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JOINT DISTRIBUTION OF THE BUSY AND IDLE PERIODS OF A DISCRETE MODIFIED $GI/GI/c/\infty$ QUEUE

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Summary. For a discrete modified $GI/GI/c/\infty$ queue, $1 \leqq c < \infty$, where the service times of all customers served during any busy period are independent random variables with not necessarily identical distribution functions, the joint distribution of the busy period, the subsequent idle period and the number of customers served during the busy period is derived. The formulae presented are in a convenient form for practical use.

The paper is a continuation of [5], where the $M/GI/c/\infty$ discrete modified queue has been studied.

Keywords: Modified $GI/GI/c/\infty$ queue, busy period, idle period, number of customers, joint distribution.

AMS Classification: 60K25

0. INTRODUCTION

During the last few years the use of discrete queueing systems with finitely or infinitely many servers in practice has been increasing. The discrete systems are used as mathematical models of, for example, mass servicing machines, electronic machines, transport problems, automated filmless blob-length measurements in track chambers in high-energy physics [3], particle counters [4], etc.

Borovkov [1, 2] studied some limit properties of queues with finitely and infinitely many servers.

For a modified queue we suppose that the service times of all customers served during any busy period are independent random variables with not necessarily identical distribution functions. The modified $M/GI/1$ queue has been investigated by Yeo [11] and Welch [10], modified $GI/M/1$ and $GI/GI/1$ queues by Pakes [7, 8] and $GI/M/1$ by Shanthikumar [9]. The modified $GI/GI/\infty$ queue has been studied in [4] in terms of the particle counter language.

The joint distribution of the busy and idle periods for the $GI/M/1$ queue has been derived by Kalashnikov [6]. For the discrete modified $M/GI/c/\infty$ queue the distribution of the busy period has been obtained by the author in [5], the independence of the busy period on the subsequent idle one being used for the purpose.

In the present note we derive the joint distribution of the busy period, idle period and the number of customers served during the busy period for the discrete modified queue $GI/GI/c/\infty$, for any $1 \leqq c < \infty$, supposing the FIFO discipline. The method used is different from that in [5]. In more detail, the case of the $GI/GI/1/\infty$ queue is treated. The formulae presented are applied to a discrete modified $GI/GI/\infty$ queue. The expressions are convenient for practical use, and the computational process can be easily programmed for a computer. Some remarks on computing and on particular cases of queues will be given in Part 3.

1. DISCRETE MODIFIED $GI/GI/c/\infty$

We suppose that the queueing system with c serves, $1 \leqq c < \infty$, is idle before the service process, and the customers arrive at the queue at moments $0 = \tau_1 < \tau_2 < \ldots$ $\ldots < \infty$. Let X_k, $k \geqq 1$, be the service time of the kth customer in the busy period, and let T_k, $k \geqq 1$, be the interarrival time between the arrivals of the k-th and $(k+1)$st customers. The busy period, B^c, is the time period until the system becomes empty for the first time. The idle period, I^c, is the time period beginning with the termination of the busy period and terminating with a new arrival. The sum, $C^c = B^c + I^c$, of the busy period and the subsequent idle period is called a cycle. Denote by ν^c the number of customers served during the busy period. For the modified queue we suppose that any busy period is produced by the sequence of the service times, $\{X_k\}_{k=1}^{\infty}$, and by the interarrival times, $\{T_k\}_{k=1}^{\infty}$. $\{X_k\}_{k=1}^{\infty}$ are assumed to be independent positive discrete random variables with the distribution laws

$$h_k(n) = P(X_k) = nh\,, \quad n \geqq 1\,, \quad k \geqq 1\,, \tag{1.1}$$

where $\sum_{n=1}^{\infty} h_k(n) = 1$, $k \geqq 1$. The sequence $\{X_k\}_{k=1}^{\infty}$ is independent of the sequence $\{T_k\}_{k=1}^{\infty}$, where T_k, $k \geqq 1$, are assumed to be positive discrete random variables with the same step $h > 0$ and with the distribution laws

$$f_k(n) = P(T_k = nh)\,, \quad n \geqq 1\,, \quad k \geqq 1\,, \tag{1.2}$$

where $\sum_{n=1}^{\infty} f_k(n) = 1$, $k \geqq 1$. Any busy period is resumed with the initial conditions independent of the previous busy periods. This discrete modified queue will be denoted by $\mathcal{G}^c = (f_1, f_2, \ldots; h_1, h_2, \ldots)$.

