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FREE VIBRATIONS FOR THE EQUATION OF A RECTANGULAR THIN PLATE

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Summary. In the paper, we deal with the equation of a rectangular thin plate with a simply supported boundary. The restoring force being an odd superlinear function of the vertical displacement, the existence of infinitely many nonzero time-periodic solutions is proved.

Keywords: Thin plate equation, periodic solution, Ljusternik-Schnirelmann theory.

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We shall investigate the existence of a nonzero periodic (in time) solution of the equation

{P}

(P1) $$u_{tt} + \Delta^2 u + f(u) = 0$$

where the unknown function $u = u(x, y, t)$, $x, y \in (0, \pi)$, $t \in R^1$ satisfies the boundary conditions

(P2) $$u(0, y, t) = u(\pi, y, t) = u_{xx}(0, y, t) = u_{xx}(\pi, y, t) = 0$$
$$\text{for all} \quad y \in [0, \pi], \quad t \in R^1,$$
$$u(x, 0, t) = u(x, \pi, t) = u_{yy}(x, 0, t) = u_{yy}(x, \pi, t) = 0$$
$$\text{for all} \quad x \in [0, \pi], \quad t \in R^1,$$

u is 2π-periodic in time, i.e.

(P3) $$u(x, y, t) = u(x, y, t + 2\pi) \quad \text{for all} \quad x, y \in (0, \pi), \quad t \in R^1.$$

The symbol Δ^2 denotes the biharmonic operator

$$\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2.$$

The function f is supposed to satisfy the condition $f(0) = 0$ and some other additional conditions.

There exists a vast literature concerning the problem of the existence of free vibrations for various kinds of equations (see e.g. [2], [4], [5]). We shall treat the problem involving two space variables. It is known that under such circumstances all techniques used up to now are not applicable in the case of the wave equation. However, if we work with the biharmonic operator the situation turns out to be better. We shall assume that the function f is monotone and odd. This fact allows us to use the Ljusternik-Schnirelmann theory in order to obtain an approximate solution of the problem $\{P\}$. Then we can pass to the limit using standard arguments of the monptone operator theory.

1. FUNCTIONAL SPACES AND NOTATION USED IN THE TEXT

For the sake of simplicity and convenience we introduce some notation:

$$Q = \{(x, y, t) \mid x, y \in (0, \pi),\ t \in (0, 2\pi)\}$$

$$D = N \times N \times Z$$

where Z denotes the set of all integers and N the set of all positive integers, while

$$q = (q_1, q_2, q_3)$$

is the notation used for elements of the set D.

Further, we introduce the system of functions

$$(1) \qquad e_{(k,l,j)}(x, y, t) = \begin{cases} \sin(kx)\sin(ly)\sin(jt) & \text{for } j \in N, \quad k, l \in N, \\ \sin(kx)\sin(ly) & \text{for } j = 0, \quad k, l \in N, \\ \sin(kx)\sin(ly)\cos(jt) & \text{for } -j \in N, \quad k, l \in N. \end{cases}$$

Let φ denote the linear hull of the system (1)

$$\varphi = \operatorname{lin}\{e_{(k,l,j)} \mid (k, l, j) \in D\}.$$

We shall use the space of periodic functions of the class L_p. The space L_p is defined as the completion of the system φ with respect to the norm

$$(2) \qquad \|u\|_p = \left(\int_Q |u|^p\right)^{1/p}$$

$$\text{for} \quad 1 \leqq p < \infty \quad \text{and} \quad \|u\|_\infty = \operatorname*{ess\,sup}_Q u \quad \text{for} \quad p = \infty.$$

For the function u belonging to L_1 we introduce the Fourier coefficients

$$a_q(u) = \int_Q u e_q \quad \text{for} \quad q \in D.$$

Further, we denote the eigenvalues of the operator

$$L = \frac{\partial^2}{\partial t^2} + \Delta^2 \tag{3}$$

with regard to the conditions (P2), (P3) as

$$\lambda_{(k,l,j)} = (k^2 + l^2)^2 - j^2 \tag{4}$$

where $(k, l, j) \in D$. We denote

$$K = \{q \mid q \in D;\ \lambda_q = 0\},$$
$$RL = \{q \mid q \in D;\ \lambda_q \neq 0\}.$$

We shall use the letters c_i for positive real numbers which are assumed to be constant in the given context.

