Leopold Herrmann

A study of an operator arising in the theory of circular plates

Aplikace matematiky, Vol. 33 (1988), No. 5, 337–353

Persistent URL: http://dml.cz/dmlcz/104315

A STUDY OF AN OPERATOR ARISING IN THE THEORY OF CIRCULAR PLATES

LEOPOLD HERRMANN

(Received April 21, 1985)

Summary. The operator $L_0: D_{L_0} \subset H \to H$, $L_0 u = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)\right]\right\}$, $D_{L_0} = \{u \in C^4([0, R]),\ u'(0) = u'''(0) = 0,\ u(R) = u'(R) = 0\}$, $H = L_{2,r}(0, R)$ is shown to be essentially self-adjoint, positive definite with a compact resolvent. The conditions on L_0 (in fact, on a general symmetric operator) are given so as to justify the application of the Fourier method for solving the problems of the types $L_0 u = g$ and $u_{tt} + L_0 u = g$, respectively.

Keywords: circular plates theory, Fourier method, thin plate equation.

AMS Classification: 35C10, 47B25, 73K12.

1. INTRODUCTION

The equations

$$D\,\Delta^2 u(x, y) = f(x, y) \tag{1.1}$$

and

$$\varrho h \frac{\partial^2 u(x, y, t)}{\partial t^2} + D\Delta^2 u(x, y, t) = f(x, y, t)\,, \tag{1.2}$$

$$\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

describe the static deflection and the transverse vibration, respectively, of a thin homogeneous elastic plate subject to a transverse load (see [8], [10], [11]). Here h is the uniform thickness of the plate, ϱ its density, D the flexural rigidity, $D = Eh^3/12(1 - \mu^2)$, E the modulus of elasticity, μ Poisson's ratio, and f the transverse load measured in units of force/area.

The middle surface of the plate is supposed to comprise initially a domain Ω in the plane of the variables x and y. The function u defines its small deflections in the direction perpendicular to the xy-plane.

If the edge of the plate is clamped the deflection along this edge is zero and the tangent planes of deflected and undeflected middle surfaces coincide along this edge. The analytical expression of the boundary conditions in this case is

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\,, \tag{1.3}$$

where $\partial/\partial\nu$ means the outward normal derivative.

When studying circular plates (of a radius R), i.e. plates for which $\Omega = \{(x, y);\ x^2 + y^2 \leqq R^2\}$, it is convenient to use polar coordinates r and ϑ instead of x and y. The biharmonic operator Δ^2 is obtained by applying twice the Laplace operator the form of which in polar coordinates is

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}\,.$$

Of particular interest is the so-called circularly symmetric case. In that case the circular plate is acted upon by a load symmetrically distributed with respect to the center of the plate, and the deflection u is considered to be independent of the angle ϑ as well (just as the initial displacement and velocity in the dynamic problem). At all points equally distant from the center of the plate the deflection will be the same, and thus it is sufficient to consider deflections only in one radial section. The equations (1.1) and (1.2) assume the form

$$L_0\, u(r) = g(r)\,, \quad 0 < r < R\,, \tag{1.4}$$

$$\frac{\partial^2 u(r, t)}{\partial t^2} + a^2 L_0 u(r, t) = g(r, t)\,, \tag{1.5}$$

$$0 < r < R\,, \quad 0 < t < T\left(a^2 = \frac{D}{\varrho h}\,,\ T > 0\right),$$

where

$$L_0 = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right\}. \tag{1.6}$$

In our particular case of a circular plate the boundary conditions (1.3) are

$$u = u_r = 0 \quad \text{for} \quad r = R \quad \left(\text{where } u_r = \frac{\partial u}{\partial r}\right). \tag{1.7}$$

The equation (1.4), together with (1.7), defines the static problem $(\mathscr{P}_{\mathrm{st}})$. The dynamic problem $(\mathscr{P}_{\mathrm{dyn}})$ is defined by the equation (1.5) supplemented with the boundary conditions (1.7) and the initial conditions

$$
\begin{aligned}
&(1.8) \qquad && u(r, 0) = \varphi(r)\,, \\
& && \qquad\qquad\qquad 0 < r < R\,, \\
&(1.9) \qquad && u_t(r, 0) = \psi(r)\,.
\end{aligned}
$$

Our aim is to investigate several properties of the operator L_0 so that we can derive results on solvability of the problems $(\mathscr{P}_{st})$ and $(\mathscr{P}_{dyn})$. A number of authors in texts on mathematical physics introduce solutions of the above problems (see, e.g., [2]) but they treat them just in a quite formal manner and in many cases they argue uncorrectly.*) We want to bring in a functional-analytical approach and derive precise existence results.

If L_0 together with suitable boundary conditions is considered to be an operator in an "appropriate" Hilbert space (namely, in the weighted space $L_{2,r}(0, R)$) we can obtain without difficulty that it is symmetric.

For many purposes, particularly to justify the use of the spectral decomposition, it is important to know whether an unbounded symmetric operator is self-adjoint and, if this is not the case, how to get a "related" operator having already this property. There are two main ways of obtaining self-adjoint extensions of symmetric operators.

First, in some cases (of the so-called essentially self-adjoint operators) we succeed in proving that a self-adjoint extension is obtained by a simple operation of closure. This is the case, for instance, for most operators of classical quantum mechanics. An essentially self-adjoint operator has its closure as the only self-adjoint extension.

The other technique is applicable to operators bounded below. For this special class of operators we get self-adjoint extensions even if the first method fails. On the other hand, however, such extensions, in general, are not determined unambiguously. An operator bounded below has either infinitely many self-adjoint extensions or exactly one self-adjoint extension and then in turn it is already essentially self-adjoint. It is worth noting that there are symmetric operators which do not admit any self-adjoint extension.

