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A UNILATERAL BOUNDARY-VALUE PROBLEM FOR THE ROD

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Summary. A unilateral boundary-value condition at the left end of a simply supported rod is considered. Variational and (equivalent) classical formulations are introduced and all solutions to the classical problem are calculated in an explicit form. Formulas for the energies corresponding to the solutions are also given. The problem is solved and energies of the solutions are compared in the perturbed as well as the unperturbed cases.

Keywords: buckling of the rod, variational inequality, Signorini problem, bifurcations.

AMS classification: 73K05, 40D20.

INTRODUCTION

In his paper [1] E. L. Reiss solved in an explicit form the problem of branching of the trivial solution of a homogeneous rod satisfying certain (bilateral) boundary-value conditions. Being inspired by his work we try to do similar work in the case of the unilateral problem. We give the explicit solutions and compare the corresponding energies for which we also obtain explicit formulae. Both the unperturbed unilateral condition and the perturbed one are considered.

The physical background of the problem is given in Section 1. In this section we also introduce the variational inequalities modelling the unilarelal problems in question and give their interpretation. As it is pointed here, we can equivalently solve the problem in its classical formulation.

In Section 2 we calculate all solutions of the problem.

The main part of the paper is Section 3 where we calculate in detail the expressions for the energies and prove their ordering.

1. PHYSICAL BACKGROUND OF THE PROBLEM AND ITS FORMULATION

An underformed rod R is meant to be a three-dimensional continuum whose projection on the x-axis is the interval $\langle 0, \pi \rangle$ and whose cross section $S(a) =$

$= \{[x, y, z] \in R; x = a\}$, $a \in \langle 0, \pi \rangle$ is symmetrical with respect to its centre which lies on the x-axis. In this paper we deal with a rod of constant cross section which is made of homogeneous material. Moreover, we suppose that during the deformation each cross section $S(a)$ remains planar.

These assumptions enable us to investigate only the deformation of the axis of the rod, i.e. of the segment $\{[x, 0, 0]; x \in \langle 0, \pi \rangle\}$. We suppose that the deformation takes place in the xz-plane.

Taking into account these assumptions and neglecting some higher order terms we are led to Kirchhoff's energy functional (see [2])

$$(1.1) \qquad E(u, w) = \int_0^\pi \left\{ \frac{\mathscr{E}\sigma}{2} (u' + \tfrac{1}{2}w'^2)^2 + \frac{\mathscr{E}J}{2} (w'')^2 - Xu - Zw \right\} dx - \\ - (\tilde{X}u + \tilde{Z}w + \tilde{M}w')|_0^\pi ,$$

where

$u(x)$, $w(x)$ denote respectively the horizontal and vertical displacements of the point $(x, 0)$ (thus the coordinates of this point after the deformation are $x + u(x)$, $w(x)$);

(X, Z) is the vector of the external force density which is a function of x;

$(\tilde{X}, \tilde{Z})$ are the vectors of the forces acting at the ends of the rod;

$\tilde{M}$ are the moments of these forces;

$\mathscr{E}$ is Young's modulus;

σ is the cross-sectional area of the rod;

$J = \int_S z^2 \, dy \, dz$ is the moment of inertia with respect to the xy-plane.

Provided $X, Z \in L^2(0, \pi)$ the functional E is defined in the space $Y = W^{1,2} \times W^{2,2}$. In what follows we denote the elements of Y by $v = (u, w)$.

Let us now formulate the boundary-value conditions. The rod is supposed to be simply supported at both ends, but the rotation of the left end is unilaterally restricted (see Fig. 1). While the left end is fixed with respect to all displacements, the right end can move horizontally. Using our notation we can write these conditions as

$$(1.2a) \qquad u(0) = 0 , \quad w(0) = 0 , \quad w'(0) \geqq \varepsilon ,$$

$$(1.2b) \qquad w''(0) \leqq 0 , \quad [w'(0) - \varepsilon] \, w''(0) = 0 ,$$

$$(1.2c) \qquad u(\pi) = -c , \quad w(\pi) = 0 ,$$

$$(1.2d) \qquad w''(\pi) = 0 ,$$

where $\varepsilon \geqq 0$, $c \in \langle 0, \pi \rangle$ are the parameters.

Applying the Lagrange principle of minimum of the potential energy we seek the state of equilibrium of the rod among the critical points of the functional E in the convex set K given by geometrical constraints (1.2a), (1.2c).

The point $v \in \mathsf{K}$ is said to be a critical point of E in the set K if it satisfies the Euler inequality

$$(1.3) \qquad v \in \mathsf{K}\colon \mathrm{D}E(v, \bar{v} - v) \geqq 0\,, \quad \forall \bar{v} \in \mathsf{K}\,,$$

where D is the Gateaux derivative.

