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TRANSONIC FLOW CALCULATION VIA FINITE ELEMENTS

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Summary. Using new results based on a convenient entropy condition, two types of algorithms for computing transonic flows are constructed. A sequence of solutions of the linearised problem with a posteriori control is constructed and its convergence to the physical solution of transonic flow in some special situations is proved.

This paper contains also numerical results and their analysis for the case of flow past NACA 230012 airfoil. Some numerical improvements of the general algorithms, based on our practical experience with this problem, are also included.

Keywords: transonic potential flow, finite elements, entropy condition.

AMS classification: 76-08, 86H05, 65N30.

INTRODUCTION

Numerical computation of steady-state transonic potential flows of an inviscid, isentropic, irrotational, ideal compressible fluid has belonged for a long time to the very difficult problems of numerical as well as applied mathematics. The problems of existence and uniqueness have been open until now. The difficulties consist in the fact that the equation for the potential of velocity is nonlinear and of mixed type. It is elliptic in the subsonic region and hyperbolic in the transonic one, and the boundary dividing these two domains is not known a priori. Generally, when passing across this boundary jumps of velocity, density and pressure may occur. Due to this fact one has to look for a weak solution of this boundary value problem. If we introduce the weak formulation similarly to the theory of elliptic equations the arising functional is not convex.

Even in this situation there exist a number of numerical models and methods for their solution. Some of them are based on finite differences but we have chosen the method of finite elements which naturally corresponds to our approach.

In our paper we broadly use the results contained in the paper "On the solvability of transonic potential flow problems" by M. Feistauer and J. Nečas. Many of the numerical experiments we have performed show the fundamental nonuniqueness of the discretized problem. Most of the diagnosed methods tend to search for non-physical solutions. For this reason we introduce the a posteriori check based on the verification of the entropy condition. We use the secant-modulus method as the based mean for the construction of the sequence of solutions that can (under some conditions) converge to the solution. When we have such a set of solutions of the linearised problems we choose, as the next step of an abstract algorithm, only those which satisfy the entropy condition. It was discovered by M. Feistauer and J. Nečas [1] that this condition brings the missing compactness into the problem. As the next step we can use the minimization of the ALTERNATIVE functional that has the indicative property.

We include into this paper the analysis of the results obtained by the algorithms constructed, using NACA 230012 airfoil.

This paper does not contain results dealing with the convergence of finite elements because this topic is a subject of the authors' forthcoming papers.

All methods described herein are rather heuristical because of a permanent lack of an existence and uniqueness results for this problem. The aim of the present paper seems to be to verify the existing theoretical results for constructions of numerical methods.

1. MATHEMATICAL FORMULATION OF THE PROBLEM

From now on we will consider an adiabatic, isentropic, irrotational, steady-state and compressible flow of a nonviscous fluid in a bounded, simply connected domain Ω, which can be described by the Full Potential Equation

$$-\operatorname{div}(\varrho \,.\, \nabla u) = 0 \quad \text{in} \quad \Omega \,, \tag{1.1}$$

where u is the velocity potential, ∇u is the vector of velocity and ϱ is the density given by the function

$$\varrho = \varrho(|\nabla u|^2) = \varrho_0 \left(1 - \frac{\varkappa - 1}{2a_0^2} |\nabla u|^2\right)^{1/(\varkappa - 1)} \tag{1.2}$$

ϱ_0 is the density corresponding to zero velocity of the flow, a_0 is the speed of sound, $\varkappa$ is the adiabatic constant ($\varkappa = 1{\cdot}4$ for air). We consider the following types of boundary conditions:

$$u = 0 \quad \text{on} \quad \Gamma_1 \tag{1.3}$$

$$\varrho \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_2 \cup \Gamma_3 \,,$$

$$g|_{\Gamma_3} = 0 \quad \text{and} \quad g|_{\Gamma_2} < 0 \,.$$

Now $\partial\Omega = \bigcap_{i=1}^{3} \Gamma_i \cup \mathfrak{N}$; $\mathfrak{N}$ is a subset of $(N - 1)$ dimensional measure zero (N is the dimension of the Euclidean space containing Ω). Using this notation we may call Γ_2 the entrance and Γ_1 the exit of the region Ω. As one can see, Γ_3 represents the "walls" of Ω.

Instead of the set of conditions (1.3) one may consider the following one:

$$(1.3') \qquad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega ,$$

where g is the given normal component of the mass flow through the boundary,

$$\int_{\partial\Omega} g \, dS = 0 \quad \text{(conservation of the mass)} .$$

At the first glance the system (1.1)–(1.3) or (1.3′) seems to be very easy to dealwith. But in fact it represents a complicated theoretical problem closely connected with many numerical difficulties as mentioned (partially) in the introduction. The main reasons are the following:

(1.4) The equation (1.1) is nonlinear and of mixed type: it is elliptic in the subsonic region and hyperbolic in the transonic one; when modelling the flow past e.g. airfoils shocks may occur, that means discontinuities (the so called shock waves) in the velocity, density and pressure.

(1.5) Most of the methods based n mere minimization of the functional corresponding to the weak formulation of the problem given by (1.1)–(1.3) or (1.3′) tend to select the solution with expansion shocks which is not admissible from the physical point of view. Therefore it is necessary to insert artificial viscosity into the process of calculation to eliminate this type of solutions.

The situation across the shock is described by at least two conditions:

$$(1.7) \qquad \varrho(\partial u/\partial n)|_- = \varrho(\partial u/\partial n)|_+ \quad \text{(Prandtl's condition)} ,$$

where $-$ and $+$ denote the quantity in front of the schok and behind it, respectively;

$$(1.8) \qquad |\nabla u\||_- > |\nabla u\||_+ \quad \text{(Entropy condition)} .$$

The condition (1.8) is satisfied by any physical solution. It expresses the fact that the density across the shock must increase. Or, in other words, that rarefaction shocks are impossible. The physical condition of bounding the velocity is represented here by

$$(1.9) \qquad |\nabla u| \leqq u_0 < (2a_0^2/(\varkappa - 1))^{1/2} \quad \text{a.e. in} \quad \Omega ,$$

where u_0 is a fixed constant. We also require

$$(1.10) \qquad |g| < \max_{[0,\text{const}]} t\,\varrho(t^2) .$$

The condition (1.8) shows that the model given by (1.1)–(1.3) does not strictly describe the situation because it does not express the macroscopic effect of the viscosity of the fluid. The a posteriori application of (1.8) is unsatisfactory because we a priori neglect the physical situation. On the other hand there are at least two good reasons to do it.

Firstly it is very well known that (1.1)–(1.3) describes the subsonic situation well and its solution is very "cheap". So one can look for a suitable modification of it to achieve good results for the transonic case as well.

Secondly, it is very difficult to avoid nonphysical solutions. The only safe way is to take into account the all fundamental aspects of the model. This leads us naturally to a complex task of solving system of four equations:

1. Continuity equation.
2. Navier-Stokes equations.
3. Energy equilibrium equation.
4. State equation.

This is a system for six unknowns. These are the pressure, density, temperature and three unknowns for the velocity. This system is undoubtedly very difficult because of its nonlinearity. It is also clear that every simplification can cause the same type of trouble as with the model (1.1)–(1.3) and the condition (1.8). The model (1.1)–(1.3) covers e.g. symmetric flows past airfoils subsonic at infinity, and flows in a nozzle with subsonic entrance. For more details and extensions see R. Glowinski [5].

The paper is organized as follows:

In Chapter 2 we introduce the notation and weak formulation of the problem (1.1)–(1.3). Chapter 3 describes the basic scheme of our approach to its solution; we introduce the secant-modulus method and the indicative functional (called the alternative). This chapter contains the basic relations and properties of the functional Φ and the bilinear form B introduced in Chapter 2. Chapter 4 consists of algorithms used for the construction of sequences of functions in the space V (definition later) giving us the change to obtain the solution. In Chapter 5 we present the proofs of convergence of the methods from Chapter 4 under some special circumstances. Chapter 6 contains the analysis of numerical results and the discussion of the selected parameters and variables. Chapter 7 contains graphs of the results obtained by the methods of Chapter 4.

2. NOTATION AND WEAK FORMULATION OF THE PROBLEM

We use standard notation that is explained in detail e.g. in R. Glowinski [5]. The Sobolev spaces $W^{m,p}(\Omega)$ are equipped with the norm $\|\cdot\|_{m,p,\Omega}$ defined as the sum of the L^p norms of the generalised derivatives up to order m: $\|\cdot\|_{m,p,\Omega} = \sum_{|k|\leq m} \|D^k \cdot\|_p$, where k is the multiindex

$$|k| = \sum_{i=1}^{N} k_i \quad \text{and} \quad D^k = \frac{\partial^{|k|}}{\partial^{k_1} x_1 \dots \partial^{k_N} x_N}\,.$$

$L^p(\Omega)$ is the space of measurable functions so that

$$\int_\Omega |v(x)|^p \,\mathrm{d}x < +\infty \quad \text{for} \quad 1 \leqq p < +\infty .$$

The space $C_0(\Omega) = \{v \in C(\Omega);\ v|_{\partial\Omega} = 0\}$ is equipped with the L^∞ norm that is defined as follows: $\|u\| = \inf_{\operatorname{meas} M = 0} \sup_{x \in \Omega \setminus M} |u(x)|$. As usual

$$\mathscr{D}(\Omega) = \{v \in C^\infty(\Omega);\ \operatorname{supp} v \subset \Omega\} \quad \text{and} \quad \mathscr{D}^+(\Omega) = \{v \in \mathscr{D}(\Omega);\ v \geqq 0\} ,$$

$W_0^{m,p}(\Omega)$ stands for the closure of $\mathscr{D}(\Omega)$ in the norm of the space $W^{m,p}(\Omega)$.