In classical boundary value problems of mathematical physics we deal with operators that are not bounded, and, of course, far less compact. Consequently, we are deprived of the possibility using nice spectral properties of compact operators which are closest to finite-dimensional (matrix) operators. The spectrum of compact operators is a countably infinite set, the only possible limit point is zero and every

*) Using the separation-of-variables method they are led to the equation

$$\Delta^2 v(r) - \lambda\, v(r) = 0\,, \quad r \in (0, R)\,, \quad (\lambda > 0)\,.$$

By analogy with the problem of a circular membrane where the Bessel equation is obtained they discard some of solutions by requiring boundedness at the origin and so they find $J_0(\lambda^{1/4} r)$ and $I_0(\lambda^{1/4} r)$. But the solution $\frac{1}{2}\pi\, Y_0(\lambda^{1/4} r) + K_0(\lambda^{1/4} r)$ is bounded at 0 (together with its derivative) as well. Here J_0, Y_0, I_0 and K_0 are the usual Bessel, Neumann, modified Bessel and MacDonald functions, respectively.

non-zero point of the spectrum is an eigenvalue of a finite multiplicity. Nevertheless, the operators frequently encountered in classical boundary value problems exhibit the property of having a compact resolvent. Such operators have similar spectral properties as the compact ones. In particular, the spectral decomposition of a self-adjoint operator with a compact resolvent is reduced to an eigenvector expansion. This property is used under the more familiar guise of separation-of-variables technique, the method which has been applied to solving boundary value problems for more than two centuries.

In fact, it is the theory of *self-adjoint operators with a compact resolvent* which ustifies the separation-of-variables method in partial differential equations.

The next section represents a short survey of concepts and results relating to the above mentioned ideas. The material is standard and can be found in many places, e.g. [4], [9], [14].

In Section 3 we derive that the closure L of the operator L_0 is a positive definite self-adjoint operator with a compact resolvent.

We go on to applications to problems $(\mathscr{P}_{\mathrm{st}})$ and $(\mathscr{P}_{\mathrm{dyn}})$ in Section 4. Roughly speaking, we look for solutions of $Lu = g$, $u \in D_L$, instead of $L_0 u = g$, $u \in D_{L_0}$ (and similarly for $(\mathscr{P}_{\mathrm{dyn}})$).

In Section 5 we find explicit forms of solutions in terms of Fourier eigenfunction expansions.

In the final section we sum up the conditions (in a general setting and relatively easily verifiable) ensuring that the solutions of (both static and dynamic) problems of mathematical physics can be obtained by means of eigenvector expansions, i.e., the Fourier method can be applied.

2. REVIEW OF SOME FUNCTIONAL-ANALYTICAL TOOLS

2.1. **Closable and essentially self-adjoint operators.** Throughout this section let H be a (complex) Hilbert space and $A: D_A \subset H \to H$ a linear operator.

A is called *closable* if there exists a closed extension of A, i.e. a linear operator $B: D_B \subset H \to H$ such that the set G_B of ordered pairs $[u, Bu]$ where $u \in D_B$ (its graph) is closed in $H \times H$, and $G_A \subset G_B$. If A is closable then there exists an operator $\bar{A}$ such that $G_{\bar{A}} = \bar{G}_A$. $\bar{A}$ is a unique minimal closed extension of A and is called the closure of A. Since $u \in D_{\bar{A}}$ is equivalent to $[u, \bar{A}u] \in G_{\bar{A}}$ we get

$$D_{\bar{A}} = \{u \in H;\ \text{there exist } u_n \in D_A,\ u_n \to u \text{ in } H \text{ for } n \to \infty\,,\ Au_n \text{ is convergent in } H\}$$

and $\bar{A}u = \lim_{n\to\infty} Au_n$.

From now on we shall be dealing with linear *densely defined operators* (i.e. defined over domains D_A which are dense in H).

If A is closable then the adjoints of A and $\bar{A}$ coincide, i.e. $A^* = (\bar{A})^*$.

If A is symmetric (i.e. A^* extends A) then it is closable and its closure $\bar{A}$ is also symmetric. In general, the closure $\bar{A}$ of a symmetric operator need not be self-adjoint. If it is so, A is said to be *essentially self-adjoint*. If A is essentially self-adjoint then it has exactly one self-adjoint extension.

A well-known test for the self-adjointness yields the following assertion.

Lemma. *If A is symmetric and the range of $\bar{A}$ is the whole of H ($R_{\bar{A}} = H$) then A is essentially self-adjoint.*

2.2. **Positive definite operators.** A *symmetric* operator A is called *bounded below* (by a constant γ) if

$$(Au, u) \geqq \gamma \|u\|^2 \quad \text{for all} \quad u \in D_A .$$

(The latter condition implies the symmetry of A in case of a complex H.) In particular, if $\gamma = 0$ ($\gamma > 0$), A is called *non-negative* (*positive definite*).

Lemma. *If A is positive definite then*

$$H = R_{\bar{A}} \oplus N_{A^*} .$$

The proof is based on the following simple facts:

– if A is symmetric then

$$H = \bar{R}_A \oplus N_{A^*} ;$$

– along with A its closure $\bar{A}$ is also positive definite and

$$R_{\bar{A}} = \bar{R}_{\bar{A}} .$$

The previous lemma and Lemma 2.1 yield

Corollary. *If A is positive definite and $N_{A^*} = \{0\}$ then A is essentially self-adjoint.*

Operators bounded below occur very often in problems of mathematical physics. The importance of the class of these operators consists in their remarkable property: each operator bounded below has a self-adjoint extension (even if it is not essentially self-adjoint). One recipe for defining such an extension (due to Friedrichs) will be touched upon in the next paragraph.