2. FORMULATION OF THE MAIN RESULTS IN THE SUPERLINEAR CASE

We start with the definition of a solution of the problem $\{P\}$.

Definition. *A solution of the problem $\{P\}$ is a function u, $u \in L_1, f(u) \in L_1$ satisfying*

$$\lambda_q\, a_q(u) + a_q(f(u)) = 0 \quad \textit{for all} \quad q \in D\,. \tag{5}$$

Recall that the following equivalence holds:

A function u is a solution of the problem $\{P\}$ only if $f(u) \in L_1$ and the equality

$$\int_0^\pi \int_0^\pi \int_{-\infty}^{+\infty} [u(x, y, t)(\Phi_{tt}(x, y, t) + \Delta^2 \Phi(x, y, t)) + f(u(x, y, t))\, \Phi(x, y, t)]\, dx\, dy\, dt = 0$$

holds for every function Φ which is sufficiently smooth and has a compact support in $[0, \pi] \times [0, \pi] \times R^1$ and satisfies the boundary conditions (P2).

Our goal is to establish the following existence theorem:

Theorem 1. *Let a function f satisfy the following assumptions:*

(S1) $\quad f \in C^1(R^1)\,,\ f(0) = 0\,,\ f$ *is increasing and odd* $(f(-u) = -f(u))$;

(S2) $\quad f(u)\, u < f'(u)\, u^2$ *for every* $u \in R^1\,,\ u \neq 0$;

(S3) $$\lim_{u \to +\infty} \frac{f(u)}{|u|^{p-2}\, u} = a$$

for a positive constant a and $p \in (2, +\infty)$.

Then for an arbitrary real number d there exists a solution u of the problem $\{P\}$, u is of the class L_p and $\|u\|_p \geq d$.

Note that the technique of the proof is applicable to the sublinear case as well (see e.g. [2] for the case of a beam equation).

For further consideration we deduce some estimates for the function f. First, let us denote

$$F(u) = \int_0^u f(s)\,\mathrm{d}s \tag{6}$$

Now we can find $\varepsilon > 0$ such that

$$\frac{a - \varepsilon}{2} > \frac{a + \varepsilon}{p} \tag{7}$$

since $p \in (2, +\infty)$. From (S3) we easily check that

$$(a - \varepsilon)\,|u|^{p-2}\,u - c_1 \leqq f(u) \leqq c_2 + (a + \varepsilon)\,|u|^{p-2}\,u \tag{8}$$

for all $u \geqq 0$, further

$$\begin{aligned} \frac{(a - \varepsilon)}{p}\,|u|^p - c_1|u| &\leqq F(u) \\ &\leqq c_2|u| + \frac{(a + \varepsilon)}{p}\,|u|^p \end{aligned} \tag{9}$$

for all $u \in R^1$ and this implies

$$\left(\frac{a - \varepsilon}{2} - \frac{a + \varepsilon}{p}\right)|u|^p - c_3|u| \leqq \tfrac{1}{2}u\,f(u) - F(u) \quad \text{for all} \quad u \in R^1\,. \tag{10}$$

Let us denote by $R(L)$ the closed subspace of L_2

$$R(L) = \{u \mid a_q(u) = 0,\ q \in K\}\,, \tag{11}$$

and by P

$$P\colon L_2 \to R(L) \tag{12}$$

the orthogonal projection (in the sense of the L_2-norm) to the space $R(L)$. To obtain further estimates we use the following lemma:

Lemma 1. *For an arbitrary real constant α, $\alpha > 1$, the sum*

$$\sum_{q \in RL} \frac{1}{|\lambda_q|^\alpha} \tag{13}$$

is convergent.