For a given queue $\mathcal{G}^c = (f_1, f_2, \ldots; h_1, h_2, \ldots)$ it is convenient to consider a sequence of discrete queues, $\{\mathcal{G}_k^c\}_{k=1}^{\infty}$, where $\mathcal{G}_k^c = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$. Suppose that the first busy period of any queue $\mathcal{G}_k^c$ is produced by sequences $\{X_n^k\}_{n=1}^{\infty}$ and $\{T_n^k\}_{n=1}^{\infty}$, where $X_n^k = X_{k+n-1}$, $T_n^k = T_{k+n-1}$, For $\mathcal{G}_k^c$ we define the corresponding busy period, B_k^c, the idle period, I_k^c, the cycle, C_k^c, and the number of customers, ν_k^c, respectively. For simplicity we put $h = 1$.

Our main aim is to determine the joint distribution, $W_k^c(n, m, p) = P(B_k^c = n, I_k^c = m, \nu_k^c = p)$, $n, m, p \geqq 1$, of the busy period, the idle one, and the number of customers served during the busy period of the discrete modified queue $\mathscr{G}_k^c = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $1 \leqq c < \infty$, for any $k \geqq 1$.

Let $n, m, p \geqq 1$ be given integers. For any i, $1 \leqq i \leqq n \wedge c$ (here $x \wedge y := \min(x, y)$), let integers $t_1, \dots, t_i \geqq 1$ with $t_1 + \dots + t_i \leqq n$ be given. Put

$$J_1 = n\,, \quad J_2 = J_1 - t_1, \dots, J_i = J_{i-1} - t_{i-1}\,,$$

and, for any $1 \leqq j_1 \leqq J_1, \dots, 1 \leqq j_i \leqq J_i$, define recursively

$$a_1 = j_1\,, \quad a_{s+1} = \max(a_s - t_s, j_{s+1}) \quad \text{for} \quad 1 \leqq s < i\,.$$

To any t_i assign t_i^* by

$$t_i^* = \begin{cases} J_i + m & \text{if} \quad t_i = J_i = a_i\,, \\ t_i & \text{otherwise}\,. \end{cases}$$

Let us introduce functions $A_k^c(n, m, p; j_1, t_1, \dots, j_i, t_i)$, for $t_1 + \dots t_i \leqq n$, $1 \leqq j_s \leqq J_s$, where $1 \leqq s \leqq i$, by

$$(1.3) \qquad A_k^c(n, m, p; j_1, t_1, \dots, j_i, t_i) = P(B_k^c = n, I_k^c = m, \nu_k^c = p \mid X_1^k = j_1\,, T_1^k = t_1, \dots, X_i^k = j_i, T_i^k = t_i^*)\,.$$

Then we have

$$(1.4) \qquad W_k^c(n, m, p) = \sum_{j_1=1}^{J_1} \sum_{t_1=1}^{a_1} h_k(j_1) f_k(t_1^*)\, A_k^c(n, m, p; j_1, t_1)\,.$$

It is clear that

$$(1.5) \qquad A_k^c(n, m, p; j_1, t_1, \dots, j_i, t_i) = \begin{cases} 1 & \text{if} \quad p = i\,, \quad t_i^* = J_i + m\,, \\ 0 & \text{if} \quad (p \neq i\,, \quad t_i^* = J_i + m) \quad \text{or} \\ & (p < i) \quad \text{or} \quad (p > n) \quad \text{or} \quad (p = i, t_i^* = t_i)\,. \end{cases}$$

If $t_i^* < J_i + m$ and $i < n \wedge c$, then

$$(1.6) \qquad A_k^c(n, m, p; j_1, t_1, \dots, j_i, t_i) = \sum_{j_{i+1}=1}^{J_{i+1}} \sum_{t_{i+1}=1}^{a_{i+1}} h_{k+i}(j_{i+1}) f_{k+i}(t_{i+1}^*)\, A_k^c(n, m, p; j_1, t_1, \dots, j_{i+1}, t_{i+1})\,.$$

Hence

$$(1.7) \qquad A_k^c(1, m, 1; 1, 1) = 1\,, \qquad W_k^c(1, m, 1) = h_k(1) f_k(1 + m)\,.$$

To know $W_k^c(n, m, p)$ it is sufficient to calculate (1.3) for all n, m, p and, therefore, in the sequel we shall concentrate ourselves to the study of the interrelations of A_k^c.