2.3. **The energy space.** Let A be an operator bounded below by γ. Choose μ such that $\mu + \gamma > 0$ and put

$$(u, v)_A = (Au, v) + \mu(u, v) ,$$

$$\|u\|_A = (u, u)_A^{1/2} , \quad u, v \in D_A ,$$

$$H_A = \{u \in H; \text{ there exist } u_n \in D_A,\ u_n \to u \text{ in } H \text{ as } n \to \infty ,$$

$$\|u_n - u_m\|_A \to 0 \text{ as } n, m \to \infty\} .$$

The space H_A becomes a Hilbert space under the inner product

(2.1) $$(u, v)_A = \lim_{n \to \infty} (u_n, v_n)_A ,$$

where $\{u_n\}$ and $\{v_n\}$ are sequences from the definition of H_A corresponding to u and v, respectively. It may be proved that the limit (2.1) always exists and is independent of the choice of the approximating sequences. D_A is dense in H_A. Choosing various μ with $\mu + \gamma > 0$ we obtain equivalent norms on the same linear space H_A. H_A is called *the energy space* corresponding to the operator A.

We could have introduced an alternative definition of the energy space via the standard procedure of completion of D_A under the energy norm $\|\cdot\|_A$. Both definitions are actually equivalent. In brief outline, here is the main idea of the proof. It relies on the following properties of the energy norm, which may be easily verified:

a) $\|u\|_A^2 \geqq (\gamma + \mu)\,\|u\|^2$ for all $u \in D_A$;

b) $\|\cdot\|_A$ is compatible with $\|\cdot\|$ in the following sense: if $u_n \in D_A$, $\|u_n - u_m\|_A \to 0$ as $n, m \to \infty$ and $\|u_n\| \to 0$, then $\|u_n\|_A \to 0$.

The completion $\hat{D}_A$ of D_A in the norm $\|\cdot\|_A$ consists of the classes $[\{u_n\}]$ of equivalent Cauchy sequences $\{u_n\}$ of elements from D_A (two Cauchy sequences $\{u_n\}$ and $\{v_n\}$ are equivalent if $\|u_n - v_n\|_A \to 0$ as $n \to \infty$). In virtue of the above properties the operator

$$J([\{u_n\}]) = \lim_{n \to \infty} u_n \text{ in } H \tag{2.2}$$

defines an isomorphism of $\hat{D}_A$ onto a subspace of H. Consequently, $\hat{D}_A$ may be considered as a subspace of H if the embedding is defined by (2.2), and in this sense we have $\hat{D}_A = H_A$.

In terms of the energy space we can simply define the so-called *Friedrichs extension* of an operator bounded below. Namely, if A is an operator bounded below by γ then the operator $\tilde{A}$ defined by

$$D_{\tilde{A}} = H_A \cap D_{A^*}, \quad \tilde{A}u = A^*u$$

is a self-adjoint extension of A. There are two important features of the Friedrichs extension:

– it is the only self-adjoint extension the domain of which is contained in H_A,

– it is bounded below and the lower bound γ remains unchanged.

The energy spaces of the operators A, $\bar{A}$ and $\tilde{A}$ coincide.

As a consequence of Lemma 2.2 (applied to $\tilde{A}$) we get $R_{\tilde{A}} = H$ for A positive definite. In other words, the problem

$Au = g$ with A positive definite admits a "generalized" solution $u = \tilde{A}^{-1}g$ for any $g \in H$.

2.4. **Operators with a compact resolvent.** A resolvent set ϱ_A of a closed densely defined operator $A: D_A \subset H \to H$ is the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is a one-to-one operator onto H (I is the identity operator in H). For each $\lambda \in \varrho_A$ the inverse operator $(A - \lambda I)^{-1}: H \to H$ (called the resolvent of A in λ) exists and is bounded.

If, moreover, $(A - \lambda I)^{-1}$ is compact (for some $\lambda \in \varrho_A$) then it is compact for any $\lambda \in \varrho_A$ and A is said to be an *operator with a compact resolvent.*

Lemma. *Let A: $D_A \subset H \to H$ be self-adjoint with a compact resolvent and let H be a Hilbert space of non-finite dimension. Then a) H is separable,*

b) in H there exists a countable complete orthonormal system $\{v_k\}_{k=1}^{\infty}$ of eigenvectors of A with the corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$,

c) the set $\{k \in \mathbb{N};\ |\lambda_k| \leqq c\}$ is finite for every $c > 0$,

d) $\lambda \in \varrho_A$ whenever $\lambda \neq \lambda_k$.

In addition, if A is bounded below by γ then $\lambda_k \geqq \gamma$ for all $k \in \mathbb{N}$. We assume λ_k to be indexed in non-decreasing order and then we have

$$\inf_{\substack{u \in D_A \\ u \neq 0}} \frac{(Au, u)}{\|u\|^2} = \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_k \leqq \ldots \quad \textit{and} \quad \lambda_k \to \infty \quad \textit{as} \quad k \to \infty .$$

The domain of an operator A satisfying the assumptions of the preceding lemma is characterized by

$$D_A = \{u \in H;\ \sum_{k=1}^{\infty} \lambda_k^2 |(u, v_k)|^2 < \infty\}$$

and

$$Au = \sum_{k=1}^{\infty} \lambda_k (u, v_k)\, v_k .$$

The latter equality is usually referred to as *the spectral decomposition* of the operator A.

Any self-adjoint positive definite operator has a unique positive definite square root $A^{1/2}$, so that $(A^{1/2})^2 = A$, and $D_{A^{1/2}} = H_A$ holds. If, moreover, A has a compact resolvent then

$$D_{A^{1/2}} = \{u \in H;\ \sum_{k=1}^{\infty} \lambda_k |(u, v_k)|^2 < \infty\}$$

and

$$A^{1/2} u = \sum_{k=1}^{\infty} \lambda_k^{1/2} (u, v_k)\, v_k .$$

2.5. **Two criteria.** We conclude this section by two results the first of which is a criterion for A to have a compact resolvent while the other is a test for the positive definiteness.

