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FINITE ELEMENT ANALYSIS OF PRIMAL AND DUAL VARIATIONAL FORMULATIONS OF SEMI-COERCIVE ELLIPTIC PROBLEMS WITH NONHOMOGENEOUS OBSTACLES ON THE BOUNDARY

TRAN VAN BON

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Summary. The Poisson equation with non-homogeneous unilateral condition on the boundary is solved by means of finite elements. The primal variational problem is approximated on the basis of linear triangular elements, and $O(h)$-convergence is proved provided the exact solution is regular enough. For the dual problem piecewise linear divergence-free approximations are employed and $O(h^{3/2})$-convergence proved for a regular solution. Some a posteriori error estimates are also presented.

Keywords: Variational inequalities, finite elements, dual formulations.

AMS Subject Classification: 65N30.

INTRODUCTION

Dual finite element analysis of unilateral elliptic problems has been presented in several papers [1], [2], [7], [8]. A relatively thorough analysis of the coercive case has been given in [7]. The semi-coercive case with homogeneous obstacles on the boundary has been studied by Hlaváček [1], [8]. It is the aim of the present paper to establish a dual finite element analysis for the latter case with non-homogeneous obstacles on the boundary. The dual problem is formulated in terms of the principle of minimum complementary energy. Making use of piecewise linear functions for the approximations of the primal problem, we establish an $O(h)$-error estimate in the energy norm, provided the exact solution is regular enough. For the approximations of the dual problem, the so called equilibrium finite element model is used (see [3]) and an error estimate is derived under the assumption of sufficient regularity of the exact solution. The convergence of the primal approximations is proved for the non-regular solution as well. A posteriori error estimates and two-sided estimates of the energy are presented.

1. DUAL VARIATIONAL FORMULATION

We consider the problem

$$-\Delta u = f \quad \text{in} \quad \Omega\,, \tag{1.1}$$

$$u \geqq g_1,\ \partial u/\partial n - g_2 \geqq 0,\ (u - g_1)(\partial u/\partial n - g_2) = 0 \quad \text{a.e.} \quad \text{on } \Gamma \equiv \partial\Omega\,, \tag{1.2}$$

where $\partial u/\partial n$ denotes the outward normal derivative of u, $f \in L^2(\Omega)$, $g_1, g_2 \in L^2(\Gamma)$ are given functions, Ω is a bounded domain with Lipschitz boundary.

Henceforth we denote by $H^k(\Omega)$ the Sobolev space $W^{k,2}(\Omega)$,

$$(f, g)_0 = \int_\Omega fg \,\mathrm{d}x\,, \quad (u, v)_{0,\Gamma} = \int_\Gamma uv \,\mathrm{d}s\,,$$

$\|u\|_k$ is the norm in $H^k(\Omega)$, $k = 0, 1, \ldots$, $|u|_k$ the seminorm in $H^k(\Omega)$, generated by all derivatives of the k-th order.

Assume that there exists a function $G \in H^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$ such that $\gamma G = g_1$ on Γ, where γ stands for the trace operator. We define the potential energy functional

$$\mathscr{L}(v) = \tfrac{1}{2}|v|_1^2 - (f, v)_0 - (g_2, \gamma v)_{0,\Gamma}$$

and the convex set

$$K = \{v \in H^1(\Omega) |\ \gamma v - g_1 \geqq 0 \text{ on } \Gamma\}\,.$$

The problem: find $u \in K$ such that

$$\mathscr{L}(u) \leqq \mathscr{L}(v) \quad \forall v \in K \tag{1.3}$$

will be called primal.

Lemma 1.1. *The problems* (1.1), (1.2) *and* (1.3) *are equivalent.*

Proof. Assume that u is a solution of (1.3). Then it is readily seen that

$$(\operatorname{grad} u, \operatorname{grad}(v - u)) \geqq (f, v - u)_0 + (g_2, \gamma(v - u))_{0,\Gamma} \quad \forall v \in K\,. \tag{1.4}$$

Inserting $v = u \pm \varphi$, where $\varphi \in C_0^\infty(\Omega)$ is an arbitrary infinitely differentiable function with a compact support in Ω, we obtain

$$(\operatorname{grad} u, \operatorname{grad} \varphi) = (f, \varphi)_0 \quad \forall \varphi \in C_0^\infty(\Omega)\,,$$

i.e., u satisfies (1.1) in the sense of distributions.

Defining

$$\langle \partial u/\partial n, w\rangle = (\operatorname{grad} u, \operatorname{grad} v) - (f, v)_0 \quad \forall w \in H^{1/2}(\Gamma)\,, \tag{1.5}$$

where $v \in H^1(\Omega)$ is any function such that $\gamma v = w$ on Γ, the derivative $\partial u/\partial n$ represents a linear functional from $H^{-1/2}(\Gamma)$, i.e., a continuous functional on $H^{1/2}(\Gamma)$. Since G

and $2u - G$ belong to K, inserting them into (1.4) and comparing the results we arrive at

$$\langle \partial u/\partial n, \gamma u - g_1 \rangle = (g_2, \gamma u - g_1)_{0,\Gamma} . \tag{1.6}$$

On the other hand, it follows from (1.4) and (1.6) that

$$\langle \partial u/\partial n, \gamma v - g_1 \rangle \geqq (g_2, \gamma v - g_1)_{0,\Gamma} . \tag{1.7}$$

Assume that $w \in H^{1/2}(\Gamma)$, $w \geqq 0$. Then $\psi = g_1 + w \in H^{1/2}(\Gamma)$. By virtue of the Trace Theorem there exists a function $v \in H^1(\Omega)$ such that $\gamma v = \psi$ on Γ, i.e., $v \in K$. From (1.7) we have

$$0 \leqq \langle \partial u/\partial n - g_2, \gamma v - g_1 \rangle = \langle \partial u/\partial n - g_2, w \rangle \quad \forall w \in H^{1/2}(\Gamma) , \quad w \geqq 0 ,$$

i.e., $\partial u/\partial n - g_2 \geqq 0$ on Γ.

Combining this with (1.6) we deduce that

$$\langle \partial u/\partial n - g_2, \gamma u - g_1 \rangle = 0 .$$

Thus u is also a solution of (1.1), (1.2), in the weak sense.

Conversely, assume that u is a solution of (1.1), (1.2). Multiplying (1.1) by an arbitrary function $v \in K$ and then integrating by parts we obtain

$$(\operatorname{grad} u, \operatorname{grad} v) - \int_\Gamma \partial u/\partial n \, \gamma v \, ds = (f, v)_0 . \tag{1.8}$$

Inserting $v = u$ into (1.8) and using (1.2), we obtain (1.6).

Let v be an arbitrary function from K. Making use of (1.8), (1.5) and then of (1.6) we can show that (1.4) holds, i.e., u is a solution of (1.3), too.

Lemma 1.2. *Assume that*

$$(f, 1)_0 + (g_2, 1)_{0,\Gamma} < 0 . \tag{1.9}$$

Then there exists a unique solution of the primal problem (1.3). *The problem* (1.3) *has a solution only if*

$$(f, 1)_0 + (g_2, 1)_{0,\Gamma} \leqq 0 . \tag{1.10}$$

Proof. (i) Existence. Let $\Gamma_0 \subset \Gamma$ be an open set of positive measure. We define

$$\bar{v} = (\operatorname{mes} \Gamma_0)^{-1} \int_{\Gamma_0} \gamma v \, ds , \quad v \in H^1(\Omega) .$$

Then $\tilde{v} = v - \bar{v}$ satisfies

$$\int_{\Gamma_0} \gamma \tilde{v} \, ds = 0 , \quad |\tilde{v}|_1 \geqq C \|\tilde{v}\|_1$$

where C is a constant independent of $\tilde{v}$. For $v \in K$ we set $w = v - G$. Then $w \in K_0$, where

$$K_0 = \{w \in H^1(\Omega) |\ \gamma v \geqq 0 \text{ on } \Gamma\}$$

and we have

$$\mathscr{L}(v) = \mathscr{L}(w + G) = \mathscr{L}_1(w) + \mathscr{L}(G),$$

where

$$\mathscr{L}_1(w) = \tfrac{1}{2}|w|_1^2 + (\text{grad } w, \text{grad } G) - (f, w)_0 - (g_2, \gamma w)_{0,\Gamma}.$$

We can show that

$$\mathscr{L}_1(w) \geqq C\|\tilde{w}\|_1^2 - C_1\|\tilde{w}\|_1 - \bar{w}[(f, 1)_0 + (g_2, 1)_{0,\Gamma}].$$

If $v \in K$ and $\|v\|_1 \to +\infty$, then $w = v - G \in K_0$ and $\|w\|_1 \to +\infty$. Hence at least one of the norms $\|\tilde{w}\|_1$ and $\|\bar{w}\|_1 = \bar{w}(\text{mes } \Omega)^{1/2}$ tends to infinity. In any case $\mathscr{L}_1(w)$ and thus also $\mathscr{L}(v)$ tends to infinity for $\|v\|_1 \to +\infty$, i.e., $\mathscr{L}(v)$ is coercive on K.