In particular $\|\cdot\|_{1,2,0}^2 = \int_\Omega (\nabla \cdot)^2 \,\mathrm{d}x$. We assume that every function from $C_0(\Omega)$, $\mathscr{D}(\Omega)$ and $W^{m,p}(\Omega)$ is extended by zero to the whole $\boldsymbol{R}^N$. Integration is performed in the N-dimensional space, $N = 2$, equipped with the Lebesgue measure, or in the $(N - 1)$ dimensional space on the boundary of the domain. We consider only real functions of the real variable.

The weak formulation of the problem (1.1)–(1.3) reads as follows

$$u \in V\colon \int_\Omega (|\nabla u|^2)\, \nabla u\, \nabla h \,\mathrm{d}x = \int_{\Gamma_2} g h \,\mathrm{d}x \quad \text{for every} \quad h \in V , \tag{2.1}$$

where $V = \{v \in W^{1,2}(\Omega);\ v = 0 \text{ on } \Gamma_1\}$ with the norm $\|\cdot\|_{1,2,0}$. The entropy condition (1.8) can be written in the form

$$-\int_\Omega \nabla u\, \nabla v \,\mathrm{d}x \leqq K \int_\Omega v \,\mathrm{d}x , \quad v \in \mathscr{D}^+(\Omega) , \tag{2.2}$$

where K is a suitable constant.

The variational inequality (2.2) is the weak form of $\Delta u \leqq K$. This proposition is discussed in J. Mandel and J. Nečas [2]. Let us denote

$$S_E = \{u \in V;\ u \text{ satisfying (2.2) and (1.9)}\} .$$

It is known cf. Feistauer, Mandel, Nečas [3] that the set S_E is a compact subset of $W^{1,2}(\Omega)$. Therefore our goal is to find the solution of (2.1) in S_E.

3. CONSTRUCTION OF MINIMIZING SEQUENCES

Let

$$\Phi(u) = 1/2 \int_\Omega \int_0^{|\nabla u|^2} \varrho(t) \,\mathrm{d}t \,\mathrm{d}x - \int_{\Gamma_2} g u \,\mathrm{d}S . \tag{3.1}$$

The function ϱ was introduced in (1.2). It is necessary for our purposes to prolong this function to the interval $[0; +\infty[$ in such a way that it has the following properties:

$$\varrho(s) = \varrho_0 \left(1 - \frac{\varkappa - 1}{2a_0^2} s\right)^{1/(\varkappa - 1)} , \quad \text{for} \quad 0 \leqq s < \lambda \quad \text{and} \tag{3.2}$$

$$\lambda \in \left]\frac{2a_0^2}{\varkappa + 1} ; \frac{2a_0^2}{\varkappa - 1}\right[; \quad \lambda \text{ is close to } 2a_0^2/(\varkappa - 1) .$$

(3.3) $$\varrho(s) \quad \text{is continuous in} \quad [0; +\infty[\,,$$
$$\varrho'(s) \quad \text{is continuous in} \quad [0; +\infty[\,.$$

(3.4) There exist two constants ϱ_1, ϱ_2 such that
$$0 < \varrho_1 \leqq \varrho(s) \leqq \varrho_2 < +\infty \quad \text{for} \quad s \in [0; +\infty[\,.$$

(3.5) $$\varrho'(s) \leqq 0 \quad \text{for} \quad s \in [0; +\infty[\,.$$

The graph of density can be found (with suitable extensions that evidently exist) in Chapter 7. A suitable extension may be constructed by prolonging the density by a constant
$$\text{e.g.} \quad \varrho_\infty = \varrho\left(\frac{1}{2}\left(\lambda + \frac{2a_0^2}{\varkappa - 1}\right)\right).$$

However we will discuss the difficulties with the selection of later. Because of the notation of the weak formulation (2.1) we denote by $w = w(u)$ the solution of the linear problem:

(3.6) $$w(u) \in V\colon \int_\Omega \varrho(|\nabla u|^2)\, \nabla w(u)\, \nabla h \, \mathrm{d}x = \int_{\Gamma_2} gh \, \mathrm{d}S\,, \quad \forall h \in V\,.$$

Let us denote the left-hand side of (3.6) by $B(u; w, h)$. This is a bilinear form symmetrical in w and h.

3.7. The properties of Φ and B

From the definition of the density we can immediately obtain the following properties for the bilinear form B:

(3.8) $$|B(u; u, h)| \leqq \text{const.}\ \|u\| \,.\, \|h\|\,,$$

(3.9) $$B(u; h, h) \geqq \text{const.}\ \|h\|^2\,.$$

It can be easily seen that $\mathrm{d}\Phi(u, h)$ (Gateaux differential at the point $\boldsymbol{u}$) exists and

(3.10) $$\mathrm{d}\Phi(u, h) = B(u; u, h) - \int_{\Gamma_2} gh \, \mathrm{d}S\,, \quad \forall h \in V\,.$$

When we take (3.5) into account and denote
$$C(u) = \Phi(u) + \int_{\Gamma_2} gu \, \mathrm{d}S\,,$$
we can prove (see M. Feistauer and J. Nečas [1]) that for every $\boldsymbol{u}$ and $\boldsymbol{h}$ from V we have

(3.11) $$1/2\, B(u; h, h) - 1/2\, B(u; u, u) - C(h) + C(u) \geqq 0\,.$$

If the condition

(3.12) $$\varrho(s) + 2s\, \varrho'(s) \geqq\ >0 \quad \text{for} \quad s \in [0; +\infty[\quad \text{and some}$$

holds then $\mathrm{d}C(\cdot, \cdot)$ satisfies

$$\text{(3.13)} \qquad \mathrm{d}C(u + h, h) - \mathrm{d}C(u, h) \geqq \text{const. } \|h\|^2 , \quad u, h \in V .$$

The condition (3.12) is equivalent to the restriction for the argument s of the density ϱ:

$$\text{(3.14)} \qquad s \in [0; \lambda^*] , \quad \text{where} \quad \lambda^* \in \left]0; \frac{2a_0^2}{\varkappa + 1}\right[.$$

This means that the condition (3.13) is satisfied only in the subset of V, which consists of the velocity potentials which correspond to the strictly subsonic flow.

3.15. Secant-modulus method

The linearisation (3.6) of the problem (2.1) is only one of the possibilities (cf. R. Glowinski [5]). The very natural form of the above linearisation leads us to the method that we sometimes call "the secant-modulus method" and that can be described as follows:

(3.16) Let $u_0 \in V$ be arbitrary ,

(3.17) $u_{n+1} \in V$: $B(u_n; u_{n+1}, h) = \int_{\Gamma_2} gh \, \mathrm{d}S , \quad \forall h \in V .$

(3.18) If $\|u_n - u_{n+1}\|_{1,2} <$ epsilon then STOP

else

$u_n := u_{n+1}$; GOTO (3.17) .

If we have (3.13) we can prove the following theorem.

3.19. Theorem. *Let us have the functional Φ defined in* (3.1), *the bilinear form B defined in* (3.6) *with the properties* (3.8) *through* (3.10), *and the density* (1.2) *with an extension of the type* (3.2)–(3.5). *Let the condition* (3.13) *hold.*

Then the sequence $\{u_n\}_{n=0}^{\infty}$ obtained by (3.16)–(3.18) *converges to u (strongly) in V and u is the only critical point of Φ in this space.*

Proof. Can be found e.g. in J. Nečas and I. Hlaváček [7].

3.20. Remark. (i) Once again we repeat that it is necessary to satisfy the condition (3.13) and this implies the subsonic flow only.

(ii) Application of this method in the case of the transonic flow was unsuccessful. We have obtained the same experiences as G. Poirier [6] when he calculated the transonic flow without any type of penalisation or other a posteriori means. Any u_0 from (3.16)–(3.18) takes us to a nonphysical solution. From this one can presume that Φ has at least two local minima and one of them is very attractive for such iterative processes. The absolute minimum of Φ can't stand for the physical solution.

Because $w(u)$ is the solution of (2.1) if $w(u) = u$, let us define the so called ALTERNATIVE functional Ψ by

$$\Psi(u) = \Phi(u) - \Phi(w(u)) . \tag{3.21}$$

Of course if $\Psi(u^*) = 0$ then u^* is the solution of (2.1). However, the following lemma also holds:

3.22. Lemma

$$\Psi(u) \geqq \text{const. } \|u - w(u)\|^2 .$$

Proof. Cf. M. Feistauer and J. Nečas [1].