Lemma. *Let A be self-adjoint and bounded below. Then A has a compact resolvent if and only if the energy space H_A is compactly embedded in H.*

Proof. We shall only prove the useful assertion, that is, A has a compact resolvent provided H_A is compactly embedded in H. Let $M \subset H$ be bounded. Choose $\lambda \in \mathbb{R} \cap \cap\, \varrho_A$. Then $A - \lambda I$ is a linear homeomorphism of D_A (which is endowed with the graph norm $|||u||| = \|Au\| + \|u\|$) onto H and $(A - \lambda I)^{-1} M$ is a bounded set in D_A. Since D_A *is embedded in* H_A the set $(A - \lambda I)^{-1} M$ is bounded in H_A and by assumption it is relatively compact in H.

The converse may be proved by using the compactness of the operator $[(A + \mu)^{1/2}]^{-1} : H \to H$.

Corollary. *Let A be non-negative and let H_A be compactly embedded in H. If 0 is not an eigenvalue of A^* $(N_{A^*} = \{0\})$ then A is positive definite (and moreover*), A is essentially self-adjoint).*

Proof. Let $\tilde{A}$ be the Friedrichs extension of A. ($\tilde{A}$ coincides with $\bar{A}$ provided A is essentially self-adjoint.) As H_A equals $H_{\tilde{A}}$ it follows from the preceding lemma that $\tilde{A}$ has a compact resolvent. Moreover, $\tilde{A}$ is non-negative. By Lemma 2.4 we get the following alternative:

– either 0 is an eigenvalue of $\tilde{A}$,

– or the smallest eigenvalue λ_1 of $\tilde{A}$ is positive and in this case $\tilde{A}$ is positive definite:

$$(\tilde{A}u, u) \geqq \lambda_1 \|u\|^2 \quad \text{for all} \quad u \in D_{\tilde{A}} .$$

Since A^* is an extension of $\tilde{A}$ the assertion follows immediately.

At this point, let us only remark that the result is no longer true if the assumption $N_{A^*} = \{0\}$ is replaced by $N_A = \{0\}$.

3. PROPERTIES OF THE OPERATOR L_0

3.1. **The operator L_0.** Let L_0 denote an operator defined by

$$D_{L_0} = \{u \in C^4([0, R]),\ u(R) = u'(R) = 0,\ u'(0) = u'''(0) = 0\} ,$$

$$L_0 u = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left\{ r \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}u}{\mathrm{d}r} \right) \right] \right\} .$$

Some alternate expressions for $L_0 u$ are the following ones:

$$L_0 u = \frac{1}{r} \left[(ru'')'' - \left(\frac{1}{r} u'' \right) \right] = u'''' + \frac{2}{r} u''' - \frac{1}{r^2} u'' + \frac{1}{r^3} u' .$$

The boundary conditions imposed at $r = 0$ reflect the fact that u should represent the restriction to a radial section of a function defined on a circle K that depends only on the radial coordinate r.

3.2. **The weighted space $L_{2,r}(0, R)$.** We shall study properties of L_0 in the framework of the weighted space $L_{2,r}(0, R)$. Here $L_{2,r}(0, R)$ stands for the space of (equi-

---

*) by Corollary 2.2.

valence classes of) complex Lebesgue-measurable functions defined on $(0, R)$ for which

$$\|u\|^2 = \int_0^R r|u(r)|^2 \, dr$$

is finite. The formula $Uu = r^{1/2}u$ defines an isometric isomorphism of $L_{2,r}(0, R)$ onto $L_2(0, R)$. In particular, we get that $L_{2,r}(0, R)$ is a Hilbert space with the inner product

$$(u, v) = \int_0^R r\, u(r)\, \overline{v(r)} \, dr .$$

3.3. **Symmetry and non-negativity of L_0.** In what follows let H denote the space $L_{2,r}(0, R)$.

We shall regard L_0 as an operator

$$L_0 \colon D_{L_0} \subset H \to H .$$

It is a matter of routine to check (using Taylor's expansion) that $L_0 u$ really belongs to H for $u \in D_{L_0}$. As a consequence of the density of $C_0^\infty(0, R)$ (the set of infinitely differentiable functions with a compact support in $(0, R)$) in $L_2(0, R)$ we get that D_{L_0} is dense in H, i.e. L_0 is densely defined.

Proposition. L_0 *is non-negative (hence symmetric).*

Proof. This property is obtained by a simple integration by parts which yields

$$(L_0 u, u) = \int_0^R \left(r|u''|^2 + \frac{1}{r} |u'|^2 \right) dr \geqq 0 \quad \text{for all} \quad u \in D_{L_0} .$$

We only need to take into account the relation

$$\lim_{r \to 0+} \left(u''(r) - \frac{1}{r} u'(r) \right) = 0$$

which is true for any $u \in D_{L_0}$ (use again Taylor's expansions).

3.4. **Compactness of the embedding of H_{L_0} in H.** The energy space H_{L_0} is given by completion of D_{L_0} in the energy norm (cf. Sec. 2.3, where we set $\mu = 1$)

$$\|u\|_{H_{L_0}} = \left(\int_0^R \left[r(|u|^2 + |u''|^2) + \frac{1}{r} |u'|^2 \right] dr \right)^{1/2} .$$

Using the Sobolev embedding theorem (see, e.g., [1]) we can describe the energy space as the collection of functions $u \in H$ that are absolutely continuous together with u' on $[\delta, R]$ for any $0 < \delta < R$, $u(R) = u'(R) = 0$ and $\|u\|_{H_{L_0}}$ is finite. (In fact, $u \in C^1([0, R])$ and moreover $u'(0) = 0$, but we do not need this result.)

Proposition. H_{L_0} *is compactly embedded in* H.