Proof.

$$\sum_{(k,l,j) \in RL} \frac{1}{|(k^2 + l^2)^2 - j^2|^\alpha} = \sum_{k,l \in N} \frac{1}{(k^2 + l^2)^\alpha} + 2 \sum_{\substack{(k,l,j) \in RL \\ j \in N}} \frac{1}{|(k^2 + l^2)^2 - j^2|^\alpha}$$

The first term on the right hand side is summable for $\alpha > \frac{1}{2}$. We are going to estimate the series

$$\sum_{\substack{(k,l,j)\in RL\\ j\in N}} \frac{1}{|k^2 + l^2 - j|^\alpha \, |k^2 + l^2 + j|^\alpha} \leqq \sum_{\substack{(k,l,j)\in RL\\ j\in N}} \frac{1}{|k^2 + l^2|^\alpha \, |k^2 + l^2 - j|^\alpha} =$$

$$= \sum_{k,l\in N} \left(\frac{1}{|k^2 + l^2|^\alpha} \sum_{\substack{j\in N\\ k^2+l^2\neq j}} \frac{1}{|k^2 + l^2 - j|^\alpha} \right) \leqq 2 \sum_{k,l\in N} \frac{1}{|k^2 + l^2|^\alpha} \sum_{m\in N} \frac{1}{m^\alpha} .$$

Now a sufficient condition for the convergence of the sum on the right hand side of the inequality is $\alpha > 1$.

We can choose α_0 such that

$$r = \frac{\alpha_0(p-2)}{p} < 1 \tag{14}$$

holds and $\alpha_0 > 1$. ∎

For $u \in R(L)$ we now have

$$\|u\|_\infty \leqq c_3 \sum_{q\in R} |a_q(u)| \leqq c_5 \left(\sum_{q\in R} |\lambda_q|^{\alpha_0} \, a_q^2(u) \right)^{1/2} \quad \text{(Hölder)} . \tag{15}$$

We use the results of the complex interpolation theory in order to obtain

$$\|u\|_p \leqq c_6 \left(\sum_{q\in RL} |\lambda_q|^r \, a_q^2(u) \right)^{1/2} \tag{16}$$

where the numbers r, p are taken from (14), (S3), respectively.

In order to approximate our problem, we use the Galerkin method. We define the sequence of spaces

$$H_n = \text{lin} \{ e_q \mid q \in D, \ |q_i| \leqq n \ \text{for} \ i = 1, 2, 3 \} \tag{17}$$

for $n \in N$. From the topological point of view, the spaces H_n are considered as finite dimensional subspaces of the space L_2. Recall that

$$\text{cl} \bigcup_{n\in N} H_n = L_2 . \tag{18}$$

We consider the approximate problem $\{P_n\}$:

$\{P_n\}$ *Find the function* $u_n \in H_n$ *satisfying*

$$\sum_{q\in D} \lambda_q \, a_q(u_n) \, a_q(w) + \int_Q f(u_n) \, w = 0 \tag{19}$$

for every $w \in H_n$.

Obviously (19) is the Euler necessary condition for the existence of a critical point of the functional h_n,

$$h_n(w) = \tfrac{1}{2} \sum_{q \in D} \lambda_q \, a_q^2(w) + \int_Q F(w) \,, \tag{20}$$

defined on H_n.

We transform the variational problem of finding the critical points of the functional h_n on the whole space H_n to the problem of the existence of critical points of an appropriate functional on the sphere S_n in H_n,

$$S_n = \{ w \mid w \in H_n, \; \|w\|_2 = 1 \} \,. \tag{21}$$

Due to the symmetry of our problem (the assumption of oddness of the function f), we can use the Ljusternik - Schnirelmann theory.

We define a new functional J_n by

$$J_n(w) = \inf_{\beta \in R^1} h_n(\beta u) \tag{22}$$

As a consequence of the condition (S3), J_n is well defined on the whole H_n. One easily checks that J_n is even and continuous on the set $H_n \setminus \{0\}$. Let us consider the set

$$M_n = \{ w \mid w \in H_n; \; J_n(w) < 0 \} \,, \tag{23}$$

then M_n is open (due to the continuity of J_n) and the following assertion holds:

Lemma 2. *J_n is of the class $C^1(M_n)$.*

Proof. According to the estimates (9), there exists a number $\beta_0 > 0$ such that

$$J_n(w) = h_n(\beta_0 w) \,.$$

Let us differentiate

$$\frac{\partial}{\partial t} (h_n(tw)) = t \sum_{q \in D} \lambda_q \, a_q^2(w) + \int_Q f(tw) \, w \,,$$

further

$$\frac{\partial^2}{\partial t^2} (h_n(tw)) = \sum_{q \in D} a_q^2(w) + \int_Q f'(tw) \, w^2 \,.$$