Let $n \geqq 2$ and suppose that we know all $A_k^c(n_1, m_1, p; j_1, t_1, \ldots, j_v, t_v)$ for any $1 \leqq n_1 < n$, $v \leqq i \wedge n_1$. Then the process of the evaluation of A_k^c will be algorithmically divided into eight steps.

First we put $t_0 = 0$ and define $\theta_s = t_0 + \ldots + t_{s-1}$ for any $1 \leqq s \leqq i$. We shall suppose that $i < p \leqq n$.

(I) Existence of "gaps": There is an integer u, $1 \leqq u \leqq i$, such that $\max_{1 \leqq s \leqq u} (\theta_s + j_s) < \theta_{u+1}$. Then

$$A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i) = 0 . \tag{1.8}$$

In the following let there be no "gaps".

(II) Existence of "busy periods stuck together": There is an integer u, $1 \leqq u \leqq i$, such that $\max_{1 \leqq s \leqq u} (\theta_s + j_s) = \theta_{u+1}$. Then

$$A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i) = \begin{cases} W_{k+u}^c(n - \theta_{u+1}, m, p - u) & \text{if } u = i , \\ A_{k+u}^c(n - \theta_{u+1}, m, p - u; j_{u+1}, t_{u+1}, \ldots, j_i, t_i) & \text{if } u < i . \end{cases} \tag{1.9}$$

Now, let there be no "busy periods stuck together".

(III) Existence of "quasi-busy periods": There are two integers u and v, $1 \leqq u \leqq v \leqq i$, such that $\theta_u + j_u \leqq \theta_{v+1}$. Then

$$A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i) = \begin{cases} A_{k+1}^c(n, m, p - 1; j_2 + t_1, t_1 + t_2, j_3, t_3, \ldots, j_i, t_i) & \text{if } u = 1 , \\ A_{k+1}^c(n, m, p - 1; j_1, t_1, \ldots, j_{u-2}, t_{u-2}, j_{u-1}, t_{u-1} + t_u, j_{u+1}, t_{u+1}, \ldots, j_i, t_i) & \text{if } u \geqq 2 . \end{cases} \tag{1.10}$$

In the following let there be no "quasi-busy periods"; hence, let $j_s > t_s + \ldots + t_i$ for any $1 \leqq s \leqq i$.

(IV) Possibility of "rearrangement": There are two integers u and v with $1 \leqq u < v \leqq i$ such that $\theta_u + j_u < \theta_v + j_v$. Then

$$A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i) = A_k^c(n, m, p; j_1, t_1, \ldots, j_{u-1}, t_{u-1}, j_v + \theta_v - \theta_u, t_u, j_{u+1}, t_{u+1}, \ldots, j_{v-1}, t_{v-1}, j_u + \theta_u - \theta_v, t_v, j_{v+1}, t_{v+1}, \ldots, j_i, t_i) . \tag{1.11}$$

Now, let there be no possibility of "rearrangement".

(V) Existence of "big step": There is $t_q \geqq 2$ for some q, $1 \leqq q \leqq i$. Let q be the minimal integer with this property. Then

$$A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i) = A_k^c(n - 1, m, p; j_1 - 1, t_1, \ldots, j_{q-1} - 1, t_{q-1}, j_q - 1, t_q - 1, j_{q+1}, t_{q+1}, \ldots, j_i, t_i) . \tag{1.12}$$

In the following let there be no "big steps".

(VI) Let $1 = i < n \leqq c$ $(1 \leqq p \leqq n,\ n \geqq 2)$. Using the simple probabilistic argument concerning the independence of successive cycles we have, for $1 \leqq j \leqq n - 1$, $1 \leqq t \leqq j$,

$$(1.13)\qquad A_k^c(n, m, p; j, t) = \begin{cases} 0 & \text{if } p = 1\,, \\ \sum_{r=1}^{[j-t+2]/2} B_k^c(n, m, p, r; j, t) & \text{if } p \geqq 2\,, \end{cases}$$

where $B_k^c(n, m, p, r; j, t) = P(C_{\mu_1}^c + \ldots + C_{\mu_{r-1}}^c \leqq j - t,\ C_{\mu_1}^c + \ldots + C_{\mu_{r-1}}^c + B_{\mu_r}^c = n - t,\ v_{\mu_1}^c + \ldots + v_{\mu_r}^c = p - 1)$, and $\mu_1 = k + 1, \mu_2 = \mu_1 + v_{\mu_1}^c, \ldots, \mu_r = \mu_{r-1} + v_{\mu_{r-1}}^c$. Here $[x]$ denotes the integer part of a real x. Therefore,