In what follows we treat the situation when

$$X = Z = 0\,,$$
$$\tilde{X} = \tilde{M} = 0\,.$$

We give the specific form of (1.3) in this particular case.

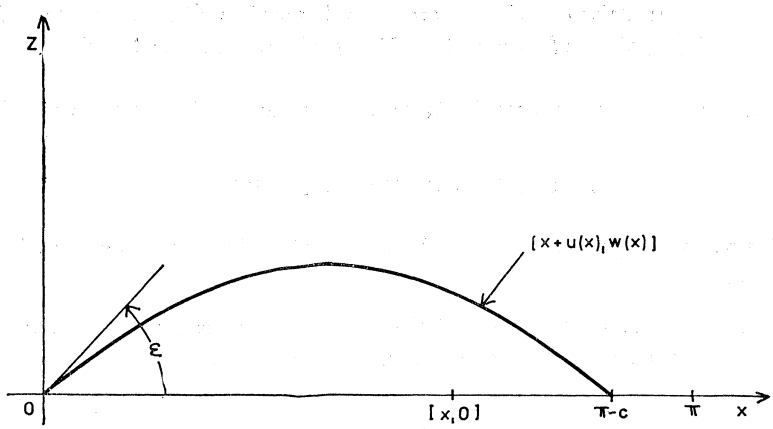

Fig. 1.

Variational formulation of the problem

Denote

$$(1.4) \qquad \mathsf{K} = \{v \in W^{1,2}(0, \pi) \times [W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)];\ u(0) = 0,\ u(\pi) = = -c,\ w'(0) \geqq \varepsilon\}\,.$$

We want to find $v \in \mathsf{K}$ satisfying the inequality

$$(1.5) \qquad \int_0^\pi \{\beta(u' + \tfrac{1}{2}w'^2)\,[(\bar{u} - u)' + w'(\bar{w} - w)'] + w''(\bar{w} - w)''\}\,\mathrm{d}x \leqq 0$$

for all $\bar{v} = (\bar{u}, \bar{w}) \in \mathsf{K}$.

Remark 1.1. We divided (1.3) by $\mathscr{E}J > 0$, denoting $\beta = \sigma/J$. The term $-\tilde{Z}w|_0^\pi$ in (1.1) vanishes since $w \in W_0^{1,2}(0, \pi)$.

It can be easily proved that any sufficiently smooth solution of (1.5) solves the following classical problem:

Find $(u, w) \in C^2(\langle 0, \pi\rangle) \times C^4(\langle 0, \pi\rangle)$ satisfying the system of equations

$$u' + \tfrac{1}{2}(w')^2 = -\frac{\lambda}{\beta} \tag{1.6a}$$

(for some $\lambda \in \mathbb{R}$),

$$w^{(IV)} + \lambda w'' = 0 \tag{1.6b}$$

together with the conditions (1.2).

In the sequel we refer to this problem as to the problem (1.6).

The following theorem permits us to work with the classical problem (1.6) instead of (1.5).

Theorem 1.1. *Any solution of the variational problem* (1.5) *is of the class* $C^2(\langle 0, \pi\rangle) \times C^4(\langle 0, \pi\rangle)$ *and solves* (1.6). *The problems* (1.5) *and* (1.6) *are then equivalent.*

We omit the proof of this regularity result – it can be done by the standard method.

2. SOLUTION OF THE PROBLEM

All solutions of the problem (1.6) can be calculated explicitly and are given in the tables below.

Remark 2.1. It is readily checked that the equation $\mu\pi \cos \mu\pi - \sin \mu\pi = 0$ has exactly one zero in each interval $(n, n + \frac{1}{2})$, $n = 1, 2 \ldots$. We denote these zeros μ_n.

Table 1_0 (Solutions of (1.6) in the case $\varepsilon = 0$)

for $c > n^2\pi/\beta$, $n \in \mathbb{N}$:

$$\begin{aligned} u(x) &= -\frac{c}{\pi} x - \frac{c - n^2\pi/\beta}{2\pi n} \sin 2nx\,, \\ w(x) &= \frac{2}{n}\left(\sqrt{\frac{c - n^2\pi/\beta}{\pi}}\right) \sin nx\,; \end{aligned} \tag{2.1}$$

for $c > \mu_n^2\pi/\beta$, $n \in \mathbb{N}$, where μ_n are the numbers defined in Remark 2.1:

$$\begin{aligned} u(x) &= -\frac{\mu_n^2}{\beta} x - \frac{2(c - \mu_n^2\pi/\beta)}{\pi \sin^2 \mu_n\pi}\Big[x(\tfrac{1}{2} + \cos^2 \mu_n\pi) - \\ &\quad - \frac{2}{\mu_n} \cos \mu_n\pi \sin \mu_n(x - \pi) + \frac{1}{4\mu_n} \sin 2\mu_n(x - \pi) - \frac{3}{2}\pi \cos^2 \mu_n\pi\Big]\,, \\ w(x) &= \frac{2}{\mu_n\pi}\left(\sqrt{\frac{c - \mu_n^2\pi/\beta}{\pi}}\right)\left[x - \pi + \pi\frac{\sin \mu_n(\pi - x)}{\sin \mu_n\pi}\right]; \end{aligned} \tag{2.2}$$

for $0 \leqq c < \pi$:

$$(2.3)\qquad \begin{aligned} u(x) &= -\frac{c}{\pi}x\,, \\ w(x) &= 0\,. \end{aligned}$$