3.23. Consequence

$$\Psi(u) \geqq 0 \quad \textit{for every} \quad u \in V .$$

3.24. Remark. In the case of the transonic flow the condition (3.13) is not longer valid and therefore neither the functional Φ nor Ψ are convex. In this case we can apply following conceptual algorithm:

(3.25) Construct a subset A of local minima of Ψ .

(3.26) Does there exist in the subset A a point u satisfying the entropy condition (2.2)?
YES: u is the physical solution. STOP.
NO: Construct a new subset $\boldsymbol{B}$ of local minima of Ψ such that $\boldsymbol{A} \subset \boldsymbol{B}$ and let $\boldsymbol{A}$ now stand for $\boldsymbol{B}$. GOTO (3.26).

We tried to use this method to compute the flow (even successfully) but it has disadvantages which have three main reasons:

(i) Ψ is not convex,
(ii) it is very difficult to construct the set $\boldsymbol{B}$ of the above definition,
(iii) this method has enormous demands for the computer time.

Later the functional Ψ will be of great advantage for us.

4. APPLICATION OF THE SECANT-MODULUS METHOD

4.1. Definition. *For $\beta \in]0, 1]$ we define the function*

$$\tilde{\varrho}_\beta\colon [0, +\infty[\to \boldsymbol{R} \quad \text{by:}$$

$$\tilde{\varrho}_\beta(s) = \varrho_0 \left(1 - \frac{\varkappa - 1}{2a_0^2} s\right)^{\beta/(\varkappa - 1)} \quad \textit{for} \quad s \leqq \lambda , \tag{4.2}$$

$$\tilde{\varrho}_\beta(s) = \tilde{\varrho}_\beta(\lambda) \qquad \textit{for} \quad s > \lambda . \tag{4.3}$$

We denote by ϱ_β: $[0; +\infty[\to \boldsymbol{R}$ *a suitable regularization of the function* $\tilde{\varrho}_\beta$ *so that*

$$\varrho_\beta \in C^1([0; +\infty[) \quad \textit{for every} \quad \beta \in]0; 1] .$$

4.4. Remark. It is clear that the following assertions again hold:

(i) for every $\beta \in]0; 1]$ there exist ϱ_1, ϱ_2 such that

$$0 < \varrho_1 \leqq \varrho_\beta(s) \leqq \varrho_2 < +\infty \quad \text{for every} \quad s \in [0; +\infty[,$$

$$\varrho'_\beta(s) \leqq 0 \text{ in } [0; +\infty[\quad \text{for every} \quad \beta \in]0; 1] ,$$

(ii) for $\beta = 0$ we can define $\varrho_\beta = 1$;

(iii) Φ^β, C^β, B^β are defined as Φ, C, B, but with the density ϱ_β.

4.5. Lemma. *For* $\beta \in]0, 1]$ *the form* $B(u; w, h)$ *and the functional* Φ^β *satisfy the conditions* (3.8)–(3.11) *and*

(i) $\Phi^\beta(u)$: $V \to \boldsymbol{R}$ *has the Gateaux differential* $\mathrm{d}\Phi^\beta(u, \cdot)$ *for every* $u \in V$,

(ii) *the form* $B^\beta(u; w, h)$ *is bilinear and symmetric in* w *and* h,

(iii) Φ^β *is coercive for every* $u \in V$.

Proof. The functional $\Phi^\beta(u)$ is continuous and differentiable on every line in V for every $\beta \in]0; 1]$. It has the Gateaux differential. The fact that B^β is bounded and V-elliptic (see condition (1.9) with B^β) is an immediate consequence of (i) from Remark (4.4).

The verification of (3.12) proceeds in the same way as in the case of the density ϱ if we consider (ii) of (4.4).

Let us verify the coercivity of Φ^β.

$$\Phi^\beta(u) = 1/2 \int_\Omega \int_0^{|\nabla u|^2} \varrho_\beta(t) \,\mathrm{d}t \,\mathrm{d}x - \int_{\Gamma_2} gu \,\mathrm{d}S \geqq 1/2\, \varrho_1 \|u\|_{1,2}^2 - \text{const.} \|u\|_{1,2}$$

for every $\beta \in]0; 1]$. This is a consequence of the theorem of traces, the equivalence of the norms of V and $W^{1,2}$, and of the boundedness of g. Q.E.D.

Remark. Similarly we can verify the coercivity of Ψ^β.

4.6. Remark. The introduction of β into the definition of the density ϱ has extended the possibility for the application of (3.16)–(3.18) and Theorem (3.19).

Let us look for such $\beta \in]0, 1[$ that Φ^β satisfies condition (3.13). We have

$$\mathrm{d}\Phi^\beta(u + h, h) - \mathrm{d}\Phi^\beta(u, h) =$$

$$= \int_\Omega \left(\varrho_\beta(|\nabla u + \nabla h|^2)(\nabla u + \nabla h) - \varrho_\beta(|\nabla u|^2)\nabla u\right) \nabla h \,\mathrm{d}x =$$

$$= \int_\Omega \int_0^1 g'(t) \,\mathrm{d}t \,\mathrm{d}x , \quad \text{where} \quad g(t) = \varrho_\beta(|\nabla u + t \nabla h|^2) \nabla h , \quad \text{for} \quad u \quad \text{and} \quad h \in V .$$

Hence we have

$$g'(t) = 2\varrho'_\beta(|\nabla u + t \nabla h|^2)((\nabla u + t \nabla h) \nabla h) + \varrho_\beta(|\nabla u + t \nabla h|^2)(\nabla h)^2 .$$

Let us denote $v := \nabla h$ and $w := \nabla u + t \nabla h$.

To satisfy the condition (3.13) we have to look for such $c > 0$ that

$$2\varrho'_\beta(|w|^2)(wv)^2 + \varrho_\beta(|w|^2)\, v^2 \geqq cv^2 \quad \text{for every} \quad w, v \in \boldsymbol{R}^N .$$

$$\varrho_\beta(|w|^2)\, v^2 + 2\varrho'_\beta(|w|^2)(wv)^2 = (2\varrho^2_\beta(|w|^2)\, w^2 + \varrho_\beta(|w|^2))\, v^2 - \tag{4.7}$$

$$- 2\varrho'_\beta(|w|^2) \sum_{i,j=1}^{N} (w_i v_j - w_j v_i) \geqq (2\varrho'_\beta(|w|^2)\, w^2 + \varrho_\beta(|w|^2))\, v^2 .$$

The above condition can be satisfied if and only if

$$\varrho_\beta(s) + 2s\, \varrho'_\beta(s) \geqq c > 0 \quad \text{for every} \quad s \in [0, +\infty[. \tag{4.8}$$

This inequality is equivalent to finding such β that

$$1 - ((\varkappa - 1)/2a_0^2)\, s - (\beta/a_0^2)\, s > 0 . \tag{4.9}$$

From this inequality we see that $\beta < 1/M - (\varkappa - 1)/2$, where $M = s/a_0^2$, since $M \in]0, \sqrt{(6/(\varkappa - 1))}[$ we have

$$\beta \in]0, (2 - \varkappa)/3[. \tag{4.10}$$

4.11. Remark. The estimate (4.10) gives us an opportunity to apply the algorithm (3.16)–(3.18) in the following manner:

(4.12) Let $\beta \in]0, (2 - \varkappa)/3[$, $u_0 \in V$ and $n > 3/(1 + \varkappa)$,

$n \in \boldsymbol{N}$ be arbitrary.

(4.13) $w \in V$: $B(u; w, h) = \int_{\Gamma_2} gh \, \mathrm{d}S$ for every $h \in V$.

(4.14) Is $\|w - u\|_{1,2} >$ epsilon ?

YES: $u := w$ and GOTO (4.13);

NO: GOTO (4.15).

(4.15) Let $\beta := \beta + 1/n$.

Is $\beta \geqq 1$?

YES: STOP,

NO: $u := w$ and GOTO (4.13).

4.16. Remark. Theorem (3.19) implies the convergences in the above algorithm only partially, namely, at the points (4.13) and (4.14). Nonetheless, as soon as $\beta > (2 - \varkappa)/3$ there exists a solution of (4.13), but theoretically it is not clear whether $u \to u^*$ for $\beta \to 1$ and u^* satisfies the condition (2.2).

The idea of this method is to construct a "*good* u_0" for the method of secant-modulus to obtain physical solution in some special cases.

The graphs of the solutions computed are found in Chapter 7.

5. MINIMIZATION OF THE FUNCTIONAL Φ IN THE DIRECTIONS GIVEN BY THE SECANT-MODULUS METHODS

Our basic approach to the construction of a sequence of points in V that can tend to the solution of the problem (2.1) and that satisfy the condition (2.2) can be written as follows:

(5.1) Let $\varepsilon > 0$ and $u_0 \in V$ be arbitrary.

(5.2) $w(u_n) \in V$: $B(u_n; w(u_n), h) = \int_{\Gamma_2} gh \, \mathrm{d}s$ for every $h \in V$.

(5.3) Let $Au_n = \{v \in V;\ \Phi(v) = \operatorname*{locmin}_{t \in \boldsymbol{R}} \Phi(u_n + t(w(u_n) - u_n)\}$,

(5.4) Let $A^{EC}u_n = \{v \in Au_n;\ v$ satisfies the condition (2.2)$\}$.