Suppose that

$$\frac{\partial}{\partial t} (h_n(t_0 w)) = 0$$

holds for some $t_0 > 0$, then

$$\frac{\partial^2}{\partial t^2} (h_n(t_0 w)) = \frac{1}{t_0^2} \left(\int_Q f'(t_0 w) \, t_0^2 w^2 - \int_Q f(t_0 w) \, t_0 w \right).$$

Now the right hand side is positive by the assumption (S2). Consequently, the function $t \to h_n(tw)$ has for $t > 0$ only one critical point – the minimum. Thus β_0 is determined as a solution of the equation

$$\beta_0 \sum_{q \in D} \lambda_q a_q^2(w) + \int_Q f(\beta_0 w)\, w = 0\,, \quad \beta_0 > 0\,.$$

The classical implicit function theorem gives the differentiability of the mapping $w \to \beta_0(w)$. ∎

We are going to show the relation between the functional J_n and the solutions of the problem $\{P_n\}$.

Lemma 3. *Let us denote the duality on L_2 by the symbol $\langle\ ,\ \rangle$. Suppose that*

$$\langle \operatorname{grad} J_n(v_n), w\rangle = \lambda \langle v_n, w\rangle$$

for some $v_n \in S_n$, $\lambda \in R^1$ and every $w \in H_n$. Then there exists a positive number β_0 such that $u_n = \beta_0 v_n$ is a solution of the problem $\{P_n\}$ and

$$h_n(u_n) = J_n(v_n) \tag{24}$$

holds.

Proof. Choose $\beta_0 > 0$ such that

$$J_n(v_n) = h_n(\beta_0 v_n)$$

(see the proof of Lemma 3). According to the definition of J_n, we now have

$$\frac{1}{t}\left(h_n(\beta_0 v_n + tw) - h_n(\beta_0 v_n)\right) \geqq \frac{1}{t}\left(J_n(\beta_0 v_n + tw) - J_n(\beta_0 v_n)\right) =$$

$$= \frac{1}{t}\left(J_n(v_n + \frac{t}{\beta_0} w) - J_n(v_n)\right)$$

for arbitrary $t > 0$. Setting $u_n = \beta_0 v_n$ and passing to the limit for $t \to 0+$ we get

$$\langle \operatorname{grad} h_n(u_n), w\rangle \geqq \frac{1}{\beta_0} \lambda \langle v_n, w\rangle \tag{25}$$

for all $w \in H_n$. We can take $w = \pm v_n$, obtaining

$$0 = \langle \operatorname{grad} h_n(u_n); \pm v_n\rangle = \frac{\lambda}{\beta_0} \|v_n\|_2\,.$$

Consequently, $\lambda = 0$ and from the validity of (25) for every $w \in H_n$ we deduce

$$\operatorname{grad} h_n(u_n) = 0\,.$$

∎

In order to find the critical points of the functional J_n on the sphere S_n, we use the following abstract theorem:

Theorem 2. *Let H be a Hilbert space of a finite dimension, let S denote the unit sphere in H (see (21)). Let us assume that $M \subset H$ is an open and symmetric set ($x \in M$ implies $-x \in M$). Further, let J be an even functional ($J(-x) = J(x)$) which is continuously differentiable on the set M. Let the following conditions be satisfied:*

(a) *There exists a number $z \in R^1$ such that the set $\{x \mid x \in S \cap M,\ J(x) \leqq z\}$ is closed in H.*

(b) *There exist numbers $z_1, z_2 \in (-\infty, z)$ and linear subspaces L^1, L^2 of the space H satisfying*

(b_i) $\dim (L^1) + \dim (L^2) > \dim (H)$

($\dim$ *is the symbol for algebraical dimension of a vector space*)

(b_{ii}) $\{x \mid x \in S \cap M;\ J(x) \leqq z_2\} \cap L^2 = \emptyset$,

(b_{iii}) $\{x \mid x \in S \cap M;\ J(x) \leqq z_1\} \supseteq (L^1 \cap S)$.

Then there exist $z_0 \in S \cap M$ and a real number λ_0 such that

$$\langle \operatorname{grad} J(z_0), x\rangle = \lambda_0 \langle z_0, x\rangle \quad \textit{for all} \quad x \in H . \tag{26}$$

Moreover, $z_2 < z_1$ and

$$J(z_0) \in [z_2, z_1] \tag{27}$$

holds.