Table 1_ε (Solutions of (1.6) in the case $\varepsilon > 0$)

for $c \geqq \pi(\varepsilon^2/4 + n^2/\beta)$, $n \in \mathbb{N}$:

$$(2.4)\qquad \begin{aligned} u(x) &= -\frac{c}{\pi}x - \frac{c - n^2\pi/\beta}{2\pi n}\sin 2nx\,, \\ w(x) &= \frac{2}{n}\sqrt{\frac{c - n^2\pi/\beta}{\pi}}\sin nx\,; \end{aligned}$$

for $c = \lambda\pi/\beta + (\varepsilon^2\pi/4)\,P(\mu\pi)$, where

$$P(y) = \frac{y^2 + \frac{1}{2}y\sin 2y - 2\sin^2 y}{(y\cos y - \sin y)^2}\,, \quad \mu = \sqrt{\lambda}\,,$$

$$\lambda \in (0, 1\rangle \cup \bigcup_{n=1}^{\infty} (\mu_n^2, (n+1)^2\rangle:$$

$$(2.5)\qquad u(x) = -\frac{\lambda}{\beta}x - \frac{\varepsilon^2}{4(\mu\pi\cos\mu\pi - \sin\mu\pi)^2}\,[x((\mu\pi)^2 + 2\sin^2\mu\pi) +$$

$$\mu\pi^2\sin\mu(x-\pi)\cos\mu(x-\pi) - 4\pi\sin\mu\pi\sin\mu(x-\pi) + \mu\pi^2\sin\mu\pi\cos\mu\pi -$$

$$-\,4\pi\sin^2\mu\pi]\,,$$

$$w(x) = \frac{\varepsilon}{\mu\pi\cos\mu\pi - \sin\mu\pi}\,[\pi\sin\mu(x-\pi) - (x-\pi)\sin\mu\pi]\,;$$

for $c = \lambda\pi/\beta + (\varepsilon^2\pi/4)\,Q(\mu\pi)$, where

$$Q(y) = \frac{y^2 + \frac{1}{2}y\,\mathrm{sh}\,2y - 2\,\mathrm{sh}^2 y}{(y\,\mathrm{ch}\,y - \mathrm{sh}\,y)^2}\,, \quad \mu = \sqrt{(-\lambda)},$$

$$\lambda \in (-\infty, 0\rangle:$$

$$(2.6)\qquad u(x) = -\frac{\lambda}{\beta}x - \frac{\varepsilon^2}{4(\mu\pi\,\mathrm{ch}\,\mu\pi - \mathrm{sh}\,\mu\pi)^2}\,[x((\mu\pi)^2 + 2\,\mathrm{sh}^2\mu\pi) +$$

$$+\,\mu\pi^2\,\mathrm{sh}\,\mu(x-\pi)\,\mathrm{ch}\,\mu(x-\pi) - 4\pi\,\mathrm{sh}\,\mu\pi\,\mathrm{sh}\,\mu(x-\pi) +$$

$$+\,\mu\pi^2\,\mathrm{sh}\,\mu\pi\,\mathrm{ch}\,\mu\pi - 4\pi\,\mathrm{sh}^2\mu\pi]\,,$$

$$w(x) = \frac{\varepsilon}{\mu\pi\,\mathrm{ch}\,\mu\pi - \mathrm{sh}\,\mu\pi}\,[\pi\,\mathrm{sh}\,\mu(x-\pi) - (x-\pi)\,\mathrm{sh}\,\mu\pi]\,.$$

Remark 2.2. The formulas (2.5) with $\lambda = n^2$, $n \in \mathbb{N}$ reduce to the formulas (2.4) with $c = \pi(\varepsilon^2/4 + n^2/\beta)$.

The expressions for c, $u(x)$, $w(x)$ in (2.6) have finite limits as λ tends to zero. For $\lambda = 0$ we have $c = \varepsilon^2\pi/10$ and

$$u(x) = -\frac{\varepsilon^2}{40\pi^4}\left[9(x-\pi)^5 - 10\pi^2(x-\pi)^3 + 5\pi^4(x-\pi) + 4\pi^5\right], \tag{2.7}$$

$$w(x) = \frac{\varepsilon}{2\pi^2}x(x-\pi)(x-2\pi)$$

Remark 2.3. The problem (1.6) has the trivial solution only in the case $\varepsilon = 0$.