(5.5) Is $A^{EC}u = 0$?
YES: $u_{n+1} = w(u_n)$ and GOTO (5.2)
NO: GOTO (5.6)

(5.6) $\Psi(u_{n+1})\colon = \min \Psi(u)$.

(5.7) Is $\Psi(u_{n+1}) < \varepsilon$?
YES: STOP
NO: $u_n := u_{n+1}$ and GOTO (5.2)

The algorithm (5.1)–(5.7) can be considered as the computing realization of Theorem 4.23 by M. Feistauer and J. Nečas [1]. We reproduce the Theorem here.

5.8. Theorem. *Let $\{u_n\}_{n=0}^{\infty}$ be the minimizing sequence of the functional Ψ that satisfies a posteriori the entropy condition* (1.8) *with some K and condition* (1.9). *Let $u_n \to u$ (weakly) in V if n tends to infinity. Then $u_n \to u$ (strongly) in V if n tends to infinity and u is the solution of the problem* (2.1) *that satisfies the condition* (2.2).

Let us make some remarks to the definition (5.1)–(5.7).

5.9. Remark. (i) It is of course necessary to take $t \in [-T, T]$ in (5.3). However, because of the coercivity of Φ, that is

$$\Phi(u) = 1/2 \int_\Omega \int_0^{|\nabla u|^2} \varrho(t) \, \mathrm{d}t \, \mathrm{d}x - \int_{\Gamma_2} gu \, \mathrm{d}S > 1/2\varrho_\infty \|u\|_{1,2}^2 - \|g\| \cdot \|u\|_{1,2},$$

we can find such a T by experiments.

(ii) The test (5.5) indicates whether the algorithm is successful. Strictly speaking, it means the suitability of the arbitrary constants and the choice of the point u under the assumption that we have existence and unicity of the problem (2.1).

If $A^{EC}u_n = 0$ and $Au_n = \{\tilde{u}\}$ and provided that

$$-\int_\Omega \nabla\tilde{u} \, \nabla h \, \mathrm{d}x > K \int_\Omega h \, \mathrm{d}x \quad \text{then because of the equality}$$

$$\tilde{u} = u_n + t(w(u_n) - u_n) \quad \text{for some} \quad t \in \boldsymbol{R}$$

we can write

$$-\int_\Omega \nabla u_n \nabla h \,\mathrm{d}x - t \int_\Omega \nabla(w(u_n) - u_n) \nabla h > K \int_\Omega h \,\mathrm{d}x \quad \text{for every} \quad h \in \mathscr{D}^+(\Omega) .$$

Let us even assume that u_n satisfies the condition (2.2) with K given above. Then

$$-t \int_\Omega \nabla(w(u_n) - u_n) \nabla h \,\mathrm{d}x > 0 .$$

If $t > 0$ then

$$-\int_\Omega \nabla w(u_n) \nabla h \,\mathrm{d}x > -\int_\Omega \nabla u_n \nabla h \,\mathrm{d}x .$$

If we define the function

$$\omega_n(u) = -\int_\Omega \nabla u \nabla h \,\mathrm{d}x / \int_\Omega h \,\mathrm{d}x \quad \text{for} \quad h \in \mathscr{D}^+(\Omega) \quad \text{and} \quad h > 0 ,$$

then under the above assumptions we have

$$\omega_n(w(u)) > \omega_n(u) \quad \text{for every} \quad h \in \mathscr{D}^+(\Omega) . \tag{5.10}$$

This inequality indicates a bad starting point for the process.

The case $t < 0$ cannot appear under such assumptions. Because Φ is coercive and $\Phi(\tilde{u}) \geqq \Phi(w(\tilde{u}))$ it means that Au_n contains at least two points, namely $\tilde{u}$ and some point $u_n + \tilde{t}(w(u_n) - u_n)$. But we have $Au_n = \{\tilde{u}\}$.

(iii) A priori we do not know the values of the function $\omega_n(u)$. Therefore it is a matter of some experience to build up the set $A^{EC}u_n$ during processing (5.1)–(5.7). The inequality (5.10) can be also used to indicate the a "good" choice of the constant K from (2.2).

If we consider the process $u_{n+1} = w(u_n)$ and u_n tend to u that does not satisfy (2.2) then $\omega_n(u_n) \nearrow k > K$ for some h, where K is a "suitable" constant use for the process. In other words, as soon as the function $\omega_n(u)$ begins to rise for some h this indicates that our sequence tends to a nonphysical solution. A bad choice of K can be indicated by the following two facts:

(a) for every $n \in \mathbf{N}$, $A^{EC}u_n = Au_n$ (K too large),

(b) there exists $n_0 \in \mathbf{N}$ such that $Au_n \neq 0$ and $A^{EC}u_n = 0$ and we have the existence of a solution of (2.1)

(K too small)

(iv) In (5.6) we suppose the existence of a single u_{n+1} at which Ψ attains its minimum. This step can be modified in other cases by choosing those points at which the function $\bar{\omega}_n(\cdot)$ attains its minimal value, where $\bar{\omega}_h(\cdot)$ is the maximum of $\omega_n(\cdot)$ over $\mathscr{D}_h^+(\Omega)$. If this test does not decide either we can choose the point closer to u_n with the maximal value of the parameter t. This point is unique. For the sake of readability we use in (5.6) the simpler form of this test.

(v) It is clear that the process (5.1)–(5.7) is deeply influenced by u_0.

(vi) The set $A^{EC}u_n$ can consist of the following points:

1) local minima of Φ in V,
2) saddle points of Φ in V,
3) minima in the directions considered (secant-modulus),

4) points of the form $u + T(w(u) - u)$ and $u - T(w(u) - u)$ in the case of T improperly chosen.

Let us consider only 1)–3).

If the process (5.1)–(5.7) is successful (it means $\Psi(u) = 0$) u will be of the category 1) or 2). The step (5.6) is fundamental only in the case 3) that appears most often during computation. This step corresponds to the direction of the "low descent" because this step prefers those of the points at which the difference of $\Phi(u) - \Phi(w(u))$ is as small as possible. We include this step in (5.1)–(5.2) when we have in mind that a nonphysical solution can correspond to the absolute minimum of Φ and that there exists large r such that Φ is convex in the ball $B(u, r)$ (u stands here for the absolute minimum). The step (5.6) can lead us outside of $B(u, r)$.

5.11. Lemma. *Let us have the functional Φ defined in* (3.1), *the bilinear form B from* (3.7) *and the sequence of points $\{u_n\}_{n=0}^{\infty}$ given by* (5.1)–(5.7). *We suppose that there exists such $n_0 \in \mathbf{N}$ that for every $n > n_0$ and for every $\tilde{u} \in A^{EC}u_n$*

$$\Phi(u_n) > \Phi(\tilde{u}) . \tag{5.12}$$

Then

for every $n > n_0$ there exists $\varepsilon_n > 0$ and $\beta_n < 0$ such that for every $u \in B_V(u_n; \varepsilon_n)$ ($B_V(\cdot, \cdot)$ is a ball in the space V) we have

$$\Phi(u + \lambda_n(w(u) - u)) - \Phi(u) \leqq \beta_n < 0 , \tag{5.13}$$

where $\lambda_n \in \mathbf{R}$ is defined by (5.3)–(5.7).

Proof. Let us write

$$\Phi(u + \lambda_n(w(u) - u)) - \Phi(u) =$$
$$= \Phi(u_n) - \Phi(u) + \Phi(u_n + \lambda_{u_n}(w(u_n) - u_n)) - \Phi(u_n) +$$
$$+ \Phi(u + \lambda_u(w(u) - u)) - \Phi(u_n + \lambda_{u_n}(w(u_n) - u_n)) .$$

Because of the continuity of Φ and the mapping $u \to w(u)$ (cf. Lemma 3.21, property 6 in [1]) it suffices to estimate the term $\Phi(u_n + \lambda_{u_n}(w(u_n) - u_n)) - \Phi(u_n)$.

Because of (5.3) and (5.12) this term is less or equal to zero. It is equal to zero only in the case of saddle points or local minima of Φ. But this is impossible because of (5.12). Q.E.D.

5.14. Theorem. *Let us have the sequence $\{u_n\}_{n=0}^{\infty}$ determined by the algorithm* (5.1)–(5.7), *the functional Φ given by* (3.1), *and let us assume that Φ is strictly convex in V.*

Then there exists a subsequence $\{u_{n_k}\}_{k=0}^{\infty}$ of $\{u_n\}_{n=0}^{\infty}$ that converges strongly to u in V, and u is the unique minimum of Φ in the space V.

Proof. In virtue of the process (5.1)–(5.7) and the Lax-Milgram theorem we see that it is possible to extract from the sequence $\{u_n\}_{n=0}^{\infty}$ a subsequence $\{u_{n_k}\}_{k=0}^{\infty}$, that weakly converges to some u in V. If we now apply Theorem 5.8 we have strong convergence of the subsequence $\{u_{n_k}\}_{k=0}^{\infty}$.

Let us assume that there exists $h \in V$ such that

$$d\Phi(u, h_0) \neq 0\,, \tag{5.15}$$

where u is the limit of the sequence $\{u_{n_k}\}_{k=0}^{\infty}$.