We give the proof (which may be found elsewhere) for the sake of completeness. Let $\{u_m\}$ be a bounded sequence in H_{L_0}. In view of the reflexivity of H_{L_0} we can assume that $u_m \to u$ weakly in H_{L_0} (replacing $\{u_m\}$ by a subseqence if necessary). We prove that $u_m \to u$ in H (strongly) which yields the compactness of the embedding of H_{L_0} in H.

For any $0 < \delta < R$ let χ_δ be the function defined by $\chi_\delta(r) = 0$ for $0 \leqq r < \delta$, $\chi_\delta(r) = 1$ for $\delta \leqq r < 1$. Further, let $\chi_\delta H_{L_0}$ denote the space of restrictions $u|_{(\delta,R)}$ of functions $u \in H_{L_0}$ to the interval (δ, R). Similarly for $\chi_\delta H$. It is clear that $\chi_\delta H_{L_0}$ is embedded in $W_2^2(\delta, R)$. Since $W_2^2(\delta, R)$ is compactly embedded in $L_2(\delta, R)$ (see [1], p. 144) and $L_2(\delta, R) = \chi_\delta H$ (with equivalent norms) we get that $\chi_\delta H_{L_0}$ is compactly embedded in $\chi_\delta H$. Using this result we infer from $u_m \to u$ weakly in H_{L_0} that the sequence $\{\chi_\delta u_m\}$ is convergent in H. Because of the inequality

$$\|u_m - u_n\| \leqq \|u_m - \chi_\delta u_m\| + \|\chi_\delta u_m - \chi_\delta u_n\| + \|\chi_\delta u_n - u_n\|$$

it is now enough to show that

$$\|u_m - \chi_\delta u_m\| \to 0 \quad \text{as} \quad \delta \to 0+$$

uniformly with respect to m. But this is a consequence of the estimates

$$\|u_m - \chi_\delta u_m\|^2 = \int_0^\delta r|u_m(r)|^2 \, dr = \int_0^\delta r \left|\int_r^R u_m'(s) \, ds\right|^2 dr \leqq$$

$$\leqq \frac{R\delta^2}{2} \int_0^R |u_m'(r)|^2 \, dr \leqq \frac{R^2\delta^2}{2} \int_0^R \frac{1}{r} |u_m'(r)|^2 \, dr \leqq \text{const. } \delta^2 \|u_m\|_{H_{L_0}}^2 .$$

The proof is complete.

3.5. **Triviality of the null-space of L_0^*.** Let $w \in N_{L_0^*}$. This means that $w \in H$ and

$$(3.1) \qquad (L_0 u, w) = \int_0^R \left[(ru'')'' - \left(\frac{1}{r} u'\right)'\right] \overline{w} \, dr = 0 \quad \text{for all} \quad u \in D_{L_0} .$$

We prove that $w = 0$, that is to say,

Proposition. $N_{L_0^*} = \{0\}$.

We divide the proof in four steps. We successively prove

(i) $w \in C^\infty((0, R])$,

(ii) w satisfies the Euler equation

$$(3.2) \qquad w'''' + \frac{2}{r} w''' - \frac{1}{r^2} w'' + \frac{1}{r^3} w' = 0 ,$$

(iii) the general solution of (3.2) is

$$(3.3) \qquad w(r) = C_1 + C_2 \left(\frac{r}{R}\right)^2 + C_3 \ln \frac{r}{R} + C_4 \left(\frac{r}{R}\right)^2 \ln \frac{r}{R}$$

where $C_1, \ldots, C_4$ are arbitrary constants, and eventually

(iv) $C_1 = C_2 = C_3 = C_4 = 0$.

Using a standard procedure let us take any function $\varphi \in C_0^\infty(0, R)$ and insert

$$u(r) = r^{-1/2}\, \varphi(r)$$

in (3.1). We make this choice in order to eliminate the term containing the third derivative of u in (3.1). We obtain

$$\int_0^R \left(r^{1/2}\varphi'''' + \tfrac{1}{2}r^{-3/2}\varphi'' - r^{-5/2}\varphi' + \tfrac{25}{16}r^{-7/2}\varphi\right) \overline{w}\, \mathrm{d}r = 0\,.$$

Now, let $0 < \delta < R$ be arbitrary. As w is a function from $N_{L_0*} \subset L_{2,r}(0, R)$, $w|_{(\delta,R)}$ belongs to $L_2(\delta, R)$. For notational convenience we keep the notation w for $w|_{(\delta,R)}$. We therefore have in the sense of distributions

$$\left(r^{1/2}\overline{w}\right)'''' = -\tfrac{1}{2}\left(r^{-1/2}\overline{w}\right)'' - \left(r^{-5/2}\overline{w}\right)' - \tfrac{25}{16}r^{-7/2}\overline{w}$$

on the interval (δ, R). The right-hand side, say g, is an element from $W_2^{-2}(\delta, R)$ (see, e.g., [1], p. 50). It is well-known that then the equation $\bar{v}'''' = g$ has a (unique) solution $v \in \mathring{W}_2^2(\delta, R)$, hence $(r^{1/2}w - v)''$ is a linear function in r and consequently $w \in W_2^2(\delta, R)$. Now, the right-hand side belongs to $L_2(\delta, R)$ and thus $w \in W_2^4(\delta, R)$. If we go on reiterating in this way we obtain $w \in W_2^q(\delta, R)$ for any $q < \infty$ (the bootstrap argument) and the embedding theorem yields $w \in C^\infty((0, R])$.

Integrating (3.1) by parts with $u = \varphi$ an arbitrary function from $C_0^\infty(0, R)$ we get

$$\int_0^R \left[\overline{(rw'')'' - \left(\frac{1}{r}w'\right)'}\right] \varphi\, \mathrm{d}r = 0\,.$$

Thus, w satisfies the Euler equation (3.2).