Proof. The proof is based on the concept of the Ljusternik - Schnirelmann category for topological spaces and is contained in [3]. ∎

We introduce the notation

$$X_n(z) = \{w \mid w \in H_n,\ a_q(w) = 0 \quad \text{for} \quad \lambda_q \geqq z\} , \tag{28}$$
$$Y_n(z) = \{w \mid w \in H_n,\ a_q(w) = 0 \quad \text{for} \quad \lambda_q < z\}$$

for $z \in R^1$. In order to apply Theorem 2 to the functional J_n, we show some helpful lemmas.

Lemma 4. *For arbitrary $z < 0$ there exists a number $Q(z) < 0$ such that*

$$[S_n \cap X_n(Q(z))] \subseteq \{w \mid w \in S_n \cap M_n,\ J_n(w) \leqq z\} . \tag{29}$$

Proof. Choose a number $z_1 < 0$ arbitrarily, let $w \in X_n(z_1) \cap S_n$ (in particular, $w \in R(L)$). For $t > 0$ we have

$$h_n(tw) = \tfrac{1}{2}t^2 \sum_{\lambda_q < z_1} \lambda_q\, a_q^2(w) + \int_Q F(tw) .$$

According to (9), we obtain

$$h_n(tw) \leqq \tfrac{1}{2}t^2 \sum_{\lambda_q < z_1} \lambda_q\, a_q^2(w) + c_2 t \|w\|_1 + c_4 t^p \|w\|_p^p .$$

Denote $A = c_6 \sum_{\lambda_q < z_1} |\lambda_q|^r a_q^2(w))^{1/2}$. Using the estimate (16) and the Hölder inequality, we get

$$h_n(tw) \leqq -\tfrac{1}{2}t^2|z_1|^{1-r} A^2 + c_8 tA + c_4 t^p A^p .$$

Now observe that

$$\inf_{t\in R^1} h_n(tw) = \inf_{t\in R^1} h_n(tAw) .$$

Consequently, for a sufficiently large $|z_1|$ the value of $J_n(w)$ is sufficiently small. We can choose z_1 in such a way that (29) is satisfied for $Q(z) = z_1$. ∎

Lemma 5. *Let $z < 0$ be an arbitrary real number. Then there exists a number $R(z) < 0$ such that*

$$\{w \mid w \in S_n \cap M_n,\ J_n(w) \leqq R(z)\} \cap Y_n(z) = \emptyset . \tag{30}$$

Proof. Choose $w \in Y_n(z)$, $w \neq 0$. Then according to (9),

$$h_n(tw) \geqq \tfrac{1}{2}t^2 z\|w\|_2^2 + c_q t^p\|w\|_p^p - c_1 t\|w\|_1 \geqq$$

$$\geqq \tfrac{1}{2}t^2 z\|w\|_2^2 + c_{10}t^p\|w\|_2^p - c_{11}t\|w\|_2 \quad \text{for} \quad t > 0 .$$

Analogously as in the proof of Lemma 4, we obtain the validity of (30) for $R(z) < 0$ sufficiently small. ∎

Now let us choose $z_1 < 0$ arbitrary. According to Lemma 4, there exists $Q(z_1)$ satisfying (29). Let $L^1 = X_n(Q(z_1))$. If n is sufficiently large, we can find $y < 0$ such that

$$\dim(Y_n(y)) > \dim(Y_n(Q(z_1)) .$$

According to Lemma 5 we find a number $z_2 = R(y)$ satisfying (30). We can set $L^2 = Y_n(y)$ and apply Theorem 2 for $H = H_n$, $M = M_n$. Then

$$\dim(L^2) + \dim(L^1) > \dim(X_n(Q(z_1))) + \dim(Y_n(Q(z_1))) = \dim(H_n)$$

holds. We have obtained the following result:

Lemma 6. *For an arbitrary number z_1, $z_1 < 0$ there exists a number z_2, $z_2 < z_1$ such that the following assertion holds:*

For every $n \in N$, n sufficiently large, there exists a solution u_n of the problem $\{P_n\}$ satisfying

$$\tfrac{1}{2}\sum_{q\in D} \lambda_q\, a_q^2(u_n) + \int_Q F(u_n) \in [z_2, z_1] . \tag{31}$$