3. ENERGY OF SOLUTIONS

In Section 2 we gave a list of all solutions of (1.6). Now we shall calculate the energies corresponding to these solutions and make a comparison of the results obtained.

Remember that the functional of energy (1.1) in our case (after being divided by $\frac{1}{2}\mathscr{E}J$) has the form

$$E(v) = \int_0^\pi [\beta(u'(x) + (w'(x))^2)^2 + (w''(x))^2]\,\mathrm{d}x\,. \tag{3.0}$$

Lemma 3.1. *Denote by $E_n(c)$, $E_{\mu_n}(c)$, $E^t(c)$ the energies corresponding to the solutions* (2.1), (2.2) *and* (2.3), *respectively.*

Let $E(\lambda)$ be the energy of (2.6), (2.5) *for $\lambda \in (-\infty, 0\rangle$, $\lambda \in (0, 1\rangle \cap \bigcup_{n=1}^{\infty} (\mu_n^2, (n+1)^2\rangle$, respectively. We have*

$$E_n(c) = n^2(2c - n^2\pi/\beta) \quad \textit{for} \quad c \geqq n^2\pi/\beta\,, \quad n \in \mathbb{N}\,, \tag{3.1}$$

$$E_{\mu_n}(c) = \mu_n^2(2c - \mu_n^2\pi/\beta) \quad \textit{for} \quad c \geqq \mu_n^2\pi/\beta\,, \quad n \in \mathbb{N}\,, \tag{3.2}$$

$$E^t(c) = c^2\beta/\pi \quad \textit{for} \quad 0 \leqq c < \pi\,, \tag{3.3}$$

$$E(\lambda) = \frac{\lambda^2\pi}{\beta} + \frac{\varepsilon^2\pi^2}{4}\frac{\mu^3(2\mu\pi - \sin 2\mu\pi)}{(\mu\pi\cos\mu\pi - \sin\mu\pi)^2} \quad \textit{for} \quad \lambda \in (0, 1\rangle \cap \bigcup_{n=1}^{\infty}(\mu_n^2, (n+1)^2\rangle\,, \tag{3.4}$$

$$E(\lambda) = \frac{\lambda^2\pi}{\beta} + \frac{\varepsilon^2\pi^2}{4}\frac{\mu^3(\operatorname{sh} 2\mu\pi - 2\mu\pi)}{(\mu\pi\operatorname{ch}\mu\pi - \operatorname{sh}\mu\pi)^2} \quad \textit{for} \quad \lambda \in (-\infty, 0\rangle\,. \tag{3.5}$$

Proof. From (3.0) and (1.6a) we obtain

$$E(v) = \int_0^\pi \left[\frac{\lambda^2}{\beta} + (w'')^2\right]\mathrm{d}x = \frac{\lambda^2}{\beta}\pi + \int_0^\pi (w'')^2\,\mathrm{d}x\,. \tag{3.6}$$

The proof of formulas (3.1), (3.3) can be found in [1]. Let us prove (3.2). We have $\lambda = \mu_n^2$ and

$$[w''(x)]^2 = \frac{4\mu_n^2}{\sin^2 \mu_n\pi}\left(\frac{c - \mu_n^2\pi/\beta}{\pi}\right)\sin^2 \mu_n(\pi - x)\,.$$

Using the equation $\mu_n\pi \cos \mu_n\pi = \sin \mu_n\pi$ one can easily deduce (3.2) from (3.6).

In the case of the solutions (2.5), (2.6) we have respectively, $\lambda = \mu^2$,

$$[w''(x)]^2 = \frac{\varepsilon^2\pi^2\mu^4}{(\mu\pi \cos \mu\pi - \sin \mu\pi)^2}\sin^2 \mu(\pi - x)\,,$$

and $\lambda = -\mu^2$,

$$[w''(x)]^2 = \frac{\varepsilon^2\pi^2\mu^4}{(\mu\pi \operatorname{ch} \mu\pi - \operatorname{sh} \mu\pi)^2}\operatorname{sh}^2 \mu(\pi - x)\,.$$

For $\lambda \neq 0$ the formulas (3.4), (3.5) are obtained by integration in (3.6). In the case $\lambda = 0$ one can easily verify that the energy of the couple (2.7) given by (3.6) equals $E(0) = \lim\limits_{\lambda\to 0} E(\lambda)$.