$$(1.14)\qquad B_k^c(n, m, p, r; j, t) = \sum W_{s_1}^c(n_1, m_1, p_1) \ldots W_{s_r}^c(n_r, m_r, p_r)\,,$$

where $s_1 = k + 1$, $s_2 = s_1 + p_1, \ldots, s_r = s_{r-1} + p_{r-1}$, and the summation is taken over the integers $n_1, m_1, p_1, \ldots, n_{r-1}, m_{r-1}, p_{r-1} \geqq 1$, $n_r \geqq n - j$, $m_r = m$, $p_r \geqq 1$ with $n_1 + m_1 + \ldots + n_r + m_r = n + m - t$ and $p_1 + \ldots + p_r = p - 1$.

If $j = n$, similarly as for $1 \leqq j \leqq n - 1$ we can obtain

$$(1.15)\qquad A_k^c(n, m, p; n, t) = \begin{cases} 1 & \text{if } t = n\,,\ p = 1\,, \\ 0 & \text{if either } t < n\,,\ p = 1 \text{ or } t = n\,,\ p \neq 1\,, \\ \sum_{r=1}^{[n-t+1]/2} B_k^c(n, m, p, r; j, t) & \text{if } p \neq 1\,,\ 1 \leqq t \leqq n - 1\,, \end{cases}$$

where

$$(1.16)\qquad B_k^c(n, m, p, r; n, t) = \sum W_{s_1}^c(n_1, m_1, p_1) \ldots W_{s_r}^c(n_r, m_r, p_r)\,;$$

here $s_1 = k + 1$, $s_2 = s_1 + p_1, \ldots, s_r = s_{r-1} + p_{r-1}$, and the summation runs over the integers $n_1, m_1, p_1, \ldots, n_{r-1}, m_{r-1}, p_{r-1}, n_r, p_r \geqq 1$, $m_r \geqq m$ with $n_1 + m_1 + \ldots + n_r + m_r = n + m - t$ and $p_1 + \ldots + p_r = p - 1$.

(VII) Let $2 \leqq i \leqq n \leqq c$. Denote by s_0 an arbitrary integer, $1 \leqq s_0 \leqq i$, such that $\theta_{s_0} + j_{s_0} \geqq \theta_s + j_s$ for $1 \leqq s \leqq i$. Then

$$(1.17)\qquad A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i) = A_{k+i-1}^c(n, m, p - i + 1; \theta_{s_0} + j_{s_0}, \theta_{i+1}) = A_{k+i-1}^c(n - \theta_{i+1} + 1, m, p - i + 1;\ \theta_{s_0} + j_{s_0} - \theta_{i+1} + 1, 1)\,.$$

It remains to deal with the case $1 \leqq i \leqq c < n$. Due to (1.6) and the above seven steps, it is sufficient to examine the case $2 \leqq i = c < n$ with $t_1 = \ldots = t_c = 1$ provided there are no "gaps", no "busy periods stuck together", no "quasi-busy periods" and no possibility of "rearrangement". In other words, let:

(VIII) $j_1 \geqq j_2 + 1 \geqq j_3 + 2 \geqq \ldots \geqq j_c + c - 1$, $j_c \geqq 2$ and $t_1 = \ldots = t_c = 1$. Since in this case the $(c + 1)$st customer finds all servers busy, using the direct probabilistic argument we can show that

$$(1.18)\qquad A_k^c(n, m, p; j_1, 1, \ldots, j_c, 1) = \begin{cases} 0 & \text{if } j_c = n - c + 1\,, \\ A_{k+1}^c(n-1, m, p-1; j_1 - 1, 1, \ldots \\ \ldots, j_c + j - 1, t) & \text{if } j_c < n - c + 1\,. \end{cases}$$

Combining all the above eight steps we may obtain all possible cases of $A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i)$ for any $n, m, p \geqq 1$, $1 \leqq i \leqq n \wedge c$, and this proves the main result:

Theorem 1. *The joint distribution of the busy period, the idle period and the number customers served during the busy period of the discrete modified multi-server queue* $\mathscr{G}_k^c = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, $1 \leqq c < \infty$, *is given by* (1.4), *where* (1.3) *is determined from* (1.5) *through* (1.18) *following the algorithmical steps from* (I) *to* (VIII).