4. PASSING TO THE LIMIT

In § 3 we have obtained the sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of the problems $\{P_n\}$ satisfying (19) and (31). Setting $w = u_n$ in (19), multiplying by $-\frac{1}{2}$ and adding to (31), we have

$$\tag{32} \tfrac{1}{2}\int_Q f(u_n)\, u_n - \int_Q F(u_n) \in [-z_1, -z_2] .$$

Now we can use the estimate (10) and get

$$\tag{33} c_{10}\int_Q |u_n|^p - c_{11}\int_Q |u_n| \leqq -z_2 .$$

Consequently ($p > 2$),

$$\tag{34} \|u_n\|_p \leqq c_{12} .$$

Further, from (S1), (S3) we deduce

$$\tag{35} \|f(u_n)\|_{p'} \leqq c_{13}$$

where $1/p + 1/p' = 1$. All our estimates are independent of n. Moreover,

$$\tag{36} \tfrac{1}{2}\int_Q f(u_n)\, u_n \geqq -z_1 + \int_Q F(u_n) \geqq -z_1 > 0 .$$

We can choose a subsequence (denoted for convenience u_n again) such that

$$\tag{37} u_n \to u \quad \text{weakly in} \quad L_p ,$$

$$\tag{38} f(u_n) \to g \quad \text{weakly in} \quad L_{p'} .$$

We can pass to the limit in (29) for $n \to \infty$ and fixed $w \in H_n$. Thus we get

$$\tag{39} \sum_{q\in D} \lambda_q\, a_q(u)\, a_q(w) + \int_Q gw = 0 .$$

Observe that for proving Theorem 1 we only need to show

$$\tag{40} \lim_{n\to\infty} \int_Q f(u_n)\, u_n = \int_Q gu .$$

In fact, due to (37), (38) and the monotonicity of f, we obtain $g = f(u)$ as a result of standard arguments of the monotone operator theory. Moreover, (36) yields

$$\int_Q f(u)\, u \geqq -z_1 .$$

The number z_1 was chosen quite arbitrary. Combining these facts with the estimate (8) we see that it is possible to choose z_1 in such a way that

$$\|u\|_p \geqq d .$$

Now we shall prove (40).

Lemma 7. (i) *For arbitrary $\varepsilon > 0$ there exists $q_0 \geqq 0$ such that*

$$\sum_{\substack{q\in D\\ |\lambda_q| \geqq q_0}} |\lambda_q|\, a_q^2(u_n) < \varepsilon \quad \textit{for every} \quad n \in N .$$

(ii) *The following equality holds:*

$$\lim_{n\to\infty} \sum_{q\in D} \lambda_q \, a_q^2(u_n) = \sum_{q\in D} \lambda_q \, a_q^2(u) .$$

(iii) *$P(u_n)$ converges to $P(u)$ strongly in the L_p-norm* (*P is the projection from* (12)).

Proof. 1. Set

$$w = \sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} \operatorname{sgn}(\lambda_q)\, a_q(u_n)\, e_q .$$

Note that $w \in R(L) \cap H_n$ so that we can insert w in (19):

$$\sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} |\lambda_q| \, a_q^2(u_n) = - \int_Q f(u_n)\, w .$$

Using the Hölder inequality and the estimates (16), (35), we obtain

$$\sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} |\lambda_q| \, a_q^2(u_n) \leqq c_{12}\Big(\sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} |\lambda_q|^r \, a_q^2(u_n)\Big)^{1/2} .$$

Consequently,

$$\sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} |\lambda_q| \, a_q^2(u_n))^{1/2} \leqq c_{12} |q_0|^{(r-1)/2} .$$

For q_0 sufficiently large, (i) is satisfied independently of n.