The set K is convex and closed in $H^1(\Omega)$. The functional $\mathscr{L}(v)$ is also lower weakly semi-continuous. Hence the existence of a minimizing element follows.

(ii) Uniqueness. Let u' and u'' be two possible solutions of (1.3). Inserting them into (1.4) and subtracting, we deduce

$$(\text{grad}\,(u' - u''), \text{grad}\,(u' - u'')) \leqq 0,$$

i.e., $u' - u'' = \text{const.}$ Denote $u'' = u' + c$ and assume that $c \neq 0$. We obtain $\mathscr{L}(u') = \mathscr{L}(u' + c)$, i.e.

$$c[(f, 1)_0 + (g_2, 1)_{0,\Gamma}] = 0,$$

and we arrive at a contradiction with (1.9). Thus $c = 0$ and the solution is unique.

(iii) Let $a \in \mathbb{R}$, $a > 0$, $a \to +\infty$. It is obvious that $a + G \in K$. If there exists a solution of (1.3), then

$$\mathscr{L}(a + G) = \mathscr{L}_1(a) + \mathscr{L}(G)$$

is bounded from below,

$$\lim_{a \to +\infty} \mathscr{L}_1(a) = -[(f, 1)_0 + (g_2, 1)_{0,\Gamma}] \lim_{a \to +\infty} a > -\infty.$$

Hence (1.10) follows. Q.E.D

Lemma 1.3. *Assume that the equality sign in* (1.10) *holds and* $w \in H^1(\Omega)$ *is a weak solution of the Neumann's problem*

(1.11) $$-\Delta w = f \text{ in } \Omega, \quad \partial w/\partial n = g_2 \text{ on } \Gamma,$$

$$\int_\Gamma \gamma w \, ds = 0.$$

Then the primal problem (1.3) *has a solution if and only if* γw *is lower bounded on* Γ. *In this case all solutions of* (1.3) *have the form* $u = w + c$, *where* c *is a constant such that* $\gamma w + c \geqq 0$ *on* Γ.

Proof. It is evident that the problem (1.11) has a solution because of the equality (1.10). Let u be a solution of (1.3). Then using (1.10) and Green's formula we obtain

$$(\partial u/\partial n - g_2, 1)_{0,\Gamma} = 0 .$$

This together with (1.2) implies that $\partial u/\partial n = g_2$ a.e. on Γ. Setting $w = u - c$ with $c = (\text{mes } \Gamma)^{-1} (\gamma u, 1)_{0,\Gamma}$, we conclude that w satisfies (1.11). However, we have

$$\gamma w = \gamma u - c \geqq -c + g_1 \geqq \text{const on } \Gamma , \quad g_1 = \gamma G \in C(\Gamma) .$$

Thus γw is lower bounded.

Conversely, let w be a solution of (1.11) and let γw be lower bounded on Γ. We choose a constant c such that $\gamma w + c \geqq g_1$. Then $w + c$ is a solution of (1.1), (1.2). Furthermore, from the above argument it is readily seen that all solutions are of the form $w + c$, provided $\gamma w + c \geqq g_1$ on Γ. Q.E.D.

Having a dual formulation in mind, we introduce the space

$$Q = \{\boldsymbol{q} \in [L^2(\Omega)]^n \mid \text{div } \boldsymbol{q} \in L^2(\Omega)\} ,$$

where the operator

$$\text{div } \boldsymbol{q} = \sum_{i=1}^{n} \partial q_i/\partial x_i$$

is taken in the sense of distributions. For $q \in Q$ we define a functional $\boldsymbol{q} . \boldsymbol{n} \in H^{-1/2}(\Gamma)$ by the relation

$$\langle \boldsymbol{q} . \boldsymbol{n}, w \rangle = \int_\Omega (\boldsymbol{q} . \text{grad } v + v \text{ div } \boldsymbol{q}) \, dx \quad \forall w \in H^{1/2}(\Gamma) , \tag{1.12}$$

where $v \in H^1(\Omega)$ is any function such that $\gamma v = w$ on Γ. ($H^{-1/2}(\Gamma)$ denotes the dual space with respect to $H^{1/2}(\Gamma)$.) We write $\boldsymbol{q} . \boldsymbol{n}|_\Gamma \geqq g_2$ on Γ if

$$\langle \boldsymbol{q} . \boldsymbol{n}, w \rangle \geqq (g_2, w)_{0,\Gamma} \quad \forall w \in H^{1/2}(\Gamma) , \quad w \geqq 0 .$$

The set of admissible functions is defined by

$$\mathscr{U} = \{\boldsymbol{q} \in Q \mid \text{div } \boldsymbol{q} + f = 0 \text{ in } \Omega , \quad \boldsymbol{q} . \boldsymbol{n}|_\Gamma \geqq g_2 \text{ on } \Gamma\} .$$

Let us consider the functional of complementary energy

$$\mathscr{S}(\boldsymbol{q}) = \tfrac{1}{2} \sum_{i=1}^{n} \|q_i\|_0^2 - \langle \boldsymbol{q} . \boldsymbol{n}, g_1 \rangle . \tag{1.13}$$

The problem to find $\lambda^0 \in \mathscr{U}$ such that

$$\mathscr{S}(\lambda^0) \leqq \mathscr{S}(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathscr{U} \tag{1.14}$$

will be called dual with respect to the primal problem (1.3).

Lemma 1.4. *The dual problem has a solution if and only if the condition* (1.10) *is satisfied. Then the solution* λ^0 *is unique.*

Proof. The condition (1.10) is necessary and sufficient for the set $\mathscr{U}$ to be nonempty. In fact, assume that $\boldsymbol{q} \in \mathscr{U}$, then

$$(g_2, 1)_{0,\Gamma} \leqq \langle \boldsymbol{q} \cdot \boldsymbol{n}, 1 \rangle = (\operatorname{div} \boldsymbol{q}, 1)_0 = -(f, 1)_0 ,$$

i.e., (1.10) is necessary.

Conversely, let (1.10) hold. If (1.9) holds, the problem (1.3) has a unique solution u. Then $\boldsymbol{q} = \operatorname{grad} u \in \mathscr{U}$, i.e., $\mathscr{U}$ is nonempty. If

$$(f, 1)_0 + (g_2, 1)_{0,\Gamma} = 0 ,$$

then Neumann's problem

$$-\Delta w = f \text{ in } \Omega , \quad \partial w/\partial n = g_2 \text{ on } \Gamma$$

has at least one solution w. Then $\boldsymbol{q} = \operatorname{grad} w \in \mathscr{U}$, i.e., $\mathscr{U} \neq \emptyset$.

The set $\mathscr{U}$ is convex and closed. The functional $\mathscr{S}$ is coercive, strictly convex and continuously differentiable. Hence the existence and uniqueness of a minimizing element follows. Q.E.D.