2. DISCRETE MODIFIED $GI/GI/1/\infty$ AND $GI/GI/\infty$ QUEUES

In the present section we apply the results of the previous section to the particular cases of $GI/GI/1/\infty$ and $GI/GI/\infty$ queues.

First we study the one-server queue. According to the above we must calculate $A_k^1(n, m, p; j, t) = P(B_k^1 = n, I_k^1 = m, v_k^1 = p | X_1^k = j, T_1^k = t^*)$ for any $1 \leqq t \leqq j \leqq n$, $m, p \geqq 1$, where

$$t^* = \begin{cases} n + m & \text{if } t = n = j\,, \\ t & \text{otherwise}\,. \end{cases}$$

Therefore, a more detailed analysis gives

$$(2.0)\qquad W_k^1(n, m, p) = \sum_{j=1}^{n} \sum_{t=1}^{j} h_k(j) f_k(t^*) A_k^1(n, m, p; j, t)\,,$$

$$(2.1)\qquad A_k^1(n, m, p; j, t) = \begin{cases} 1 & \text{if } t = n = j\,, \quad p = 1\,, \\ 0 & \text{if } p > n \quad \text{or} \quad (p = 1,\ 1 \leqq j \leqq n - 1,\ n \geqq 2) \\ & \quad \text{or} \quad (p \geqq 2,\ t = n = j)\,, \end{cases}$$

$$(2.2)\qquad \left.\begin{aligned} A_k^1(1, m, 1; 1, 1) &= 1\,, \\ W_k^1(1, m, 1) &= h_k(1) f_k(1 + m)\,. \end{aligned}\right\}$$

Now, let $2 \leqq p \leqq n$, $1 \leqq j \leqq n - 1$. Then

$$(2.3)\qquad A_k^1(n, m, p; j, t) = \sum_{r=1}^{n-1} B_k^1(n, m, r; j, t)\,,$$

where $B_k^1(n, m, p, r; j, t) = P(C_{\mu_1}^1 + \ldots + C_{\mu_r}^1 = m + n - t,\ C_{\mu_1}^1 + \ldots + C_{\mu_{r-1}}^1 + B_{\mu_r}^1 \leqq n - t,\ B_{\mu_1}^1 + \ldots + B_{\mu_r}^1 = n - j,\ v_{\mu_1}^1 + \ldots + v_{\mu_r}^1 = p - 1)$ and $\mu_1 =$

$= k + 1$, $\mu_2 = \mu_1 + v^1_{\mu_1}, \ldots, \mu_r = \mu_{r-1} + v^1_{\mu_{r-1}}$. Therefore

$$B^1_k(n, m, p, r; j, t) = \sum W^1_{s_1}(n_1, m_1, p_1) \ldots W^1_{s_r}(n_r, m_r, p_r)\,, \tag{2.4}$$

where $s_1 = k + 1$, $s_2 = s_1 + p_1, \ldots, s_r = s_{r-1} + p_{r-1}$. Here the summation is taken over all integers $n_1, m_1, p_1, \ldots, m_{r-1}, p_{r-1}, n_r, p_r \geqq 1$, $m_r \geqq m$ with $n_1 + \ldots$ $\ldots + n_r = n - j$, $m_1 + \ldots + m_r = m + j - t$, $p_1 + \ldots + p_r = p - 1$.

This proves the following theorem:

Theorem 2. *The joint distribution of the busy and idle periods and the number of customers served during the busy period of the discrete modified queue* $GI/GI/1/\infty$ $\mathcal{G}^1_k = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, *is given by* (2.0) *where* A^1_k *is calculated from* (2.1) *through* (2.4).

Let us now have a discrete modified $GI/GI/\infty$ queue. Define $W^\infty_k(n, m, p)$ as the joint distribution of the busy period B^∞_k, the idle period I^∞_k, the cycle C^∞_k, the number of customers v^∞_k served during the busy period of the discrete modified queueing system with infinitely any servers $\mathcal{G}^\infty_k = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$, $k \geqq 1$. Then

$$W^\infty_k(n, m, p) = W^n_k(n, m, p)\,, \quad n, m, p, k \geqq 1\,. \tag{2.5}$$

Here $W^n_k(n, m, p) = \sum_{j=1}^{n} \sum_{t=1}^{j} h_k(j) f_k(t^*) A^n_k(n, m, p; j, t)$, and to calculate $A^n_k(n, m, p;$ $j, t)$ we use only the first six steps of the algorithm of the previous section.