Every point $v \in V$ for which there exists $h_0 \in V$ such that (5.15) holds can be characterized by the existence of $\beta_V < 0$ such that

$$\Phi(v + \lambda_V(w(v) - v)) - \Phi(v) \leqq \beta_V < 0\,.$$

However there also exists $\varepsilon_V > 0$ such that for every $\tilde{u} \in B_V(v, \varepsilon_V)$ we have

$$\Phi(\tilde{u} + \lambda_{\tilde{u}}(w(\tilde{u}) - \tilde{u})) - \Phi(\tilde{u}) \leqq \beta_V < 0\,,$$

because in the opposite case we could find in an arbitrary ball $B_V(v, \varepsilon)$ a point $\hat{u}$ with the property

$$\Phi(\hat{u} + \lambda_{\hat{u}}(w(\hat{u}) - \hat{u})) - \Phi(\hat{u}) > 0\,.$$

However, this would imply

$$d\Phi(\hat{u}, h) = 0 \quad \text{for every} \quad h \in V\,.$$

Such a point can be only one.

So we can (for an arbitrary $\varepsilon > 0$) find $n_0 \in N$ such that $u_{n_m} \in B_V(u, \varepsilon)$ for every $m \geqq n$.

This and the inequality

$$\Phi(w(u)) \leqq \Phi(u) \quad \text{for every} \quad u \in V\,,$$

implies

$$\Phi(u_{n_m+j}) - \Phi(u_{n_m}) \leqq \Phi(u_{n_m+1}) - \Phi(u_{n_m}) \leqq \beta_u < 0\,.$$

Hence we clearly see that the sequence $\{\Phi(u_{n_k})\}_{k=0}^{\infty}$ is not a Cauchy sequence. This is impossible because $\Phi(u_{n_k}) \to \Phi(u)$. So u must be the unique minimum of Φ in V.

Q.E.D.

5.15. Consequence. *The sequence $\{u_n\}_{n=0}^{\infty}$ defined by the algorithm (5.1)–(5.7) contains a subsequence minimizing the alternative functional ψ.*

5.16. Remark. The problem

$$\text{minimize} \quad \Phi(u) - \int_{\Gamma_2} gu\, dS \quad \text{over} \quad S_E$$

has always a solution, but it is the solution of the transonic flow problem (2.1) if for every $v \in V$ there exists $\varepsilon > 0$ such that for every $t \in [0, \varepsilon]$,

$$u + tv \in S_E$$

(cf. [3]).

Therefore we had to suppose the subsonic situation in Theorem 5.15 to be able to prove the convergence because in the transonic case we can not generally assume the existence of a point $u \in S_E$ such that

$$\mathrm{d}\Phi(u, h) = 0 \quad \text{for every} \quad h \in V.$$

The assumptions which would enable us to prove Theorem 5.15 in transonic case could be rather restrictive. This means that we should suppose that the method (5.1)–(5.7) behaves like a "gradient-type" method. However we construct it like a "search – select type" method to minimize the alternative functional over a "small" set.

6. NUMERICAL REALIZATION

To finish the description of the minimization of the functional Φ in the directions given by the secant-modulus method it is necessary to solve the step (5.3).

This is the contents of the following paragraph.

6.1. Definition. *We say that a set $M_{f,\alpha}$ is an admissible set of a continuous function f: $[A, B] \to \mathbf{R}$ if and only if*

(i) $M_{f,\alpha}$ *is nonempty;*

(ii) $M_{f,\alpha} = \{x, y \in M_f \ |x - y| \geqq d > 0\}$, *where*

$\quad M_f = \{x \in [A, B];\ f(x) = \operatorname{locmin}_{t \in [A,B]} f(t)\}.$

We say that a function f: $[A, B] \to \mathbf{R}$ is admissible if there exists its admissible set.

The construction of an admissible set can be done as follows: let us have $d > 0$ and $0 < \text{eps} < d$, numbers A, B and an admissible function. Then we can define

(6.2) $C := A + d/2$
$D := A + d$
Is $D - B > 0$?
YES: STOP
NO: is $f(C) < \text{MIN}(f(A), f(D))$?
YES: GOTO (6.3),
NO: GOTO (6.4).

(6.3) Use "Line search" type algorithm (6.5) to find the minimum of f in $[A, A + d]$. Take it into the list. Let $A := D$. GOTO (6.2).

(6.4) Is $f(A) < f(B)$?
YES: Is $f(A - \text{eps}) > f(A)$?
YES: $D := C$,
GOTO (6.3),
NO: $A := D$,
GOTO (6.2).

NO: Is $f(D + \text{eps}) > f(D)$?
YES: $A := C$,
GOTO (6.3),
NO: $A := D$,
GOTO (6.2).

6.5. Definition. (Line search). Let f: $[A, B] \to \boldsymbol{R}$ and $[A, B] \subset \boldsymbol{R}$.

(6.6) $a := A$
$b := B$
$i := 1$

(6.7) $c = (a + b)/2$

(6.8) Is $f'(c) > 0$?
YES: $b := c$;
GOTO (6.9),
NO: $a := c$;
GOTO (6.9).

(6.9) Is $|a - b| \geqq \text{eps} > 0$?
YES: $i := i + 1$ and GOTO (6.7)
NO: STOP.

6.10. Theorem. *Let* f: $[A, B] \to \boldsymbol{R}$ *be a strictly convex and continuous function. Let us suppose that* $f'(x)$ *exists for every* $x \in]A, B[$. *Then the sequence* $\{a_i\}_{i=1}^{\infty}$ *given by* (6.6)–(6.9) *converges to the minimum of* f *in* $[A, B]$.

Proof. We clearly see that there exists only one point a_0 such that $f(a_0) = \min\limits_{[A,B]} f(t)$. For the sequence $\{a_i\}_{i=1}^{\infty}$ we have $|a_i - a_{i+1}| = (B - A)/2^i$, $i = 1, 2, 3, \ldots$.

As this is a Cauchy sequence, there exists $\bar{a}$ such that $\lim\limits_{k \to \infty} a_{i_k} = \bar{a}$, where $\{a_{i_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_i\}_{i=1}^{\infty}$.

Let us prove that $\bar{a} = a_0$. We have to discuss two cases:

(i) a_0 is a boundary point, e.g. $a_0 = A$. From the convexity of f it is clear that $f'(t) > 0$ in $]A, B[$. This fact and (6.6)–(6.9) give $a = A$ and $b = a_i$. So we can write $a_i \to A = a_0$.

(ii) a_0 is an inner point. Then $f'(a_0) = 0$ and we have

$$f'(t) > 0 \quad \text{for} \quad t \in]a_0, B[\,,$$
$$f'(t) < 0 \quad \text{for} \quad t \in]A, a_0[\,.$$

Let us suppose that there exists $i_0 \in \boldsymbol{N}$ such that $a_{i_0} = a_0$. It means that $a_l > a_{i_0}$ for every $l > i_0$. Let $a_l \neq a_0$ for every $l \in \boldsymbol{N}$ and suppose e.g. $a_l > a_0$. It is clear that there exists an index $n_1 \in \boldsymbol{N}$ such that $a_{n_1+1} < a_0 < a_{n_1}$. By the next step we obtain the existence of an index $n_2 \in \boldsymbol{N}$ such that $a_{n_2} < a_{n_0} < a_{n_2+1}$. Repeating this process we obtain the sequence $\{a_{n_k}\}$ satisfying

$$\begin{array}{lllll}
1) & a_{n_k} \to \bar{a} & \text{for} & k \to +\infty\,, \\
2) & a_{n_{2l}} < a_0 & \text{for} & l = 1, 2, 3, \ldots, \\
 & a_{n_{2l}} \to \bar{a} & \text{for} & l \to +\infty\,, \\
3) & a_{n_{2l+1}} > a_0 & \text{for} & l = 1, 2, 3, \ldots, \\
 & a_{n_{2l+1}} \to \bar{a} & \text{for} & l \to +\infty\,.
\end{array}$$

The condition 2) implies $\bar{a} \leqq a_0$ while the conditiin 3) yields $\bar{a} \geqq a_0$ Consequently, $a_0 = \bar{a}$. Q.E.D.

6.11. Remark. (i) The assumption about the existence of f' is not absolutely necessary.

(ii) We can use also other schemes instead of (6.6)–(6.9). See e.g. E. Polak [4].

We have solved the variational equation (2.1) using the finite element method. We have used linear elements defined on a polygonal domain Ω_h divided into triangles. The triangulation of the domain with NACA 230012 airfoil which was used to test our methods can be found in Chapter 7. We do not write here the equations constructed with the use of the finite elements because this is quite standard.

6.12. Remark. While the discretization of (2.1) is simple the proof of convergence $u_n \to u^*$, where u^* is the solution of (2.1) satisfying the condition (2.2), is rather nontrivial and needs some special conditions to be satisfied, e.g.:

$$\max \{|L_h^x(v) - v(x)|;\ x \in J_h\} \to 0\,,$$

where for every $v \in C^\infty(\Omega)$, J_h is the set of nodes of the triangulation of the polygonal domain Ω_h, and

$$L_h^x(v) \int_\Omega u_h^x = -\int_\Omega \nabla p_h(v)\, \nabla u_h^x\, \mathrm{d}x\,,$$

where u_h^x is the base function at the point x and p_h is the operator of Lagrangiean interpolation. Details can be found in J. Mandel and J. Nečas [2].