Theorem 1.1. *Let* (1.10) *hold and let the primal problem* (1.3) *have a solution* u. *Then the solution* λ^0 *of the dual problem satisfies the relations*

$$\lambda^0 = \operatorname{grad} u , \tag{1.15}$$

$$\mathscr{S}(\lambda^0) + \mathscr{L}(u) + (g_1, g_2)_{0,\Gamma} = 0 . \tag{1.16}$$

Proof. Since $v \in K$ if and only if $w = v - G \in K_0$ and $\mathscr{L}(v) = \mathscr{L}_1(w) + \mathscr{L}(G)$, we have

$$\mathscr{L}(u) = \operatorname*{Inf}_{v \in K} \mathscr{L}(v) = \operatorname*{Inf}_{w \in K_0} \mathscr{L}_1(w) + \mathscr{L}(G) = \mathscr{L}_1(w^0) + \mathscr{L}(G) ,$$

where $w^0 = u - G$.

We introduce the parameters $\mathscr{N}_i = \partial w/\partial x_i$ and symbols $M = [L^2(\Omega)]^n$, $\mathscr{W} = K_0 \times M$. Then we may write

$$\operatorname*{Inf}_{w \in K_0} \mathscr{L}_1(w) = \operatorname*{Inf}_{[w, \mathscr{N}] \in \mathscr{W}} \ \operatorname*{Sup}_{\mu \in M} \mathscr{H}(w, \mathscr{N}; \mu) , \tag{1.17}$$

where

$$\mathscr{H}(w, \mathscr{N}; \mu) = \tfrac{1}{2} \sum_{i=1}^{n} \|\mathscr{N}_i\|_0^2 + (\mathscr{N}, \operatorname{grad} G) - (f, v)_0 +$$

$$+ (\mu, \operatorname{grad} w - \mathscr{N}) - (g_2, \gamma w)_{0,\Gamma} .$$

In fact, we have

$$\operatorname*{Sup}_{\mu\in M}(\mu, \text{grad } w - \mathcal{N}) = \begin{cases} 0 & \text{if } \mathcal{N} = \text{grad } w, \\ +\infty & \text{if } \exists i, \mathcal{N}_i \neq \partial w/\partial x_i. \end{cases}$$

Therefore

$$\operatorname*{Inf}_{[w,\mathcal{N}]\in W} \operatorname*{Sup}_{\mu\in M} \mathcal{H}(w, \mathcal{N}; \mu) = \operatorname*{Inf}_{[w,\mathcal{N}]\in\mathcal{W}, \mathcal{N}=\text{grad} w} \mathcal{H}(w, \mathcal{N}; \mu) = \operatorname*{Inf}_{w\in K_0} \mathcal{L}_1(w) = \mathcal{L}_1(w^0).$$

Let us consider the problem dual to (1.17), i.e.,

$$\operatorname*{Sup}_{\mu\in M} \operatorname*{Inf}_{[w,\mathcal{N}]\in\mathcal{W}} \mathcal{H}(w, \mathcal{N}; \mu)$$

First of all, we may write

$$-\mathcal{S}(\mu) = \operatorname*{Inf}_{[w,\mathcal{N}]\in\mathcal{W}} \mathcal{H}(w, \mathcal{N}; \mu) \leqq \operatorname*{Inf}_{[w,\mathcal{N}]\in\mathcal{W}, \mathcal{N}=\text{grad} w} \mathcal{H}(w, \mathcal{N}; \mu) = \mathcal{L}_1(w^0) \quad \forall\mu\in M,$$

and consequently,

(1.18) $$\operatorname*{Sup}_{\mu\in M}[-\mathcal{S}(\mu)] \leqq \mathcal{L}_1(w^0).$$

On the other hand,

(1.19) $$-\mathcal{S}(\mu) = \operatorname*{Inf}_{[w,\mathcal{N}]\in\mathcal{W}} \{\mathcal{H}_1(\mathcal{N}, \mu) + \mathcal{H}_2(w, \mu)\},$$

where

$$\mathcal{H}_1(\mathcal{N}, \mu) = \tfrac{1}{2}\|\mathcal{N}\|^2 + (\mathcal{N}, \text{grad } G) - (\mu, \mathcal{N}),$$

$$\mathcal{H}_2(w, \mu) = -(f, w)_0 - (g_2, \gamma w)_{0,\Gamma} + (\mu, \text{grad } w).$$

It is easy to show that

(1.20) $$\operatorname*{Inf}_{\mathcal{N}\in M} \mathcal{H}_1(\mathcal{N}, \mu) = -\|\mu - \text{grad } G\|^2.$$

Next, $\mathcal{H}_2(w, \mu)$ is a linear continuous functional in $H^1(\Omega)$. If there exists a function $w_0 \in K_0$ such that $\mathcal{H}_2(w_0, \mu) < 0$. then we deduce that

$$\operatorname*{Inf}_{w\in K_0} \mathcal{H}_2(w, \mu) = -\infty,$$

because $tw_0 \in K_0$ for all positive t and $\mathcal{H}_2(tw_0, \mu)$ tends to $-\infty$ for $t \to +\infty$. Therefore, the infimum is finite only if

(1.21) $$\mathcal{H}_2(w, \mu) \geqq 0 \quad \forall w \in K_0.$$

Choosing $w = \pm\varphi$, $\varphi \in C_0^\infty(\Omega) \subset K_0$, we obtain

$$0 = \mathcal{H}_2(\varphi, \mu) = -(f, \varphi)_0 + (\mu, \text{grad } \varphi) \quad \forall\varphi \in C_0^\infty(\Omega),$$

i.e., $\mu \in Q$ and $\text{div } \mu + f = 0$ in Ω. Then we may write

$$(\mu, \text{grad } w) = (f, w)_0 + \langle \mu \,.\, \mathbf{n}, \gamma w\rangle \quad \forall w \in H^1(\Omega)$$

and therefore

(1.22) $$\mathscr{H}_2(w, \mu) = \langle \mu \,.\, \boldsymbol{n}, \gamma w \rangle - (g_2, \gamma w)_{0,\Gamma} \quad \forall w \in H^1(\Omega)\,.$$

Combining this with (1.21), we obtain $\mu \,.\, \boldsymbol{n} \geqq g_2$ on Γ. Thus the infimum of $\mathscr{H}_2(w, \mu)$ on K_0 is infinite only if $\mu \in \mathscr{U}$. Conversely, if $\mu \in \mathscr{U}$, then (1.22) and (1.21) hold and thus the infimum is finite. Consequently, we obtain

(1.23) $$\operatorname*{Inf}_{w \in K_0} \mathscr{H}_2(w, \mu) = \begin{cases} 0 & \text{if} \quad \mu \in \mathscr{U}\,, \\ -\infty & \text{if} \quad \mu \notin \mathscr{U}\,. \end{cases}$$

Finally, we arrive at

$$-\mathscr{S}(\mu) = \begin{cases} -\mathscr{S}(\mu) + (f, G)_0 - \tfrac{1}{2}|G|_1^2\,, & \mu \in \mathscr{U}\,, \\ -\infty\,, & \mu \notin \mathscr{U}\,, \end{cases}$$

which yields

(1.24) $$\operatorname*{Sup}_{\mu \in M} [-\mathscr{S}(\mu)] = \operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] + (f, G)_0 - \tfrac{1}{2}|G|_1^2\,.$$

Setting $\hat{\boldsymbol{q}} = \operatorname{grad} w^0$, we can show that $\hat{\boldsymbol{q}} + \operatorname{grad} G = \lambda^0$. In fact, it follows from (1.6) that

$$-(f, w^0)_0 = -|w^0|_1^2 - (\operatorname{grad} G, \operatorname{grad} w^0) + (g_2, \gamma w^0)_{0,\Gamma}\,;$$

therefore we have

$$\mathscr{L}_1(w^0) = \tfrac{1}{2}|w^0|_1^2 + (\operatorname{grad} G, \operatorname{grad} w^0) - (f, w^0)_0 - (g_2, \gamma w^0)_{0,\Gamma} =$$
$$= -\tfrac{1}{2}\|\hat{\boldsymbol{q}}\|^2 = -\mathscr{S}(\hat{\boldsymbol{q}} + \operatorname{grad} G) - \tfrac{1}{2}|G|_1^2 + (f, G)_0\,.$$

