{\partial u_1} = 2(n + k) = 2k^{-1}(2x^2 + k^2) , \quad \text{so} \quad x = \frac{k}{\sqrt{2}} \tan\left(2\sqrt{(2)}\, u_1\right) ,$$

$n = k \tan^2\left(2\sqrt{(2)}\, u_1\right)$ and the point (x, y, z) has a plane trajectory. The motion is realized as a rolling of two isometric ruled surfaces.

Beside the point the rolling determined by two congruent paraboloids of revolution may serve as an elementary example – the focus of the moving one has a plane trajectory.

References

[1] *G. Koenigs:* Leçons de Cinématique, Paris 1897, Note III by *G. Darboux*: Sur les mouvements algébriques.

[2] *W. Blaschke:* Kinematik und Quaternionen. Berlin 1960.

[3] *A. Karger:* Two-parametric motions in E_3. Apl. mat. 32 (1987), 96–119.

Souhrn

DARBOUXOVA VĚTA O ROVINNÝCH TRAJEKTORIÍCH DVOUPARAMETRICKÉHO PROSTOROVÉHO POHYBU

ADOLF KARGER

Na základě práce [1] je v článku dokázána klasifikační věta pro dvouparametrické prostorové pohyby mající alespoň 5 bodů s rovinnými trajektoriemi. Kromě pohybů s nekonečně mnoha rovinnými trajektoriemi jsou to tzv. Darbouxovy kvadratické pohyby, určené jistým tenzorem třetího řádu. Převedení tohoto tenzoru do kanonického tvaru je klíčem k důkazu klasifikační věty. Druhá část práce se problémem rovinných trajektorií zabývá z diferenciálně-geometrického hlediska a obsahuje některé aplikace. Článek se též zabývá některými realizacemi homogenního prostoru všech prostorových shodností s použitím duálních kvaternionů a duálních matic.

Резюме

ПРОБЛЕМА ДАРБУ О ПЛОСКИХ ТРАЕКТОРИЯХ ДВУПАРАМЕТРИЧЕСКИХ ПРОСТРАНСТВЕННЫХ ДВИЖЕНИЙ

Adolf Karger

В работе доказана теорема о классификации двупараметрических пространственных движений, имеющих по крайней мере 5 плоских траекторий. Доказательство основано на работе [1] и на приведении тензора 3-го порядка к каноническому виду. Часть работы занимается проблемой плоских траекторий с точки зрения дифференциальной геометрии и содержит некоторые приложения.

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