We have tested three algorithms mentioned above:

(i) (3.25)–(3.27),

(ii) (4.12)–(4.15),

(iii) (5.1)–(5.7).

These are the main reasons to do it:

1) By (i) we have tested the suitability of the functional Ψ.

2) By (ii) we have tested the ability to prepare a "suitable" u_0 for other algorithms (e.g. (iii)).

3) We have chosen the algorithm defined by (iii) for the reasons mentioned before Theorem 5.8. Because we consider it to be fundamental (with respect to the present theory) we will comment only on its numerical realization. When doing this we mention the aspects of the other two algorithms.

6.13. Numerical analysis of the algorithm (5.1)—(5.7)

By many numerical experiments we have verified the following input parameters to be the most important:

(i) the starting point u_0,

(ii) the value of λ from (3.2), which we choose near the point $2a_0^2/(\varkappa - 1)$,

(iii) the constant K from the definition of the entropy condition (2.2),

(iv) the length of $[-T, T]$ as mentioned in Remark 5.9(i).

Now we discuss these parameters.

Since we do not know the constant K a priori the starting point u has the key role. To define this point we use (4.11)–(4.15) with the constants $\beta \in [(2 - \varkappa)/3, (2 - \varkappa)/3 + \text{eps}]$; eps > 0, near the value $(2 - \varkappa)/3$. Given a partition of $[0, 1]$ we choose the maximal value of β so that u_β satisfies the condition (2.2) with some K. The graphs of such initial solutions are given in Section 7.

It is clear from the realization of (5.1)–(5.7) that if we choose the initial solution so that it does not satisfy the condition (2.2) a priori it is impossible to obtain a physical solution. When such a situation occurs in the course of the process we may use the following modification:

Let $\omega(u_n) = M_n > K$ and $\omega(\tilde{u}_0) > K$, where $\omega(\cdot) = \max\limits_{\substack{h \in \mathscr{D}_h^+(\Omega) \\ h > 0}} \omega_n(\cdot)$ and $\mathrm{d}\Phi(\tilde{u}_0, v) = 0$ for every $v \in V$. Let us define

$$r(u_n) = \max_{h \in \mathscr{D}_h^+(\Omega), h > 0} \left[(\omega_h(\tilde{u}_0) - \omega_h(u_n)) \int_\Omega h \, \mathrm{d}x / \int_\Omega (\nabla h)^2 \, \mathrm{d}x \right].$$

On the line $u_n + t(w(u_n) - u_n)$ for $t \in \boldsymbol{R}$ we find such two points $\bar{u}_n$, $\bar{\bar{u}}_n$ satisfying

$$\|\bar{u}_n - u_n\|^2 = \|\bar{\bar{u}}_n - u_n\| = r(u_n).$$

Then we can choose

$$\omega(u_{n+1}) = \min \{\omega(\bar{u}_n); \omega(\bar{\bar{u}}_n)\}.$$

6.14. Remark. (i) the number $r(u_n)$ can stand for $\|u_0 - u_n\|^2$. As we do not know the number $\omega_h(u_0)$ a priori we have to approximate its value by experiments. This can be done if we consider the process $u_{n+1} = w(u_n)$ and $u_0 = \text{const.}$ in Ω.

The initial solution u_0 can be constructed by other methods as well. The construction of a piecewise linear function in Ω with a "correctly" prescribed jump of velocity is one of the possibilities. However our experience show that this is undoubtedly the most complicated way, because the prescription of the jump (we mean "where" and "value") is very difficult.

The value of the parameter is much more important than one can think at the first glance.

We have found two reasons:

1) this value influences the enumeration of Ψ,

2) this value influences the number of iterations necessary to find the solution of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ (from the discretization of the equation (5.1) by the finite element methods).

The following table shows the dependence between the value of λ and the value of Ψ:

λ	Ψ
2·0	−112·97
3·5	−3·66
4·9	31·59

maximum speed of the flow 577 ms^{-1}

The relation between the accuracy eps of the solution of $\boldsymbol{Ax} = \boldsymbol{b}$ and the value of λ is given in the following table:

Entropy constant	Entrance velocity	eps λ	1	10^{-1}	10^{-2}	10^{-3} P	10^{-4}
4·0	$M = 0{\cdot}5$	2·0	34	51	70	88	105
4·0	$M = 0{\cdot}5$	3·5	35	60	77	94	111
4·0	$M = 0{\cdot}5$	4·9	60	97	105	121	153
4·0	$M = 0{\cdot}5$	4·95	100	141	158	174	198
10·0	$M = 0{\cdot}55$	4·8	97	>200			

For every $m \geqq P$: we have $\|u^m - u^{m+1}\| \leqq$ eps; $(u^m)_{m=1}^n$ stands for the solution of the above linear system.

The constant K from the entropy condition (2.2) was found by numerical experiments. We have found that if

$K < 2{\cdot}8$ the algorithm eliminates all the minima in the directions given by the secant-modulus method,

$K > 10$ the algorithm gives all of them.

The optimal value we have used was $K = 4$.

It is not problem to find the length of the interval $[-T, T]$. The problem is to find the smallest suitable T. Again using experiments we have found that $T \in [20, 30]$ seems to be optimal.

6.14. Method (4.11)—(4.15)

The corresponding graphs can be found in the next chapter. This graphs show the solutions for some characteristic values of β.

The solution for $\beta = 0{\cdot}19$ is similar to the solution of the Laplace equation. For β that tends to 1, $u_\beta \to u^*$ but this solution does not satisfy the entropy condition (2.2). When testing this method we chose the entrance velocity 200 ms^{-1}. The maximal velocity of the flow was 758 ms^{-1}. Even a lower entrance velocity and an other step for β did not give better results. The sense of this method was mentioned when discussing u_0.

6.15. Implementation remarks

We have tested both algorithms on the domain whose triangulation is mentioned in Chapter 7. It is a channel with NACA 230012 airfoil. We have tested entrance velocities ranging from $M = 0{\cdot}2$ to $M = 0{\cdot}7$. Due to this airfoil we have computed the following results:

Entrance velocity	Character of the flow
$\leqq M = 0{\cdot}43$	subsonic flow
$> M = 0{\cdot}43$ and $\leqq M = 0{\cdot}56$	transonic flow
$> M = 0{\cdot}56$	channel blocked

In the case of the entrance velocity ranging from $M = 0{\cdot}43$ to $M = 0{\cdot}56$ we usually obtained physical solutions when we found correct constants and variables needed by both processes.