It is readily seen that $\hat{\boldsymbol{q}} + \operatorname{grad} G = \operatorname{grad} u \in \mathscr{U}$; hence

$$\operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] \geqq -\mathscr{S}(\operatorname{grad} u) = \mathscr{L}_1(w^0) + \tfrac{1}{2}|G|_1^2 - (f, G)_0\,.$$

On the other hand, from (1.18) we have

$$\operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] + (f, G)_0 - \tfrac{1}{2}|G|_1^2 = \operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] \leqq \mathscr{L}_1(w^0)\,.$$

Combining these results, we obtain

(1.25) $$\operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] = \mathscr{L}_1(w^0) + \tfrac{1}{2}|G|_1^2 - (f, G)_0 = -\mathscr{S}(\operatorname{grad} u)\,.$$

Thus we arrive at

$$-\mathscr{S}(\lambda^0) = -\operatorname*{Inf}_{\mu \in \mathscr{U}} \mathscr{S}(\mu) = \operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] = -\mathscr{S}(\operatorname{grad} u)\,.$$

Then (1.15) is a consequence of the uniqueness of the solution λ^0.

Finally, from (1.25) we have

$$\mathscr{S}(\lambda^0) + \mathscr{L}(u) + (g_1, g_2)_{0,\Gamma} = \mathscr{S}(\lambda^0) + \mathscr{L}_1(w^0) + \tfrac{1}{2}|G|_1^2 - (f, G)_0 = 0\,,$$

i.e., (1.16) holds. Q.E.D.

2. FINITE ELEMENT APPROXIMATION OF THE PRIMAL PROBLEM

We consider the case $n = 2$. Assume that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and that (1.9) holds.

We divide Ω into triangles T, forming a triangulation $\mathcal{T}_h$ according to the standard finite element method. Let h denote the longest side of all triangles in $\mathcal{T}_h$. Denote by V_h the space of all continuous functions on $\bar{\Omega}$, piecewise linear on $\mathcal{T}_h$. Assume that g_{1h} is the Lagrange linear interpolate of g_1 with the nodes determined by $\mathcal{T}_h$ on Γ. We introduce

$$K_h = \{v_h \in V_h | \; v_h \geqq g_{1h} \text{ on } \Gamma\} .$$

In general, $K_h \not\subset K$.

We say that $u_h \in K_h$ is a finite element approximation of the solution u if

$$\mathscr{L}(u_h) \leqq \mathscr{L}(v_h) \quad \forall v_h \in K_h . \tag{2.1}$$

In the same way as in the proof of Lemma 1.2 (instead of G we use the Lagrange linear interpolate G_h of G) it is possible to show that (2.1) has a unique solution.

To estimate the distance between u and u_h it is useful to prove the following modification of a Falk's result [4].

Lemma 2.1. *We have*

$$\begin{aligned} |u - u_h|_1^2 \leqq (f, u - v_h)_0 + (f, u_h - v)_0 + (g_2, \gamma u - \gamma v_h)_{0,\Gamma} + \\ + (g_2, \gamma u_h - \gamma v)_0 + (\operatorname{grad} u, \operatorname{grad}(v - u_h)) + \\ + (\operatorname{grad} u, \operatorname{grad}(v_h - u)) + (\operatorname{grad}(u_h - u), \operatorname{grad}(v_h - u)) \end{aligned} \tag{2.2}$$

for each $v \in K$ and $v_h \in K_h$.

Proof. From (1.4) we obtain

$$|u|_1^2 \leqq (f, u - v)_0 + (g_2, \gamma u - \gamma v)_{0,\Gamma} + (\operatorname{grad} u, \operatorname{grad} v) \quad \forall v \in K .$$

Moreover, using (2.1) we can show that

$$|u_h|_1^2 \leqq (f, u_h - v_h)_0 + (g_2, \gamma u_h - \gamma v_h)_{0,\Gamma} + (\operatorname{grad} u_h, \operatorname{grad} v_h) \quad \forall v_h \in K_h .$$

Thus for any $v \in K$, $v_h \in K_h$ we obtain

$$\begin{aligned} |u - u_h|_1^2 = |u|_1^2 + |u_h|_1^2 - 2(\operatorname{grad} u_h, \operatorname{grad} u) \leqq \\ \leqq (f, u - v)_0 + (f, u_h - v_h)_0 + (g_2, \gamma u - \gamma v)_{0,\Gamma} + (g_2, \gamma u_h - \gamma v)_{0,\Gamma} + \\ + (\operatorname{grad} u, \operatorname{grad} v) + (\operatorname{grad} u_h, \operatorname{grad} v_h) - 2(\operatorname{grad} u, \operatorname{grad} u_h) = \\ = (f, u - v_h)_0 + (f, u_h - v)_0 + (g_2, \gamma u - \gamma v_h)_{0,\Gamma} + (g_2, \gamma u_h - \gamma v)_{0,\Gamma} + \\ + (\operatorname{grad} u, \operatorname{grad}(v_h - u)) + (\operatorname{grad} u, \operatorname{grad}(v - u_h)) + \\ + (\operatorname{grad}(u_h - u), \operatorname{grad}(v_h - u)) . \quad \text{Q.E.D.} \end{aligned}$$

Theorem 2.1. *Let* $u \in H^2(\Omega)$ *and* $\gamma u \in H^2(\Gamma_m)$ *for each side* Γ_m, $m = 1, \ldots, N$, *of the polygonal boundary* Γ, $g_1 \in H^2(\Gamma_m)$, $m = 1, \ldots, N$. *Then*

$$(2.3) \quad |u - u_h|_1 \leqq Ch\{\|u\|_2^2 + (\|g_2\|_{0,\Gamma} + \|u\|_2) \sum_{m=1}^{N} (\|u\|_{2,\Gamma_m} + \|g_1\|_{2,\Gamma_m})\}^{1/2},$$

where C is independent of h.

Proof. Integrating by parts, we obtain

$$(\text{grad}\, u, \text{grad}\,(v_h - u)) = \int_\Gamma \partial u/\partial n(\gamma v_h - \gamma u)\, dS - (f, u - v_h)_0,$$

$$(\text{grad}\, u, \text{grad}\,(v - u_h)) = \int_\Gamma \partial u/\partial n(\gamma v - \gamma u_h)\, dS - (f, u_h - v)_0.$$

Then (2.2) implies that

$$(2.4) \quad |u - u_h|_1^2 \leqq \tfrac{1}{2}|u - u_h|_1^2 + \tfrac{1}{2}|v_h - u|_1^2 + \|\partial u/\partial n\|_{0,\Gamma} \|v_h - u\|_{0,\Gamma} +$$
$$+ \|g_2\|_{0,\Gamma} \|v_h - u\|_{0,\Gamma} + \int_\Gamma (\partial u/\partial n - g_2)(\gamma v - \gamma u_h)\, dS.$$

We take $v_h = u_I$, i.e. the linear Lagrange interpolate of u with the nodes of $\mathscr{T}_h$. Then $u_I \in K_h$ and

$$(2.5) \quad |u - u_I|_1 = ch\|u\|_2, \quad \|u - u_I\|_{0,\Gamma_m} \leqq Ch^2\|u\|_{2,\Gamma_m},$$
$$\|\partial u/\partial n\|_{0,\Gamma} \leqq C\|u\|_2.$$