The indicial equation of (3.2) is $\lambda^2(\lambda - 2)^2 = 0$. (We arrive at this equation formally if we look for a solution in the form r^λ.) The fundamental system of solutions is formed by the functions 1, r^2, $\ln r$, $r^2 \ln r$ (see, e.g., [5], p. 85). Hence the general solution of the equation (3.2) can be written in the form (3.3).

To prove (iv) we shall pick out suitable functions u to insert them in (3.1). If $u \in D_{L_0}$ and $L_0\, u(r) = 0$ for $r \in (0, R/2)$ then

$$\begin{aligned} &[\overline{w}(ru'')' - r\overline{w}'u'']\,|_{r=R} - \overline{w}\left[(ru'')' - \frac{1}{r}u'\right]|_{r=R/2} + \\ &+ u\left[(r\overline{w}'')' - \frac{1}{r}\overline{w}'\right]\Big|_{r=R/2} + r(\overline{w}'u'' - u'\overline{w}'')\,|_{r=R/2} = 0\,. \end{aligned} \tag{3.4}$$

We pause briefly to give the explicit formulas for the derivatives of w. Namely,

$$w'(r) = \left(2C_2 + C_4 + 2C_4 \ln\frac{r}{R}\right)\frac{r}{R^2} + \frac{C_3}{r}\,,$$

$$w''(r) = \left(2C_2 + 3C_4 + 2C_4 \ln \frac{r}{R}\right) \frac{1}{R^2} - \frac{C_3}{r^2},$$

$$w'''(r) = \frac{2C_4}{rR^2} + \frac{2C_3}{r^3}.$$

Let us now take $u \in D_{L_0}$ such that $u = 1$ for $r \in [0, R/2]$. If, moreover, $u''(R) = u'''(R) = 0$ then all brackets in (3.4) vanish except for the third one and this in turn gives $\bar{C}_4 = 0$. If we had taken $u''(R) = 0$ but $u'''(R) \neq 0$ instead then (3.4) would have become $R\, \bar{w}(R)\, u'''(R) = 0$ and consequently $\bar{C}_1 + \bar{C}_2 = 0$. Likewise, if $u''(R) \neq 0$ and $u'''(R) = 0$ then

$$(\bar{w}(R) - R\, \bar{w}'(R))\, u''(R) = 0 \quad \text{implies} \quad \bar{C}_1 - \bar{C}_2 - \bar{C}_3 = 0\,.$$

Now, let us choose $u \in D_{L_0}$ such that $u(r) = r^2$ for $r \in [0, R/2]$ and $u''(R) = u'''(R) = 0$. We get $\bar{w}'(R/2) - (R/2)\, \bar{w}''(R/2) = 0$ and in virtue of this, $\bar{C}_3 = 0$. Summarizing, we easily see that $C_1 = C_2 = C_3 = C_4 = 0$, hence $w = 0$ and the proposition is proved.

3.6. **Properties of L_0.** We are now in a position to state the fundamental properties of the operator L_0. They are direct consequences of Propositions 3.3–3.5 and of the general lemmas of Section 2.

Let L be the closure of L_0, i.e. $L = \bar{L}_0$.

Theorem. 1) L_0 *(and L) is positive definite,*

2) L_0 *is essentially self-adjoint (in particular, $L = L_0^*$),*

3) L^{-1} *is compact.*

Proof. The first two assertions follow from Corollary 2.5. The compactness of the embedding of H_L $(= H_{L_0})$ in H implies that L has a compact resolvent (Lemma 2.5) and owing to 1) this is equivalent to 3). The proof is complete.

4. APPLICATION TO THE STATIC AND DYNAMIC PROBLEMS IN THE THEORY OF PLATES

4.1. **Definitions of solutions.** The properties of the operator L_0 established in the preceding theorem make it possible to derive some existence theorems for static and dynamic problems formulated in Section 1. In Section 5 we will continue to employ these properties to obtain the solution representations.

To begin with, let us give a precise meaning to a solution of the above problems.

By a solution of the problem $(\mathscr{P}_{st})$ given by (1.4) and (1.7) we mean a function $u \in D_L$ satisfying the equation

$$Lu = g\,.$$

More explicitly, a function $u \in H$ is a solution of $(\mathscr{P}_{\text{st}})$ if and only if there exists a sequence of functions $u_n \in D_{L_0}$ (see Sec. 3.1) such that $u_n \to u$ in H and $L_0 u_n \to g$ in H.

Other equivalent definitions are possible. For instance, since $L = L_0^*$ we get: $u \in H$ is a solution of $(\mathscr{P}_{\text{st}})$ if and only if

$$u \in D_{L_0^*}(= D_L) = \left\{ u \in H_{L_0};\int_0^R \frac{1}{r}\left|(ru'')'' - \left(\frac{1}{r}u'\right)'\right|^2 \mathrm{d}r < \infty \right\}$$

and

$$L_0^* u\,(= Lu) = \frac{1}{r}\left[(ru'')'' - \left(\frac{1}{r}u'\right)'\right] = g\,,$$

where the derivatives are taken in the sense of distributions on $(0, R)$. The latter equality, $L_0^* u = g$, means the same as the relation

$$(4.1) \qquad (u, L_0 v) = (g, v)$$

for all $v \in D_{L_0}$, where $(\cdot, \cdot)$ means the scalar product in H.

By a solution of the problem $(\mathscr{P}_{\text{dyn}})$ given by (1.5), (1.7)–(1.9) we mean a function $u \in C([0, T]; D_L) \cap C^1([0, T]; H_L) \cap C^2([0, T]; H)$ satisfying

$$u_{tt} + a^2 Lu = g\,, \quad t \in (0, T)\,,$$

$$u(0) = \varphi\,,$$

and

$$u_t(0) = \psi\,.$$

4.2. **Theorem.** *Let $g \in H$. Then the problem $(\mathscr{P}_{\text{st}})$ has a unique solution.*

Proof follows immediately from Lemma 2.2.