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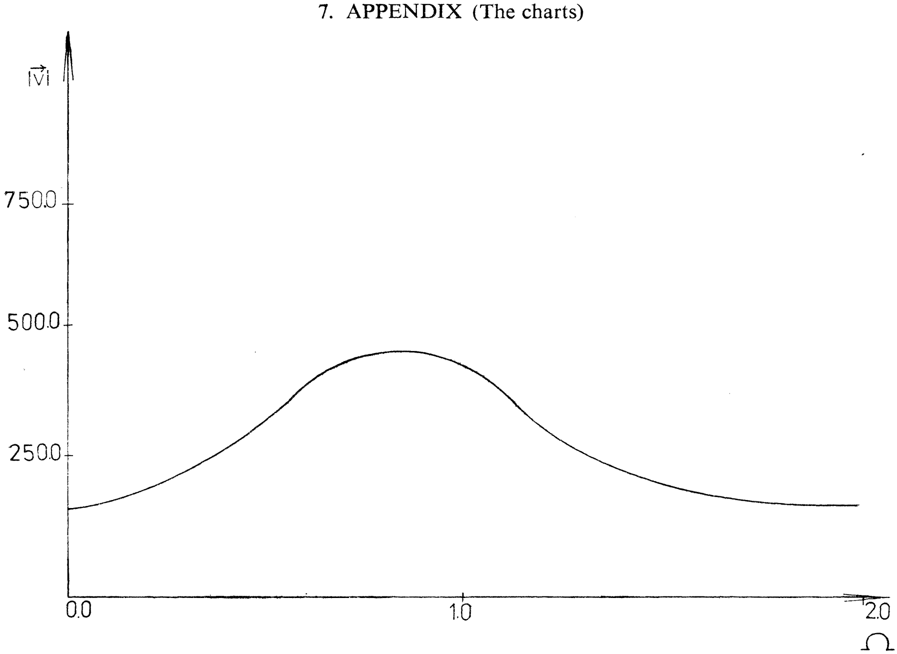
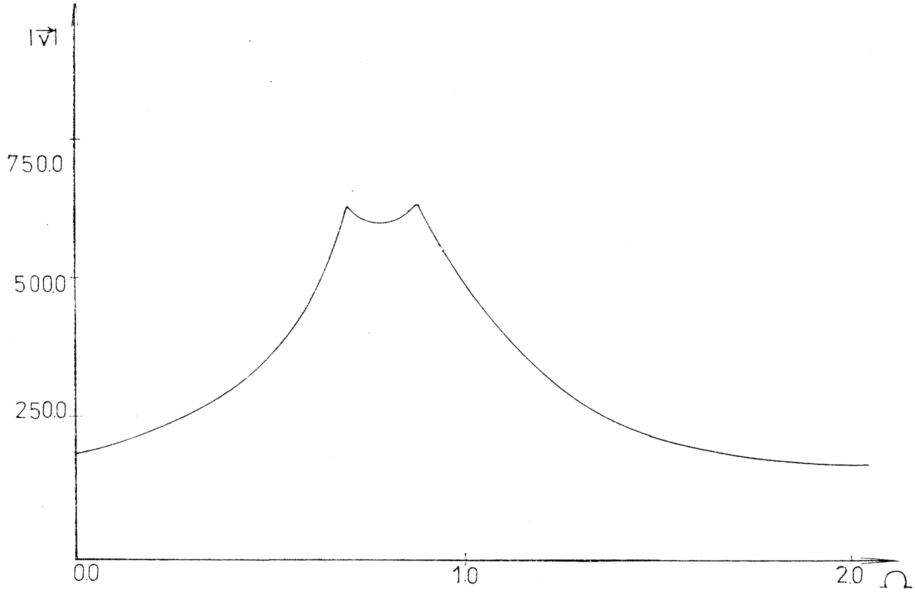
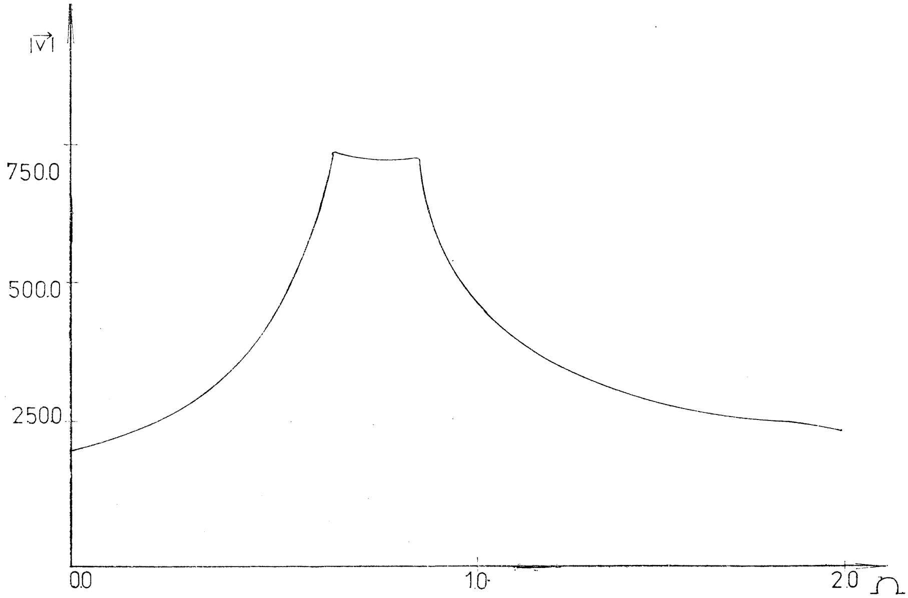
7. APPENDIX (The charts)



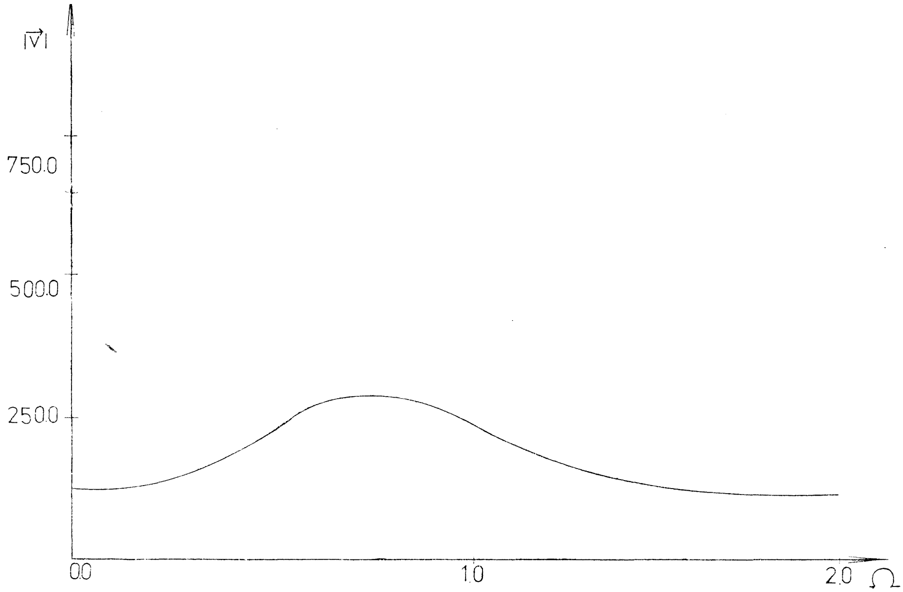
Graph of the solution ∇u_β to the equation.
$d\Phi^\beta(u, h) = 0$ for every $h \in V$ and $\beta = 0{\cdot}19$, $g = 150\ \mathrm{ms}^{-1}$, the 15th iteration.



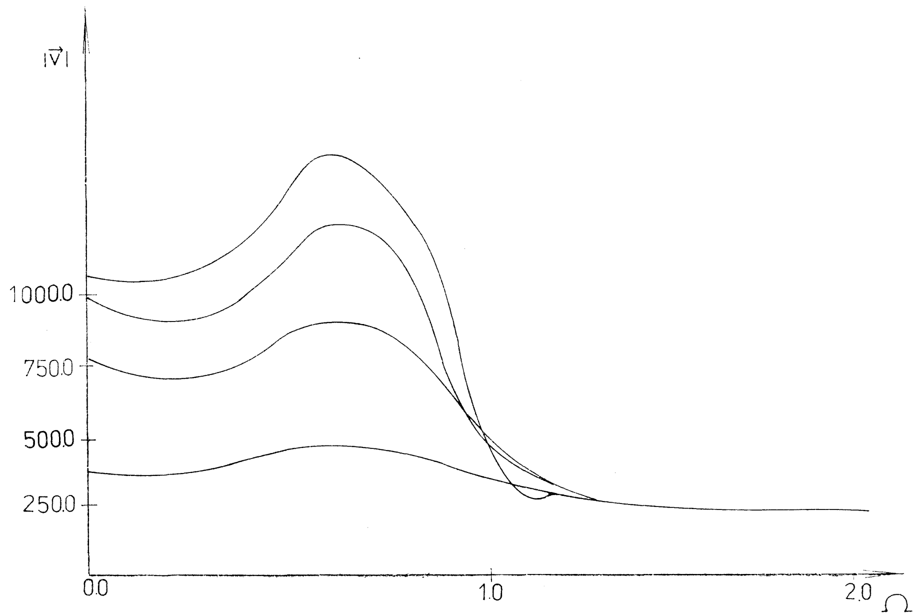
Graph of the solution ∇u_β to the equation.
$B^{\beta_2}(u_{\beta_1};\ w(u_{\beta_1}), h) = \int_{\partial\Omega} gh\, \mathrm{d}S$ for every $h \in V$, $\beta_1 = 76$, $\beta_2 = 81$, $g = 150\ \mathrm{ms}^{-1}$, max. of velocity: $672\ \mathrm{ms}^{-1}$.



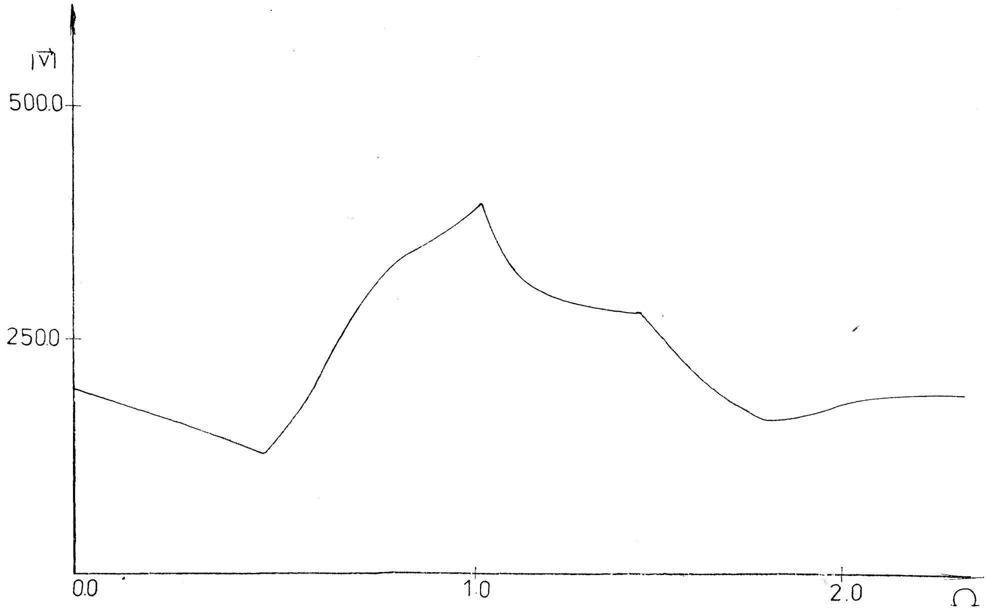
Graph of the solution $\nabla w(u_3)$ to the equation.
$B(u_3;\ w(u_3), h) = \int_{\partial\Omega} gh\, \mathrm{d}S$ for every $h \in V$, 3rd iteration of the secant-modulus $u_0 = u_{0.96}$, $g = 150\ \mathrm{ms}^{-1}$, max. velocity: $740\ \mathrm{ms}^{-1}$.