Lemma 3.2. *Define the function* $c(\lambda)$ *as*

$$c(\lambda) = \frac{\lambda\pi}{\beta} + \frac{\varepsilon^2\pi}{4}\,Q(\mu\pi)\,,\quad \mu = \sqrt{(-\lambda)}\quad \textit{if}\quad \lambda \leqq 0\,,$$

$$c(\lambda) = \frac{\lambda\pi}{\beta} + \frac{\varepsilon^2\pi}{4}\,P(\mu\pi)\,,\quad \mu = \sqrt{\lambda}\quad \textit{if}\quad \lambda \in (0, 1\rangle \cup \bigcup_{n=1}^{\infty} (\mu_n^2, (n+1)^2\rangle$$

and let $E(\lambda)$ *be the function defined by* (3.4), (3.5).

Then the functions c and E have the following properties:

1. *They are continuously differentiable in the set*

$$(-\infty, 1\rangle \cup \bigcup_{n=1}^{\infty} (\mu_n^2, (n+1)^2\rangle$$

and their derivatives satisfy the equation

$$E'(\lambda) = 2\lambda\, c'(\lambda)\,. \tag{3.7}$$

2. *The function c is increasing in the interval* $(-\infty, 1\rangle$ *and it is strictly convex in each interval* $(\mu_n^2, (n+1)^2\rangle$.

3. *There exist numbers* $\lambda_n \in (\mu_n^2, (n+1)^2)$ *such that c is decreasing in* $(\mu_n^2, \lambda_n\rangle$ *and increasing in* $\langle\lambda_n, (n+1)^2\rangle$, *and we have*

$$c(n^2) < c(\lambda_n)\quad \textit{for all}\quad n \in \mathbb{N}\,. \tag{3.8}$$

4. *The function E is decreasing in* $(-\infty, 0\rangle$ *and in each interval* $(\mu_n^2, \lambda_n\rangle$. *It is increasing in the interval* $\langle 0, 1\rangle$ *as well as in each* $\langle\lambda_n, (n+1)^2\rangle$.

For the shape of the functions c, E see Figs. 2, 3.

Proof. As far as Assertion 2 of Lemma 3.2 and the inequalities (3.8) are concerned, we have not succeeded in finding their analytic proof. So we conjecture them, relying on the results of numerical computations.

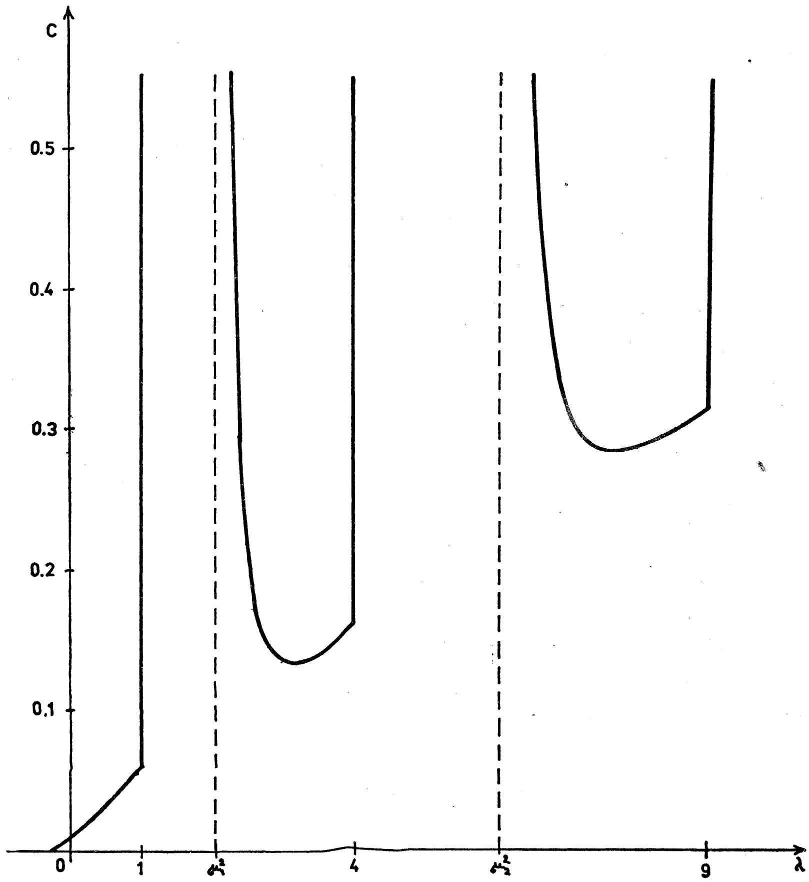

Fig. 2 — Function c

$$c(n^2) = \pi\left(\frac{\varepsilon^2}{4} + \frac{n^2}{\beta}\right); \qquad \varepsilon = 0{\cdot}2,\ \beta = 100,\ n = 1, 2, 3, \ldots .$$