To estimate the integral

$$I = \int_\Gamma (\partial u/\partial n - g_2)(\gamma v - \partial\gamma u_h)\, dS,$$

we choose a function w defined on Γ by

$$w = \sup\{g_1, u_h\}.$$

Since $w \in H^{1/2}(\Gamma)$, there exists a function $v^0 \in H^1(\Omega)$ such that $\gamma v^0 = w$ on Γ. Moreover, $v^0 \in K$ by definition. Thus we have

$$\int_\Gamma (\partial u/\partial n - g_2)(\gamma v^0 - \gamma u_h)\, dS = \int_\Gamma (\partial u/\partial n - g_2)(w - \gamma u_h)\, dS =$$
$$= \int_{\Gamma_-} (\partial u/\partial n - g_2)(g_1 - \gamma u_h)\, dS,$$

where

$$\Gamma^- = \{x \in \Gamma \mid u_h(x) < g_1(x)\}.$$

The inequalities $\gamma u_h \geqq g_{1h}$ and $\partial u/\partial n - g_2 \geqq 0$ on Γ imply

$$(2.6) \quad \int_{\Gamma_-} (\partial u/\partial n - g_2)(g_1 - \gamma u_h)\, dS \leqq \int_{\Gamma_-} (\partial u/\partial n - g_2)(g_1 - g_{1h})\, dS \leqq$$

$$\leqq \|\partial u/\partial n - g_2\|_{0,\Gamma} \|g_1 - g_{1h}\|_{0,\Gamma} \leqq C(\|u\|_2 + \|g_2\|_{0,\Gamma})\, h^2 \sum_{m=1}^{N} \|g_1\|_{2,\Gamma_m}.$$

Combining (2.4), (2.5) with (2.6) we come to the assertion of the theorem. Q.E.D.

In Theorem 2.1 we have assumed very strong regularity of the solution u. In general, such regularity is unrealistic. In the sequel we shall show the convergence of u_h to u without any assumption on regularity. First of all we prove a general abstract result.

Theorem 2.2. *Let V be a Hilbert space with a norm $\|u\|$ and a seminorm $|u|$. Let $h \in (0, 1]$ be a real parameter. Assume that K and $K_h \subset V$ are closed and convex subsets of V for each h. Let F be a functional on V, differentiable and coercive on K and K_h. Moreover, assume that the second differential of F (in the sense of Gâteau) exists and satisfies the relation*

$$(2.7) \qquad \alpha_0 |z|^2 \leqq D^2F(u; z, z) \leqq C\|z\|^2 \quad \forall u, z \in V,$$

where α_0, C are positive constants. Assume that from $v_h \in K_h$ and $\|v_h\| \to +\infty$, $F(v_h) \to +\infty$ follows. Denote by u and u_h the minimizing elements of F on K and K_h, respectively, and let them be unique. Assume that there exist elements $v_h \in K_h$ such that

$$(2.8) \qquad \|u - v_h\| \to 0 \quad \text{for} \quad h \to 0\,;$$

$$(2.9) \qquad \text{if} \quad v_h \in K_h\,, \quad v_h \rightharpoonup v \quad (\text{weakly}) \text{ in } V, \quad \text{then} \quad v \in K\,.$$

Then

$$|u - u_h| \to 0\,,$$

$$u_h \rightharpoonup u \quad (\text{weakly}) \text{ in } V$$

holds for $h \to 0$.

Proof. From (2.7) and the coercivity of F, the existence of minimizing elements u and u_h follows. Let $v_h \in K_h$ satisfy (2.8). Using Taylor's Theorem (see e.g. [9], chapt. 2) we obtain

$$F(v_h) = F(u) + DF(u, v_h - u) + \tfrac{1}{2}D^2F(u + \vartheta_h(v_h - u); v_h - u, v_h - u)\,,$$

where $\vartheta_h \in (0,1)$ for each h. Making use of (2.7), we arrive at

$$(2.10) \qquad \lim_{h \to 0} F(v_h) = F(u)\,.$$

From the definition of u_h we obtain

$$F(u_h) \leqq F(v_h)\,,$$

and therefore

$$F(u_h) \leqq C_1 = \text{const.} < +\infty \quad \forall h\,.$$

Then the coerciveness of F implies the boundedness of the norms $\|u_h\|$ for all h. Consequently, there exist a subsequence $\{u_{h'}\}$ and an element $u^* \in V$ such that $u_{h'} \rightharpoonup u^*$ in V, and then $u \in K$ follows from (2.9). Since F is convex and differentiable, it is lower weakly semicontinuous. We have

$$(2.11) \qquad F(u^*) \leqq \liminf_{h' \to 0} F(u_{h'}) \leqq \limsup_{h' \to 0} F(u_{h'}) \leqq \lim_{h' \to 0} F(v_{h'}) = F(u)\,.$$

Since $u^* \in K$, by definition of u we obtain $F(u) \leqq F(u^*)$. The uniqueness of u implies $u^* = u$ and

$$(2.12) \qquad u_h \rightharpoonup u \ \text{(weakly) in } V.$$

From the identity

$$F(u_h) = F(u) + DF(u, u_h - u) + \tfrac{1}{2}D^2F(u + \gamma_h(u_h - u), u_h - u, u_h - u)\,,$$

where $\gamma_h \in (0, 1)$, we have

$$\tfrac{1}{2}\alpha_0|u_h - u|^2 \leqq F(u_h) - F(u) - DF(u, u_h - u)\,.$$

Then $|u_h - u| \to 0$ for $h \to 0$ follows from (2.12) and (2.11). Q.E.D.

Next, we have to recall some results from [6].

Lemma 2.2. *Let us denote*

$$K_0 = \{v \in H^1(\Omega) |\ \gamma v \geqq 0 \ \text{on } \Gamma\}\,.$$

Then K_0 is the closure of the set

$$\mathscr{E}_+(\Omega) = \{v \in C^\infty(\bar{\Omega}) |\ \gamma v \geqq 0 \ \text{on } \Gamma\}$$

in the space $H^1(\Omega)$.

Lemma 2.3. *Let φ be a continuous function defined in the interval $[a, b]$ $(-\infty < a < b < +\infty)$. Let D_n: $a = x_0^n < x_1^n < \ldots < x_n^n = b$ be a partition of $[a, b]$, $\nu(D_n) = \max_{i=1,\ldots,n} |x_i^n - x_{i-1}^n| \to 0$ for $n \to \infty$. Let $\{\psi_n\}$, $n = 1, 2, \ldots$ be a sequence of continuous linear functions with the nodes x_i^n such that $\psi_n(x_i^n) \geqq \varphi(x_i^n)$ $\forall i = 0, 1, \ldots, n$; $n = 1, 2, \ldots$. Assume that $\psi_n \to \psi$ a.e. in $[a, b]$. Then $\psi \geqq \varphi$ a.e. in $[a, b]$.*

Now we are able to prove convergence without any regularity assumption.

Theorem 2.3. *Let (1.9) hold and let there exist a function $G \in H^{1+\varepsilon}(\Omega)$ such that $\gamma G = g_1$ on Γ, $\varepsilon > 0$. Then*

$$\|u - u_h\|_1 \to 0 \quad \textit{for} \quad h \to 0\,.$$

Proof. It is possible to apply Theorem 2.2 with $F = \mathscr{L}$, $V = H^1(\Omega)$, K and K_h defined as in Section 1. It is readily seen that (2.7) holds with $\alpha_0 = C = 1$. Let $v_h \in K_h$, $\|v_h\|_1 \to \infty$. Since $H^{1+\varepsilon}(\Omega) \subset C(\bar{\Omega})$, G is continuous in $\bar{\Omega}$ and $g_1 \in C(\Gamma)$. Denote the linear Lagrange interpolate of G on $\mathscr{T}_h$ by G_h^I. Then $\|G - G_h^I\|_1 \to 0$ for $h \to 0$ and $\|G_h^I\|_1 \leqq C_2 = \text{const}$ for all $h \in (0, 1]$. Setting $w_h = v_h - G_h^I$, we deduce $\|w_h\|_1 \to +\infty$ for $h \to 0$.