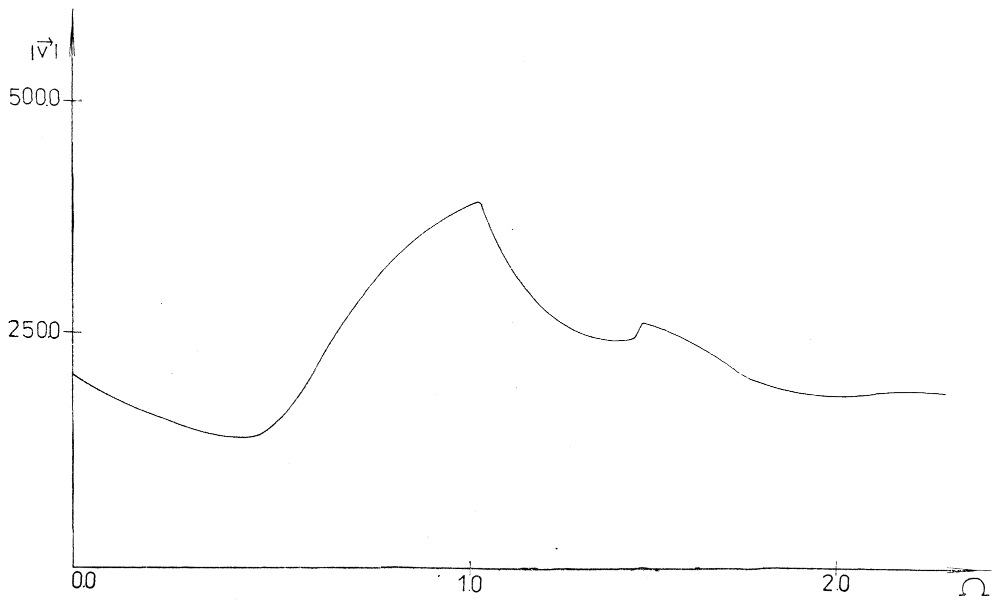
Graph of the solution $\nabla w(u_{10})$ to the equation.
$B(u_{10}; w(u_{10}), h) = \int_{\partial\Omega} gh \, \mathrm{d}S$ for every $h \in V$, for strictly subsonic flow 10th iteration of the secant-modulus $g = 100 \text{ ms}^{-1}$, max. velocity: 254 ms^{-1}.



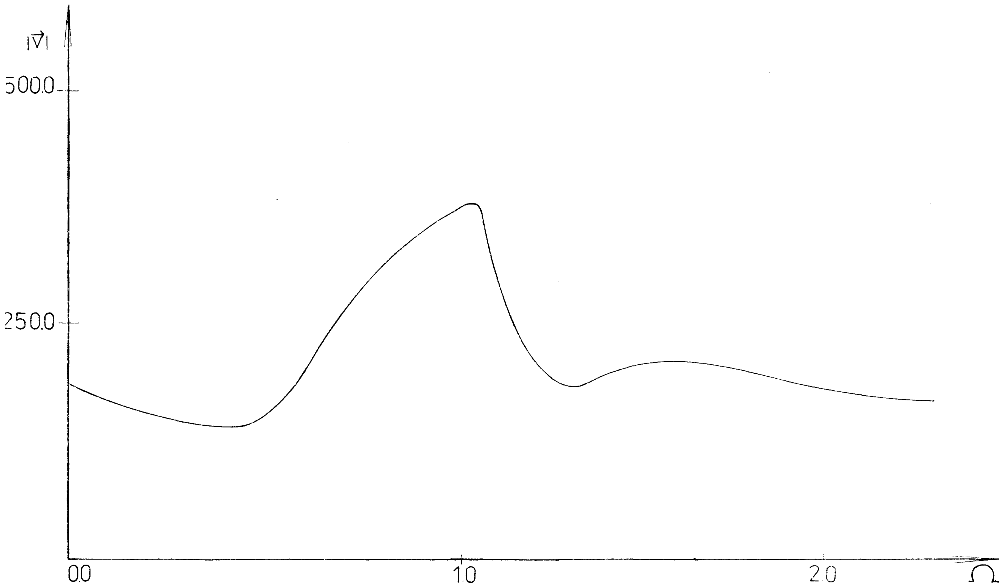
Blocking of the channel. Graph of the solution $\nabla w(u_i)$ to the equation.
$B(u_i; w(u_i), h) = \int_{\partial\Omega} gh \, \mathrm{d}S$ for every $h \in V$, 1st, 5th, 10th, 15th iteration of the secant-modulus $g = 255 \text{ ms}^{-1}$, max. velocity: 1513 ms^{-1}.



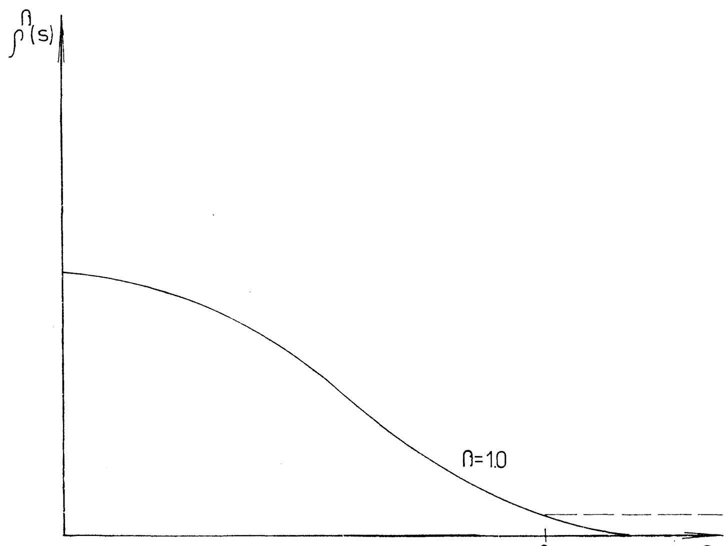
Graph of the velocity ∇u. Method (5.1)—(5.7). 3rd interation
$g = 160\ ms^{-1}$, $\lambda = 4{\cdot}5$, $K = 4$, $T = 20$ max. velocity: 410 ms^{-1}.



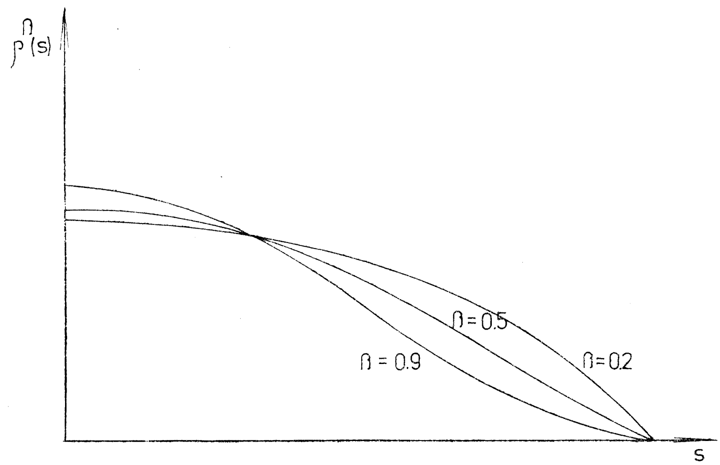
Graph of the velocity ∇u. Method (5.1)—(5.7).
4th iteration $g = 160\ ms^{-1}$, $\lambda = 4{\cdot}5$, $K = 4$, $T = 20$ max. velocity: 409 ms^{-1}.



Graph of the velocity ∇u. Method (5.1)—(5.7).
5th iteration $g = 160\ \mathrm{ms}^{-1}$, $\lambda = 4{\cdot}5$, $K = 4$, $T = 20$ max. velocity: $407\ \mathrm{ms}^{-1}$.



Graph of the density ϱ.



Graph of the density ϱ_β.
Triangulation of the domain with NACA 230012 airfoil.

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Souhrn

VÝPOČET TRANSONICKÉHO PROUDĚNÍ UŽITÍM METODY KONEČNÝCH PRVKŮ

Petr Klouček, Josef Málek

V této práci je podána řada nových algoritmů, které lze užít k řešení transonického proudění. Práce obsahuje základní schéma přístupu, použije-li se metoda konečných prvků: implementaci podmínky entropie. Protože v definicích algoritmů je řada konstant, které je nutné experimentálně určit, obsahuje práce rozbor numerických výsledků v závislosti na různých hodnotách těchto konstant.

Резюме

ВЫЧИСЛЕНИЕ ОКОЛОЗВУКОВОГО ТЕЧЕНИЯ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

PETR KLOUČEK, JOSEF MÁLEK

В статье изложены новые алгоритмы для решения околозвукового потенциального течения и приведены иллюстрирующие численные результаты.

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