4.3. **Theorem.** *Let $g \in C([0, T]; H_L)$, $\varphi \in D_L$ and $\psi \in H_L$. Then the problem $(\mathscr{P}_{\text{dyn}})$ has a unique solution.*

Proof is based on Lemma 2.4. The method is the same as, e.g., in [12], p. 52.

5. SERIES REPRESENTATION OF SOLUTIONS

5.1. **Eigenfunction expansions.** So far we have only been taking up existence questions. The solutions of the both problems $(\mathscr{P}_{\text{st}})$ and $\mathscr{P}(_{\text{dyn}})$ can be expressed in the form of Fourier expansions if we use the spectral decomposition of the operator L.

Let $\{\lambda_k\}_{k=1}^{\infty}$ be the non-decreasing sequence of eigenvalues of L (every eigenvalue counted according to its multiplicity, which is finite) and $\{v_k\}_{k=1}^{\infty}$ the corresponding complete orthonormal system in H $(= L_{2,r}(0, R))$. The existence of such sequences is ensured by Lemma 2.4 and Theorem 3.6. We have, moreover,

$$0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_k \leqq \ldots$$

and ∞ is the only limit point of $\{\lambda_k\}_{k=1}^{\infty}$.

Using the spectral decomposition of the operator L, i.e.

$$Lu = \sum_{k=1}^{\infty} \lambda_k(u, v_k)\, v_k \quad \text{for all} \quad u \in D_L$$

(the convergence in the sense of H), we obtain that the solution u of $(\mathscr{P}_{st})$ is representable by the series

$$u(r) = \sum_{k=1}^{\infty} \frac{g_k}{\lambda_k} v_k(r)\,, \tag{5.1}$$

where

$$g_k = (g, v_k)\,.$$

By definition (see Sec. 4.1) the solution u is the limit (in H) of $u_n \in D_{L_0}$ such that $L_0 u_n \to g$. Since $v_k \in D_{L_0}$ for all $k \in \mathbb{N}$ (see Sec. 5.2) we may take

$$u_n = \sum_{k=1}^{n} \frac{g_k}{\lambda_k} v_k\,.$$

Analogously, the solution of the problem $(\mathscr{P}_{dyn})$ can be represented in the form

$$u(r, t) = \sum_{k=1}^{\infty} \left[\varphi_k \cos(\lambda_k^{1/2} t) + \frac{\psi_k}{\lambda_k^{1/2}} \sin(\lambda_k^{1/2} t) + \right.$$
$$\left. + \frac{1}{\lambda_k^{1/2}} \int_0^t \sin(\lambda_k^{1/2}(t - \tau))\, g_k(\tau)\, \mathrm{d}\tau \right] v_k(r)\,, \tag{5.2}$$

where

$$\varphi_k = (\varphi, v_k)\,, \quad \psi_k = (\psi, v_k)$$

and

$$g_k(t) = (g(\cdot, t), v_k)\,.$$

Under the assumptions of Theorem 4.3 the series in (5.2) converges in H uniformly with respect to $t \in [0, T]$, and the same is true for the series arising by differentiating twice term-by-term with respect to t or multiplicating by λ_k.

5.2. **The form of eigenfunctions and the equation for eigenvalues.** What actually remains is to find the concrete form of $\{v_k\}$ and $\{\lambda_k\}$. For the sake of completeness we briefly indicate the necessary formulas, referring, e.g., to [2] for details.

The fundamental system of solutions to the equation $Lv - \lambda v = 0$ is formed by the functions $\mathrm{J}_0(\lambda^{1/4} r)$, $\mathrm{Y}_0(\lambda^{1/4} r)$, $\mathrm{I}_0(\lambda^{1/4} r)$, $\mathrm{K}_0(\lambda^{1/4} r)$, where J_0, Y_0, I_0 and K_0 are Bessel, Neumann, modified Bessel and MacDonald functions (of index zero), respectively. (For the literature on these functions we refer the reader to [3], [7], [13].) The functions $\alpha \mathrm{Y}_0 + \beta \mathrm{K}_0$ do not belong to H_L for any constants α and β, $|\alpha| + |\beta| > 0$, unless $\alpha/\pi = \beta/2$, but $\pi \mathrm{Y}_0 + 2\mathrm{K}_0 \notin D_L$.

The (orthonormal) eigenfunctions assume the form

$$v_k(r) = \frac{1}{R|\mathrm{J}_0(\mu_k)|\, \mathrm{I}_0(\mu_k)} \left[\mathrm{J}_0\left(\mu_k \frac{r}{R}\right) - \frac{\mathrm{J}_0(\mu_k)}{\mathrm{I}_0(\mu_k)} \mathrm{I}_0\left(\mu_k \frac{r}{R}\right)\right], \quad k \in \mathbb{N}\,,$$

where $\{\mu_k\}_{k=1}^{\infty}$ is the sequence of positive roots of the equation

$$J_0(\mu)\, I_0'(\mu) - J_0'(\mu)\, I_0(\mu) = 0$$

and

$$\lambda_k = \left(\frac{\mu_k}{R}\right)^4, \quad k \in \mathbb{N}.$$

Let us recall that (for any complex z)

$$J_0(z) = 1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} - \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \ldots,$$

$$I_0(z) = 1 + \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \ldots$$

and, by [6], $\mu_1 \doteq 3.190$, $\mu_2 \doteq 6.306$, $\mu_3 \doteq 9.425$.