1. Let us first prove (3.7).

Let $\lambda < 0$. Denoting $a = 1/\beta\pi^3$, $b = \varepsilon^2/4\pi$, we have $c(\lambda) = \pi^2(-ay^2 + b\,Q(y))$, where $y = \pi\sqrt{(-\lambda)}$,

$$E(\lambda) = ay^4 + b\,\frac{y^3(\operatorname{sh} 2y - 2y)}{(y \operatorname{ch} y - \operatorname{sh} y)^2}.$$

Easy calculation yields

$$- 2y^2 \frac{dQ}{dy}(y) = \frac{d}{dy} \frac{y^3(\operatorname{sh} 2y - 2y)}{(y \operatorname{ch} y - \operatorname{sh} y)^2},$$

which means that

$$\pi^2 \frac{dE}{dy}(y) = -2y^2 \frac{dc}{dy}(y),$$

the last equation being (3.7) for $\lambda < 0$.

The case $\lambda > 0$ can be treated in a similar way.

To complete the proof of Assertion 1 we compute the limits and derivatives of the functions c and E at the point $\lambda = 0$ using the definition of the derivative of a function and some standard analytic methods.

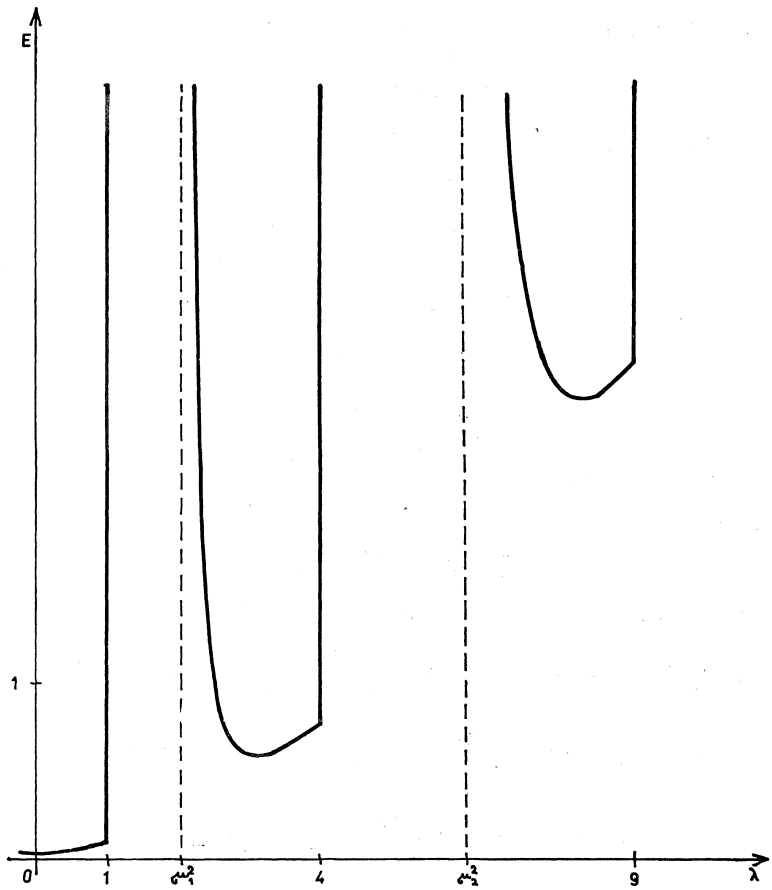

Fig. 3 — Function E

$\varepsilon = 0{\cdot}2, \ \beta = 100.$

3. One can easily compute that $c'(n^2) > 0$ for all n and, since obviously $c(\lambda) \to +\infty$ as $\lambda \to \mu_n^+$, the existence of the points λ_n follows from the convexity of the function c (Assertion 2).

4. Follows directly from Assertion 1, 2, 3.

Theorem 3.1. *For any integer n the following estimates hold*:

I) $\forall c > (n + 1)^2 \pi/\beta$:

$$E_n(c) < E_{\mu_n}(c) < E_{n+1}(c) < E^t(c)\,; \tag{3.9}$$

II) $\forall \lambda \in (\mu_n^2, (n + 1)^2)$:

$$E_n(c(\lambda)) < E(\lambda)\,; \tag{3.10}$$

III) $\forall \lambda \in (\mu_n^2, (n + 1)^2)$, $c(\lambda) > \pi\left(\dfrac{\varepsilon^2}{4} + \dfrac{(n + 1)^2}{\beta}\right)$:

$$E(\lambda) < E_{n+1}(c(\lambda))\,; \tag{3.11}$$

IV) $\forall \lambda', \lambda'' \in (\mu_n^2, (n + 1)^2\rangle$, $\lambda' < \lambda''$, $c(\lambda') = c(\lambda'')$:

$$E(\lambda') < E(\lambda'')\,. \tag{3.12}$$

Proof. I) For a fixed c denote $G(y) = y^2(2c - y^2\pi/\beta)$. It can be easily checked that G is an increasing function in the interval $\langle 0, \sqrt{(c\beta/\pi)}\rangle$. Since $n < \mu_n < n + 1 < \sqrt{(c\beta/\pi)}$ and according to Lemma 3.1 we have

$$\begin{aligned} E_n(c) &= G(n)\,, \\ E_{\mu_n}(c) &= G(\mu_n)\,, \\ E^t(c) &= G(\sqrt{(c\beta/\pi)})\,, \end{aligned}$$

the inequalities (3.9) hold.