Remark. The joint distribution of the cycle and the number of customers served during the busy period, $P^c_k(i, p) = P(C^c_k = i,\ v^c_k = p)$, $i \geqq 2$, $p \geqq 1$, of the discrete modified queue $\mathcal{G}^c_k = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, for any $c = 1, 2, \ldots, \infty$, is given by

$$P^c_k(i, p) = \sum_{\substack{n+m=i \\ n,m \geqq 1}} W^c_k(n, m, p)\,. \tag{2.6}$$

3. CONCLUSION

We see that the actual computation of the joint distribution $W^c_k(n, m, p)$ is relatively simple when we have either only few servers (for example, if $c = 1, 2, 3, \ldots$; we note that even in this case the joint distribution was unknown) or infinitely many servers. In other cases the result of Theorem 1 may be simply programmed for a computer if we wish to obtain either numerical results or analytical expressions.

Here we note only that the following simplifications hold. If $1 \leqq i \leqq n \leqq c < \infty$, then

$$A^c_k(n, m, p; j_1, t_1, \ldots, j_i, t_i) = A^{c+1}_k(n, m, p; j_1, t_1, \ldots, j_i, t_i)\,, \tag{3.1}$$

$$W^c_k(n, m, p) = W^{c+1}_k(n, m, p) = W^\infty_k(n, m, p)\,, \tag{3.2}$$

and they enable us to simplify the computation for a queue with a large number of servers available.

If $h_k(j)$ $(k \geqq 1)$ are non-zero only for few integers j, then the calculation is again simple. In fact, it suffices to evaluate, for example, only a few values of $W_k^c(n, m, p)$ (that is, only $W_k^c(n, m, p)$ with $h_k(j) > 0$). Analogously we proceed for the other quantities $A_k^c(n, m, p; j_1, t_1, \ldots, j_i, t_i)$.

We say that the discrete modified queue $\mathscr{G}^c = (f_1, f_2, \ldots; h_1, h_2, \ldots)$ is of order l if $h_l = h_{l+1} = \ldots$. If $l = 1$, then we obtain the usual non-modified queue. If $\mathscr{G}^c$ is of order at least three, then A_k^c and W_k^c do not depend on the superscript k. Moreover, in this case we may simply determine the joint distribution of the busy and idle periods without knowing v_k^c (the procedure is analogous to that described in the first section).

If $l > 1$, then the computation of W_k^c for the queues $\mathscr{G}_k^c$ with $1 \leqq k \leqq l$, $1 \leqq c \leqq \infty$, may be organized so that first of all we calculate all the necessary expressions for $k = l$, then we continue for $k = l - 1$, etc, for $k = 1$.

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Súhrn

ZDRUŽENÉ ROZDELENIE PERIÓDY OBSADENOSTI A PRESTOJA DISKRÉTNEHO MODIFIKOVANÉHO SYSTÉMU $GI/GI/c/\infty$

ANATOLIJ DVUREČENSKIJ

Pre diskrétny modifikovaný systém hromadnej obsluhy $GI/GI/c/\infty$, $1 \leqq c < \infty$, kde doby obslúh zákazníkov obslúžených v každej periόde obsadenosti sú nezávislé náhodné veličiny s nie

nutne identickými funkciami rozdelenia, určí sa združené rozdelenie periódy obsadenosti, prestoja a počtu zákazníkov, obslúžených za periódu obsadenosti. Predložené formuly sú v tvare vhodnom pre praktické použitie.

Práca je pokračovaním príspevku [5], kde sa študoval diskrétny modifikovaný systém $M/GI/c/\infty$.

Резюме

СОВМЕСТНОЕ РАСПРЕДЕЛЕНИЕ ПЕРИОДОВ ЗАНЯТОСТИ И ПРОСТОЯ ДИСКРЕТНОЙ МОДИФИЦИРОВАННОЙ СИСТЕМЫ МАССОВОГО ОБСЛУЖИВАНИЯ $GI/GI/c/\infty$

Anatolij Dvurečenskij

Для дискретной модифицированной системы массового обслуживания типа $GI/GI/c/\infty$, $1 \leqq c < \infty$, где времена обслуживания всех заказчиков в каждом периоде занятости — независимые случайные величины с необязательно идентичными функциями распределения, выводится совместное распределение периодов занятости и простоя и числа заказчиков, обслуженных за период занятости. Формулы предложены в виде, удобном для практического пользования.

Работа является продолжением результатов работы [5], где изучалась система $M/GI/c/\infty$.

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