6. GENERALIZATION

Let us point out that the proofs of Theorems 4.2 and 4.3 do not depend on the particular form of L_0 and can be carried out with any operator L_0 whenever it has the following properties:

1) *$L_0: D_{L_0} \subset H \to H$, D_{L_0} is dense in H, H is a Hilbert space,*

2) *L_0 is symmetric and non-negative,*

3) *the energy space of L_0 is completely embedded in H,*

4) *if $w \in H$ and $(L_0 u, w) = 0$ for all $u \in D_{L_0}$ then $w = 0$ (in other words, the null-space of the adjoint operator L_0^* is trivial).*

In this way, for any such general operator (often encountered in various problems of mathematical physics) we get the existence and uniqueness of a solution (more precisely, a "generalized solution") of the problems given by $L_0 u = g$ and $u_{tt} + L_0 u = g$, $u(0) = \varphi$, $u_t(0) = \psi$, respectively. These solutions can be developed in the Fourier series with respect to the system of eigenvectors (cf. (5.1) and (5.2)).

Remark. Other types of (generalized) solutions to the above mentioned problems can be defined for which the representation by means of the Fourier series is still valid. For example, the solutions with finite energy, the so-called energy solutions, are sometimes of our preference. The energy solution of $(\mathscr{P}_{st})$ is defined as a function $u \in H_{L_0}$ satisfying (4.1) for all $v \in D_{L_0}$. The energy solution of $(\mathscr{P}_{dyn})$ is defined as a function $u \in C([0, T]; H_{L_0}) \cap C^1([0, T]; H)$ satisfying

$$\int_0^T [-(u_t, w_t) + a^2(u, L_0 w) - (g, w)]\, dt = (\psi, w(0)), \quad u(0) = \varphi$$

for all

$$w \in C([0, T];\ H_{L_0}) \cap C^1([0, T]; H), \quad w(T) = 0.$$

The energy solution of $(\mathscr{P}_{st})$ exists under the assumption that $g \in H$ (or, more generally, g is an (anti)linear continuous functional over H_{L_0}), while the energy solution of $(\mathscr{P}_{dyn})$ exists if $g \in C([0, T]; H)$, $\varphi \in H_{L_0}$ and $\psi \in H$. These solutions are given again by the Fourier series (5.1) and (5.2), respectively, where the convergence is to be understood in the corresponding topologies.

References

[1] *R. A. Adams:* Sobolev spaces. Academic Press 1975.

[2] *B. M. Budak, A. A. Samarskii, A. N. Tikhonov:* A collection of problems on mathematical physics. International series of monographs in pure and applied mathematics 52, Pergamon Press, Oxford 1964. (Russian: Izd. Nauka, Moscow 1980, 3rd ed.)

[3] *A. Erdélyi et al.:* Higher transcendental functions, Vol 2. McGraw Hill 1953.

[4] *W. G. Faris:* Self-adjoint operators. Lecture Notes in Mathematics 433, Springer-Verlag 1975.

[5] *P. Hartman:* Ordinary differential equations. J. Wiley and Sons 1964.

[6] *E. Jahnke, F. Emde:* Tables of functions. Dover Publ. 1945 (4th ed.).

[7] *W. Magnus, F. Oberhettinger:* Formeln und Sätze für die speziellen Funktionen der mathematischen Physik. Springer-Verlag 1948 (2nd ed.).

[8] *D. E. Mc Farland, B. L. Smith, W. D. Bernhart:* Analysis of plates. Spartan Books 1972.

[9] *S. G. Mikhlin:* Linear partial differential equations. (Russian.) Vysš. škola, Moscow 1977.

[10] *S. Timoshenko, D. H. Young, W. Weaver Jr.:* Vibrations problems in engineering. John Wiley and Sons 1974 (4th ed.). (Russian: Mashinostrojenije, Moscow 1985.)

[11] *S. Timoshenko, S. Woinowski-Krieger:* Theory of plates and shells. McGraw Hill 1959.

[12] *O. Vejvoda et al.:* Partial differential equations: time-periodic solutions. Martinus Nijhoff 1982.

[13] *G. N. Watson:* A treatise on the theory of Bessel functions. Cambridge Univ. Press 1958 (2nd ed.).

[14] *J. Weidmann:* Linear operators in Hilbert spaces. Graduate texts in Mathematics 68, Springer-Verlag 1980.

Souhrn

VYŠETŘENÍ OPERÁTORU Z TEORIE KRUHOVÝCH DESEK

LEOPOLD HERRMANN

V článku se dokazuje, že operátor $L_0\colon D_{L_0} \subset H \to H$, $L_0 u = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)\right]\right\}$, $D_{L_0} = \{u \in C^4([0, R]),\ u'(0) = u'''(0) = 0,\ u(R) = u'(R) = 0\}$, $H = L_{2,r}(0, R)$ je v podstatě samoadjungovaný, positivně definitní s kompaktní rezolventou. Jsou ukázány podmínky na operátor L_0 (i v případě obecného symetrického operátoru), zaručující aplikovatelnost Fourierovy metody pro řešení úloh typu $L_0 u = g$ a $u_{tt} + L_0 u = g$.

Резюме

ИССЛЕДОВАНИЕ ОПЕРАТОРА, ВОЗНИКАЮЩЕГО В ТЕОРИИ КРУГОВЫХ ПЛАСТИНОК

LEOPOLD HERRMANN

В работе доказывается, что оператор L_0: $D_{L_0} \subset H \to H$, $L_0 u = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)\right]\right\}$, $D_{L_0} = \{u \in C^4([0, R]), u'(0) = u'''(0) = 0, u(R) = u'(R) = 0\}$, $H = L_{2,r}(0, R)$ существенно само-сопряжённый и положительно определённый с компактной резольвентой, и найдены условия на оператор L_0 (даже в общем случае симметрического оператора), которые обеспечивают применимость метода Фурье к решению проблем типа $L_0 u = g$ или $u_{tt} + L_0 u = g$.

Author's address: RNDr. *Leopold Herrmann*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.