Let $\Gamma_0 \subset \Gamma$ be an arbitrary open set of positive measure. We define $\bar{w}_h = (\text{mes}\,\Gamma_0)^{-1} (\gamma w_h, 1)_{0,\Gamma_0}$. Then $\tilde{w}_h = w_h - \bar{w}_h$ satisfies $(\gamma \tilde{w}_h, 1)_{0,\Gamma_0} = 0$. Applying the generalized Friedrichs Theorem, we obtain $C_3\|\tilde{w}_h\|_1 \leqq |\tilde{w}_h|_1$. Taking into account the boundedness of $\|G_h^I\|$, we may write

$$\begin{aligned}(2.13)\qquad \mathscr{L}(v_h) &= \mathscr{L}_1(w_h) + \mathscr{L}(G_h^I) \geqq \\ &\geqq C_4\|\tilde{w}_h\|_1^2 - C_5\|\tilde{w}_h\|_1 - \bar{w}_h[(f, 1)_0 + (g_1, 1)_{0,\Gamma}] + C_6\,.\end{aligned}$$

Since $\|w_h\|_1 \to \infty$, at least one of the norms $\|\tilde{w}_h\|_1$ and $\|\bar{w}_h\|_1 = \bar{w}_h(\text{mes}\,\Omega)^{1/2}$ must tend to infinity. Therefore, (2.13) and (1.9) imply $\mathscr{L}(v_h) \to +\infty$ for $h \to 0$.

Next, we have to verify the conditions (2.8) and (2.9). We start with (2.8). We know that $u - G \in H^1(\Omega)$ and $\gamma(u - G) = \gamma u - g_1 \geqq 0$ on Γ, so that $u - G \in K_0$. There exist $\varphi_k \in \mathscr{E}_+(\bar{\Omega})$ such that $\varphi_k \to u - G$ in $H^1(\Omega)$ for $k \to \infty$. Let us dcnote by φ_{kh}^I the Lagrange linear interpolate of φ_k on $\mathscr{T}_h$. We have

$$\|\varphi_k - \varphi_{kh}^I\|_1 \to 0 \quad \text{for} \quad h \to 0\,.$$

Setting $v_h = G_h^I + \varphi_{kh}^I$, we obtain $v_h \in K_h$ and $\|v_h - u\|_1 \to 0$ for $h \to 0$, i.e., (2.8) holds.

Let $v_h \in K_h$ and $v_h \rightharpoonup v$ (weakly) in $H^1(\Omega)$. Then $v_h \to v$ (strongly) in $L^2(\Gamma)$, since the mapping $\gamma\colon H^1(\Omega) \to L^2(\Gamma)$ is completely continuous. Here it is possible to choose a subsequence $\{v_h'\}$ such that $v_h' \to v$ a.e. on Γ. Then $v \in K$ by virtue of Lemma 2.3.

All assumptions of Theorem 2.2 are satisfied. Consequently,

$$(2.14)\qquad |u_h - u|_1 \to 0\,, \quad u_h \rightharpoonup u \text{ (weakly) in } H^1(\Omega) \text{ for } h \to 0\,.$$

Making use of Rellich's Theorem we deduce that $u_h \to u$ (strongly) in $L_2(\Omega)$. Combining this with (2.14), we arrive at the assertion of Theorem 2.3. Q.E.D.

Up to now, the primal problem has been approximated only on polygonal domains and $O(h)$ convergence has been proved provided the solution u is regular enough. The same rate of convergence, however, can be obtained with somewhat weaker regularity for another class of domains. To this end, we consider a convex domain $\Omega \subset \mathbb{R}^2$ whose boundary has Lipschitz continuous derivatives with respect to the arc parameter (the domain Ω may be a polygonal domain as well). For any $h \in (0, 1\rangle$ we denote by Ω^h the polygonal domain inscribed in Ω, whose sides are either less than h or fully contained in Γ. We triangulate Ω^h according to the standard finite element method. We denote this triangulation by $\mathscr{T}(\Omega^h)$. For each $\mathscr{T}(\Omega^h)$ we consider the space V_h of all continuous piecewise linear functions on Ω^h with the nodes of $\mathcal{T}(\Omega^h)$. If $v_h^* \in V_h$, we construct function v_h defined on Ω as follows:

$$v_h = v_h^* \quad \text{on} \quad \Omega^h ,$$

$$v_h(Q) = v_h^*(P) ,$$

where P is the projection of Q onto $\partial\Omega^h$ in the direction normal to $\partial\Omega^h$ if $Q \in S^h = \Omega \dot{-} \Omega^h$; i.e., v_h is defined on S^h as the constant extension in the direction normal to $\partial\Omega^h$.

We define V_h' as the set of all such functions v_h and $K_h' = \{v_h \in V_h' \mid v_h(a_i) \geqq g_1(a_i)$ for each node a_i on $\Gamma\}$. We say that $u_h' \in K_h'$ is an approximation of the primal problem (1.3) if

$$\mathcal{L}(u_h') \leqq \mathcal{L}(v_h) \quad \forall v_h \in K_h' . \tag{2.15}$$

The proof of existence of a solution of (2.15) is quite analogous to that of Lemma 1.2.

Next, we follow the procedure suggested in [5] to show the rate of convergence of u_h' to u.

Theorem 2.4. *Let Ω be a convex domain whose boundary has Lipschitz continuous derivatives with respect to the arc parameter. Assume that g_1 can be extended onto a neighbourhood of Γ, $u \in H^2(\Omega)$, $g_1, u \in W^{1,\infty}$ in a neighbourhood of Γ, g_2, $\partial u/\partial n \in L^\infty(\Gamma)$, and the number of points where the changes from $u = g_1$ to $u \geqq g_1$ occur is finite. Then*

$$|u - u_h'| = O(h) .$$

Proof. Setting $a(u, v) = (\operatorname{grad} u, \operatorname{grad} v)$, we may write

$$\begin{aligned} a(u - u_h', u - u_h') &= a(u - u_h', u - v_h) + a(u - u_h', v_h - u_h') \leqq \\ &\leqq a(u - u_h', u - v_h) + a(u, v_h - u_h') - a(u_h', v_h - u_h') + \\ &+ (f, v_h - u_h')_0 + (g_2, v_h - u_h')_{0,\Gamma} - (f, v_h - u_h')_0 - (g_2, v_h - u_h')_{0,\Gamma} \leqq \\ &\leqq a(u, v_h - u_h') + a(u - u_h', u - v_h) - (f, v_h - u_h')_0 - (g_2, v_h - u_h')_{0,\Gamma} , \end{aligned}$$

since u_h' is a solution of (2.15), i.e.,

$$a(u_h', v_h - u_h') \geqq (f, v_h - u_h')_0 + (g_2, v_h - u_h')_{0,\Gamma}$$

holds for each $v_h \in K_h'$. From this and the fact that

$$a(u, v_h - u_h') = (f, v_h - u_h')_0 + \int_\Gamma \partial u/\partial n(v_h - u_h') \, \mathrm{d}S$$

we find out that

$$a(u - u_h', u - u_h') \leqq a(u - u_h', u - v_h) + \int_\Gamma \partial u/\partial n(v_h - u_h') \, \mathrm{d}S \tag{2.16}$$

holds for all $v_h \in K_h'$.