2. Let

$$w = \sum_{\substack{q\in D\\ |q_i|\leqq n}} \operatorname{sgn}(\lambda_q)\, a_q(u)\, e_q .$$

We can now insert w in (39) obtaining

$$\sum_{\substack{q\in D\\ |q_i|\leqq n}} |\lambda_q| \, a_q^2(u) = - \int_Q g w .$$

Using (16) we get

$$\sum_{\substack{q\in D\\ |q_i|\leqq n}} |\lambda_q| \, a_q^2(u) \leqq c_{13} .$$

Consider now the difference

$$\Big|\sum_{q\in D} \lambda_q \, a_q^2(u) - \sum_{q\in D} \lambda_q \, a_q^2(u_n)\Big| \leqq \Big| \sum_{\substack{q\in D\\ |\lambda_q|\leqq q_0}} \lambda_q (a^2(u) - a_q^2(u_n))\Big| +$$

$$+ \sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} |\lambda_q| \, a_q^2(u_n) + \sum_{\substack{q\in D\\ |\lambda_q|\geqq q_0}} |\lambda_q| \, a_q^2(u) .$$

The first term on the right hand side converges to zero because it contains a finite number of members only, the second term is small according to part (i), the third is the rest of a convergent series.

3. Analogously we can estimate the difference

$$\|P(u) - P(u_n)\|_p \leqq c_{14}(\sum_{q \in RL} |\lambda_q|^r a_q^2(u_n - u))^{1/2}$$

using (16). As in 2, observe that

$$\|P(u) - P(u_n)\|_p \to 0$$

since

$$a_q^2(u_n - u) \leqq c_{14}(a_q^2(u_n) + a_q^2(u)) .$$

According to (19), we have

$$\sum_{q \in D} \lambda_q a_q^2(u_n) = - \int_Q f(u_n) u_n = - \int_Q f(u_n) P(u_n) . \tag{41}$$

Using Lemma 7 and passing to the limit for $n \to \infty$ in (41), we get

$$\sum_{q \in D} \lambda_q a_q^2(u) = - \int_Q g P(u) . \tag{42}$$

Further, we can set $w = u_n - P(u_n)$ in (39) and thus we obtain

$$\int_Q g(u_n - P(u_n)) = 0 .$$

Passing to the limit for $n \to \infty$ we have

$$\int_Q gu = \int_Q g P(u) . \tag{43}$$

Combining (41), (42), (43) we obtain the desired result (40). Thus Theorem 1 has been proved.

5. POSSIBLE EXTENSIONS AND OTHER COMMENTS

1. In the same way as has been presented, we can treat a more general problem

$\{P'\}$

$$u_{tt}(x, y, t) + c^2 \Delta^2 u(x, y, t) + f(x, y, t, u(x, y, t)) = 0 ; \tag{P1)'$$

u defined on $(0, a) \times (0, b) \times R^1$ satisfying

$$\begin{aligned} u(0, y, t) = u(a, y, t) = u_{xx}(0, y, t) = u_{xx}(a, y, t) = 0 , \\ u(x, 0, t) = u(x, b, t) = u_{yy}(x, 0, t) = u_{yy}(x, b, t) = 0 ; \end{aligned} \tag{P2)'$$

$$u(x, y, t) = u(x, y, t + T) \tag{P3)'$$

where the period T satisfies

$$T = r \cdot \frac{2a^2}{|c| \pi} ,$$

r is an arbitrary positive rational number. If the function f satisfies the conditions (S1)–(S3) uniformly with respect to x, y, t and a/b is rational, then Theorem 1 is valid for the problem $\{P'\}$ as well.

2. If we drop the assumption that f is odd, we can use the dual action method as in [2]. In such a way we are again able to show the existence of a weak periodic solution of the problem $\{P'\}$.

3. In the nonautonomous case, i.e. when the right hand side of the equation (P1)′ is a nonzero function, we can apply the technique presented in [1]. Let us note that we would be able to treat the sublinear case only.

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Souhrn

VOLNÉ VIBRACE PRO ROVNICI TENKÉ OBDÉLNÍKOVÉ DESKY

EDUARD FEIREISL

V práci jsou vyšetřována nenulová časově periodická řešení rovnice tenké obdélníkové desky s volně podepřenými okraji. Je ukázána existence nekonečné posloupnosti takových řešení za předpokladu, že působící síla závisí nelineárně na vertikální výchylce.

Резюме

СВОБОДНЫЕ КОЛЕБАНИЯ ДЛЯ УРАВНЕНИЯ ТОНКОЙ ПРЯМОУГОЛЬНОЙ ПЛАСТИНЫ

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В статье изучаются периодичекие во времени решения уравнения тонкой прямоугольной пластины со свободно опертой границей. Доказано существование бесконечной последовательности ненулевых решений при предположении, что действующая сила зависит нелинейно от поперечного сдвига.

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