II) By calculation we obtain

$$E_n(c(n^2)) = E(n^2) \quad \forall n\,. \tag{3.13}$$

Denoting $c = c((n + 1)^2)$ we have for $\lambda \in (\mu_n^2, (n + 1)^2)$:

$$\begin{aligned} E(\lambda) - E_n(c(\lambda)) &= E((n + 1)^2) - E_n(c) - \int_\lambda^{(n+1)^2} [2\sigma\, c'(\sigma) - 2n^2\, c'(\sigma)]\, d\sigma = \\ &= E_{n+1}(c) - E_n(c) + 2\int_\lambda^{(n+1)^2} c(\sigma)\, d\sigma - 2[(\sigma - n^2)\, c(\sigma)]_\lambda^{(n+1)^2} = \\ &= \frac{\pi}{\beta}[n^4 - (n + 1)^4] + \int_\lambda^{(n+1)^2} 2c(\sigma)\, d\sigma + 2(\lambda - n^2)\, c(\lambda) \geqq \\ &\geqq \frac{\pi}{\beta}[n^4 - (n + 1)^4] + \frac{\pi}{\beta}\int_\lambda^{(n+1)^2} 2\sigma\, d\sigma + 2(\lambda - n^2)\frac{\pi}{\beta}\lambda = \end{aligned}$$

$$= \frac{\pi}{\beta}[n^4 - (n+1)^4 + (n+1)^4 - \lambda^2 + 2\lambda^2 - 2n^2\lambda] =$$

$$= \frac{\pi}{\beta}[n^4 - 2n^2\lambda + \lambda^2] = \frac{\pi}{\beta}(n^2 - \lambda)^2 > 0 .$$

To complete the proof of (3.10), let us now prove the inequality

$$c(\lambda) > \frac{\pi}{\beta}\lambda ,$$

which has been used in the above estimates. Obviously, it is sufficient to prove $P(\mu\pi) > 0$.

We have

$$y^2 + y \sin y \cos y - 2 \sin^2 y > y^2 - y - 2 > 0$$

for $y = \mu\pi > 2$, and therefore also for $\mu > \mu_1 > 1$. To estimate the difference $E(\lambda) - E_n(c(\lambda))$ we also needed $\lambda - n^2 > 0$. This is true since we have $\lambda \in (\mu_n^2, (n+1)^2\rangle$.

IV) Let

$$\lambda', \lambda'' \in (\mu_n^2, (n+1)^2\rangle, \lambda' < \lambda'', c(\lambda') = c(\lambda'') = c .$$

Assertion 2 of Lemma 3.2 implies $c(\lambda) < c\ \forall\lambda \in (\lambda', \lambda'')$. Consequently,

$$E(\lambda'') - E(\lambda') = \int_{\lambda'}^{\lambda''} 2\lambda\, c'(\lambda)\, \mathrm{d}\lambda = - \int_{\lambda'}^{\lambda''} 2\, c(\lambda)\, \mathrm{d}\lambda + [2\lambda\, c(\lambda)]_{\lambda'}^{\lambda''} =$$

$$= - \int_{\lambda'}^{\lambda''} 2\, c(\lambda)\, \mathrm{d}\lambda + 2(\lambda'' - \lambda')\, c = 2 \int_{\lambda'}^{\lambda''} [c - c(\lambda)]\, \mathrm{d}\lambda > 0 ,$$

and (3.12) is proved.

III) The behaviour of the function c in the interval $(\mu_n^2, (n+1)^2)$ indicates that there exists a point $\bar{\lambda}$ of this interval with the following properties:

a) $c(\bar{\lambda}) = c((n+1)^2)$,

b) $c(\sigma) \leqq c((n+1)^2)\ \forall\sigma \in \langle\bar{\lambda}, (n+1)^2\rangle$,

c) c is decreasing in $(\mu_n^2, \bar{\lambda}\rangle$.