By $u_I' \in K_h'$ we denote the function such that $u_I' = u_I$ on Ω^h, where u_I is the linear Lagrange interpolate of u on $\mathcal{T}(\Omega^h)$, and such that u_I' is defined as a constant extension of u_I in the direction normal to $\partial\Omega^h$. In (2.16) we choose $v_h = u_I'$. We consider two sets, namely

$$\Gamma_0 = \{x \in \Gamma \mid u(x) = g_1(x)\} ,$$
$$\Gamma_1 = \{x \in \Gamma \mid u(x) > g_1(x)\} .$$

By Γ_0^h and Γ_1^h we denote the sets of all sides $S^h \subset \partial\Omega^h$ for which the corresponding arcs $\hat{S}^h \subset \Gamma$ are parts of Γ_0 and Γ_1, respectively. If $S^h \subset \Gamma_0^h$, then because of $\partial u/\partial n - g_2 = 0$ on $\hat{S}^h$ we have

$$(\partial u/\partial n - g_2, u_I' - u_h)_{0,\hat{S}^h} = 0 .$$

If $S^h \subset \Gamma_1^h$, then $\partial u/\partial n - g_2 \geqq 0$ and $u_I' = g_{1h}^I$ on S^h, where g_{1h}^I is the linear Lagrange interpolate of g_1 on $\partial\Omega^h$ extended by a constant in the direction normal to $\partial\Omega^h$. Thus $g_{1h}^I \leqq u_h'$ implies that

$$(\partial u/\partial n - g_2, u_I' - u_h')_{0,\hat{S}^h} = (\partial u/\partial n - g_2, g_{1h}^I - u_h')_{0,\hat{S}^h} \leqq 0 .$$

Since the number of transition points is finite, it is possible to consider h so small that on each $\hat{S}^h$ there is only one such point. Assume that there exists S^h such that $\text{mes}\,(\Gamma_i \cap \hat{S}^h) > 0$, $i = 0, 1$. Then we can find one point Q on $\hat{S}^h$ such that $u(Q) = g_1(Q)$, and Q divides the arc $\hat{S}^h$ into two parts $\hat{S}_0^h$ and $\hat{S}_1^h$ such that $u = g_1$ on $\hat{S}_0^h$ and $u > g_1$ on $\hat{S}_1^h$. Hence $\partial u/\partial n - g_2 \geqq 0$ on $\hat{S}_0^h$ and $\partial u/\partial n - g_2 = 0$ on $\hat{S}_1^h$. Because of $u(Q) = g_1(Q)$, and $u, g_1 \in W^{1,\infty}$ in a neighbourhood of Γ, we have $u_I' - g_{1h}^I = O(h)$ on $\hat{S}^h$. Thus we obtain

$$(\partial u/\partial n - g_2, u_I' - u_h')_{0,\hat{S}^h} = (\partial u/\partial n - g_2, u_I' - u_h')_{0,\hat{S}_0^h} =$$
$$= (\partial u/\partial n - g_2, u_I' - g_{1h}^I)_{0,\hat{S}_0^h} + (\partial u/\partial n - g_2, g_{1h}^I - u_h')_{0,\hat{S}_0^h} \leqq$$
$$\leqq (\partial u/\partial n - g_2, g_{1h}^I - u_h')_{0,\hat{S}_0^h} = O(h^2) .$$

From the above obtained results we deduce that

$$(2.17) \quad (\partial u/\partial n - g_2, u_I' - u_h') = \sum_{S^h \subset \partial\Omega^h} (\partial u/\partial n - g_2, u_I' - u_h')_{0,\hat{S}^h} = O(h^2) .$$

By the properties of the interpolation we have

$$\|u - u_I'\|_{1,\Omega^h} = O(h) .$$

Since $\text{mes}\,(S^h) = O(h)$, $\nabla u \in L^\infty(S^h)$, $\nabla u_I'$ is bounded on S^h independently of h, we obtain (see [5], Lemma 6.1)

$$\|u - u_I'\|_{1,S^h} = O(h) .$$

Finally, we arrive at

$$(2.18) \quad \|u - u_I'\|_{1,\Omega} = O(h) .$$

Then the assertion of the theorem follows from (2.16), (2.17), (2.18) and the fact that

$$a(u - u'_h, u - u'_I) \leqq |u - u'_h|_1 \, |u - u'_I|_1 \leqq \tfrac{1}{2}|u - u'_h|_1^2 + \tfrac{1}{2}|u - u'_I|_1^2$$

and

$$a(u - u'_h, u - u'_h) = |u - u'_h|_1^2 .$$

Remark 2.1. In the same way as in the proof of Theorem 2.3, an analogous result can be proved.

3. APPROXIMATION OF THE DUAL PROBLEM

We introduce an equivalent formulation of the dual problem (1.14). To this end, we construct a function $\lambda^f \in Q$ such that

$$\operatorname{div} \lambda^f + f = 0 \quad \text{in} \quad \Omega ,$$

$$\lambda^f \,.\, \boldsymbol{n} = g_2 - F \quad \text{on} \quad \Gamma ,$$

where

$$F = (\operatorname{mes} \Gamma)^{-1} \left[(f, 1)_0 + (g_2, 1)_{0,\Gamma}\right]$$

(F is a non-positive constant). Such a function can be defined as follows: $\lambda^f = \operatorname{grad} w$, where w is a solution of the problem

$$-\Delta w = f \text{ in } \Omega , \quad \partial w / \partial n = g_2 - F \text{ on } \Gamma .$$

(The solution exists by virtue of the equation

$$(f, 1)_0 + (g_2 - F, 1)_{0,\Gamma} = 0.)$$

It is clear that the problem: to find

$$\boldsymbol{q}^0 \in \mathscr{U}_0 = \{\boldsymbol{q} \in Q | \operatorname{div} \boldsymbol{q} = 0 \text{ in } \Omega , \quad (\boldsymbol{q} + \lambda^f) \,.\, \boldsymbol{n}|_\Gamma \geqq g_2 \text{ on } \Gamma\}$$

such that

(3.1) $$J(\boldsymbol{q}^0) \leqq J(\lambda) \quad \forall \lambda \in \mathscr{U}_0 ,$$

where

$$J(\boldsymbol{q}) = \tfrac{1}{2}\|\boldsymbol{q}\|^2 + (\lambda^f, \boldsymbol{q}) - \langle \boldsymbol{q} \,.\, \boldsymbol{n}, g_1 \rangle ,$$

is equivalent to the dual problem (1.14).

The solutions satisfy the relation $\lambda^0 = \boldsymbol{q}^0 + \lambda^f$. Let us introduce the convex set

$$\mathscr{U}_0^h = \{\boldsymbol{q} | \boldsymbol{q} \in \mathscr{N}_h, \ \boldsymbol{q} \,.\, \boldsymbol{n}|_\Gamma \geqq F_0\} = \mathscr{U}_0 \cap \mathscr{N}_h ,$$

where $\mathscr{N}_h \subset Q$ is a subspace of the space of piecewise linear divergence-free vector-functions (see [3]).

We say that $\lambda^f + \mathbf{q}^h$, $\mathbf{q}^h \in \mathscr{U}_0^h$ is an approximation of (1.14) if

$$J(\mathbf{q}^h) \leqq J(\mathbf{q}) \quad \forall \mathbf{q} \in \mathscr{U}_0^h . \tag{3.2}$$

The problem (3.2) has a unique solution, because $\mathscr{U}_0^h$ is convex and nonempty, $J(\mathbf{q})$ is continuously differentiable and strictly convex on $[L_2(\Omega)]^2$.

To estimate the distance between $\lambda^h = \lambda^f + \mathbf{q}^h$ and λ^0, it is suitable to recall some results of [1], namely:

Lemma 3.1. *Let there exist* $\mathbf{W}^h \in \mathscr{U}_0^h$ *such that* $2\mathbf{q}^0 - \mathbf{W}^h \in \mathscr{U}_0$. *Then*

$$\|\mathbf{q}^0 - \mathbf{W}^h\| \geqq \|\mathbf{q}^0 - \mathbf{q}^h\| . \tag{3.3}$$

Proof. Making use of Lemma 2.1 in [7], where $B = \{\mathbf{q} \in Q \mid \operatorname{div} \mathbf{q} = 0 \text{ in } \Omega\}$, $\mathscr{F} = J$, $M = \mathscr{U}_0$, $M_h = \mathscr{U}_0^h$, $\alpha_0 = c = 1$, the assertion (3.3) follows.