Using IV) we now obtain

$$E_{n+1}(c(\bar{\lambda})) - E(\bar{\lambda}) = E_{n+1}(c((n+1)^2)) - E(\bar{\lambda}) = E((n+1)^2) - E(\bar{\lambda}) > 0 .$$

If we denote by $F(\lambda)$ the difference on the right-hand side, we have

$$F'(\lambda) = 2(n+1)^2\, c'(\lambda) - 2\lambda\, c'(\lambda) = 2[(n+1)^2 - \lambda]\, c'(\lambda) .$$

Taking into account our assumption we can write

$$c > \pi\left(\frac{\varepsilon^2}{4} + \frac{(n+1)^2}{\beta}\right) = c((n+1)^2) ,$$

and therefore

$$\lambda \in (\mu_n^2, \bar{\lambda}) .$$

Hence

$$F(\lambda) > F(\bar{\lambda}) > 0,$$

and (3.11) is proved.

CONCLUSIONS

1) *The case* $\varepsilon = 0$.

Table 1 shows that for $c \leqq \pi/\beta$ the problem (1.6) has but the trivial solution. This means that unless the right end of the rod is displaced by more than π/β from its original position, no buckling occurs. When c exceeds π/β, we obtain the first "buckled" solution, namely the couple

$$(3.14) \qquad \begin{aligned} u(x) &= -\frac{c}{\pi}x - \frac{c - \pi/\beta}{2\pi}\sin 2x , \\ w(x) &= 2\left(\sqrt{\frac{c - \pi/\beta}{\pi}}\right)\sin x . \end{aligned}$$

The value $c = \pi/\beta$ is then the first bifurcation point of our problem.

With c increasing we pass through other bifurcation points, as more solutions emerge (see Fig. 2). However, as a result of the estimates (3.9), all these new solutions have energies higher than (3.14).

In short, the equilibrium state in the case $\varepsilon = 0$ is described by the trivial solution ("no buckling") for $c \leqq \pi/\beta$, and by the couple (3.14) for values of the parameter greater than π/β. As a matter of fact, the lowest energy solution is the same as in the case of the bilateral problem studied in [1].

2) *The case* $\varepsilon > 0$.

Here the bifurcation diagram is somewhat more complicated. Assertion 2 of Lemma 3.2 implies that for values of $c < c(\lambda_1)$ we have a unique branch of solutions given by (2.6) for $c \leqq c(0)$, by (2.5) for $c(0) \leqq c \leqq c(1)$, and by (3.14) for $c \geqq c(1)$ (see Fig. 2). According to Assertion 4 of the same lemma the energy of this solution decreases for c smaller than $c(0)$ and increases with c on the interval $\langle c(0), c(1)\rangle$. Since the function E_1 is increasing (see formula (3.1)). the energy E is increasing in $\langle c(1), c(\lambda_1)\rangle$ as well.

As the value of c passes through the first bifurcation point $c(\lambda_1)$, more solutions of the problem begin to emerge. Namely, the couple (3.14) is at this point joined by two new solutions of the form (2.5) (see Fig. 2). At the point $c(4)$ one of these branches is replaced by a solution of the form (2.4) (with $n = 2$).

Once again, two new branches given by formulas (2.5) bifurcate from the point $c(\lambda_2)$, and so on. Inequalities (3.9)–(3.12) enable us to compare the energies of all solutions. As in the previous case, the equilibrium state is given by (3.14) for c greater than $c(1) = \pi(\varepsilon^2/4 + 1/\beta)$.

In both cases $\varepsilon = 0$, $\varepsilon > 0$ the lowest-energy solutions, except for the trivial one, have no zeros in the open interval $(0, \pi)$. (The buckled rod has no points in common with the x-axis, except for its ends.)

References

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Souhrn

JEDNOSTRANNÁ OKRAJOVÁ ÚLOHA PRO PRUT

MIROSLAV BOSÁK

Autor se zabývá problémem větvení řešení jedné úlohy o prostě podepřeném prutu s jednostrannou okrajovou podmínkou na levém konci. Variační formulace úlohy je převedena na ekvivalentní úlohu klasickou, která je explicitně vyřešena. Explicitní tvar řešení, výpočet vzorců, vyjadřujících energii jednotlivých řešení a srovnání těchto energií tvoří hlavní přínos článku. Výpočet řešení, energií i jejich srovnání jsou provedeny v případě úlohy s porušenou i homogenní okrajovou podmínkou.

Резюме

ОДНОСТОРОННЯЯ КРАЕВАЯ ЗАДАЧА О СТЕРЖНЕ

MIROSLAV BOSÁK

Автор рассматривает проблему ветвления решений для задачи о стержне с односторонним граничным условием на левом конце. Вариационная постановка приведена к эквивалентной краевой задаче в классической формулировке, которая решается в явном виде.

Даны формулы для решений и выражения для энергий, соответствующих этим решениям. Вычисление решений и энергий и их сравнение осуществлены в случае однородного и в слусдвинутого краевого условия.

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