Lemma 3.2. *Let* $\mathbf{q}^0 \in [H^2(\Omega)]^2$, $\mathbf{q}^0 . \mathbf{n} \in H^2(\Gamma_m)$ *for each side* Γ_m *of the polynomial boundary* Γ, $m = 1, \ldots, N$. *Then for* h *sufficiently small there exists a piecewise linear function* ψ_h *on* Γ *with the nodes defined by* $\mathscr{T}_h$, *such that*

$$\int_\Gamma \gamma\psi_h \, \mathrm{d}S = \int_\Gamma \mathbf{q}^0 . \mathbf{n} \, \mathrm{d}S = 0 , \tag{3.4}$$

$$F_0 \leqq \gamma\psi_h \leqq 2\mathbf{q}^0 . \mathbf{n} - F_0 \quad \text{on} \quad \Gamma , \tag{3.5}$$

$$\|\gamma\psi_h - (r_h\mathbf{q}^0) . \mathbf{n}\|_{0,\Gamma} \leqq Ch^2 \sum_{m=1}^{N} |\mathbf{q}^0 . \mathbf{n}|_{2,\Gamma_m} , \tag{3.6}$$

where $r_h\mathbf{q}^0$ *is the projection of* $\mathbf{q}^0$ *onto* $\mathscr{N}_h$ *(see* [6, 7]*), and* $|\cdot|_{2,\Gamma_m}$ *denotes the semi-norm generated by the second derivatives.*

Proof is analogous to that of Lemma 4.2 in [1].

Definition. We say that a *system* $\{\mathscr{T}_h\}$, $h \to 0$ *of triangulations of the domain* Ω *is* $\alpha - \beta$*-regular,* if there exist positive α and β independent of h and such that (i) the minimal angle of all triangles in $\mathscr{T}_h$ is not less than α for any h and (ii) the ratio between the lengths of any two sides of $\mathscr{T}_h$ is less than β.

Theorem 3.1. *Assume that* Ω *is a polygonal bounded domain, and* (1.9) *and the assumptions of Lemma* 3.2 *hold. Denote* $\lambda^0 = \mathbf{q}^0 + \lambda^f$, $\lambda^h = \mathbf{q}^h + \lambda^f$, *where* λ^f *is constructed as above,* $\mathbf{q}^h$ *and* $\mathbf{q}^0$ *are solutions of* (3.2) *and* (3.1), *respectively,* λ^0 *is the solution of* (1.14). *Then for any* $\alpha - \beta$*-regular system of triangulations*

$$\|\lambda^h - \lambda^0\| = \|\mathbf{q}^h - \mathbf{q}^0\| \leqq Ch^{3/2}\{|\mathbf{q}^0|_{2,\Omega} + \sum_{m=1}^{N} |\mathbf{q}^0 . \mathbf{n}|_{2,\Gamma_m}\} , \tag{3.7}$$

holds, where $|\cdot|_{2,\Omega}$ *is the semi-norm generated by the second derivatives.*

Proof. Let ψ_h be the function from Lemma 3.2. We set

$$\varphi = (r_h\mathbf{q}^0) . \mathbf{n} - \psi_h .$$

The identity

$$\int_\Gamma \varphi \, \mathrm{d}s = 0$$

implies that there exists a function $\mathbf{w}_h \in \mathcal{N}_h$ such that

$$\mathbf{w}_h \, . \, \mathbf{n} = \varphi \quad \text{on} \quad \Gamma \,, \tag{3.8}$$

$$\|\mathbf{w}_h\| \leqq Ch^{-1/2} \|\varphi\|_{0,\Gamma} \tag{3.9}$$

(see [7], Lemma 5.3). Here we have used the relation

$$\int_\Gamma [\mathbf{q}^0 \, . \, \mathbf{n} - (r_h \mathbf{q}^0) \, . \, \mathbf{n}] \, \mathrm{d}S = 0$$

(see [3]). The function $\mathbf{W}^h = r_h \mathbf{q}^0 - \mathbf{w}_h$ satisfies the assumptions of Lemma 3.1. In fact, $\mathbf{W}^h \in N_h$ and

$$\mathbf{W}^h \, . \, \mathbf{n} = (r_h \mathbf{q}^0) \, . \, \mathbf{n} - \varphi = \psi_h \geqq F_0 \quad \text{on} \quad \Gamma \,,$$

i.e., $\mathbf{W}^h \in \mathcal{U}_0^h$. Moreover, the inequality

$$\mathbf{W}^h \, . \, \mathbf{n} \leqq 2\mathbf{q}^0 \, . \, \mathbf{n} - F_0$$

implies $2\mathbf{q}^0 - \mathbf{W}^h \in \mathcal{U}_0$.

Making use of the estimate (see [3], Theorem 3.1)

$$\|\mathbf{q} - r_h \mathbf{q}\| \leqq Ch^2 |\mathbf{q}|_{2,\Omega} \quad \forall \mathbf{q} \in [H^2(\Omega)]^2$$

and of (3.8), (3.6), we arrive at

$$\|\mathbf{q}^0 - \mathbf{W}^h\| \leqq \|\mathbf{q}^0 - r_h \mathbf{q}^0\| + \|r_h \mathbf{q}^0 - \mathbf{W}^h\| \leqq Ch^2 |\mathbf{q}^0|_{2,\Omega} + \|\mathbf{w}_h\| \leqq$$
$$\leqq Ch^2 |\mathbf{q}^0|_{2,\Omega} + C_1 h^{3/2} \sum_{m=1}^{N} |\mathbf{q}^0 \, . \, \mathbf{n}|_{2,\Gamma_m} \, .$$

The assertion of the theorem follows from Lemma 3.1 and the fact that $\lambda^0 - \lambda^h = \mathbf{q}^0 - \mathbf{q}^h$.

Remark 3.1. Using the results of [8], we can show the convergence of the dual approximations without any assumptions about the regularity. Namely, let Ω be a convex polygonal domain such that the sum of any two neighbouring internal angles is not less than π. Assume that $\lambda^f \in [H^1(\Omega)]^2$ and (1.9) holds. Then for an arbitrary $\alpha - \beta$-regular system of triangulations, $\|\lambda^h - \lambda^0\| \to 0$ holds for $h \to 0$. The proof is parallel to that in [8].

4. A POSTERIORI ERROR ESTIMATES AND TWO-SIDED ESTIMATES OF ENERGY

It is readily seen from (1.4) that for all $v \in K$ we may write

$$2[\mathcal{L}(v) - \mathcal{L}(u)] = |v|_1^2 - |u|_1^2 - 2(f, v - u)_0 - 2(g_2, v - u)_{0,\Gamma} \geqq \tag{4.1}$$
$$\geqq |v|_1^2 - |u|_1^2 - 2(\operatorname{grad} u, \operatorname{grad}(v - u)) = |v - u|_1^2 \, .$$

Making use of (1.16), for all $\lambda \in \mathscr{U}$ we obtain

$$-\mathscr{L}(u) = (g_1, g_2)_{0,\Gamma} + \mathscr{S}(\lambda^0) \leqq \mathscr{S}(\lambda) + (g_1, g_2)_{0,\Gamma} . \tag{4.2}$$

Theorem 4.1. *Let $\tilde{u}_h$ be an arbitrary approximation of the primal problem such that $\tilde{u}_h \in K$ and let $\tilde{\lambda}^h = \lambda^f + {}^{\sim}\mathbf{q}^h$, where ${}^{\sim}\mathbf{q}^h \in \mathscr{U}_0^h$ is an arbitrary approximation of the dual problem. Then*

$$|\tilde{u}_h - u|_1^2 \leqq \|\tilde{\lambda}^h - \operatorname{grad} \tilde{u}_h\|^2 + 2\int_\Gamma (\tilde{\lambda}^h \cdot \mathbf{n} - g_2)(\tilde{u}_h - g_1) \, \mathrm{d}S = E(\tilde{u}_h, \tilde{\lambda}^h) , \tag{4.3}$$

$$\|\tilde{\lambda}^h - \operatorname{grad} u\|^2 \leqq E(\tilde{u}_h, \tilde{\lambda}^h) . \tag{4.4}$$

Proof. From (4.1) and (4.2) we deduce that

$$|u_h - u|_1^2 \leqq 2[\mathscr{L}(\tilde{u}_h) + \mathscr{S}(\tilde{\lambda}^h) + (g_1, g_2)_{0,\Gamma}] =$$
$$= |\tilde{u}_h|_1^2 - 2(f, \tilde{u}_h)_0 - 2(g_2, \gamma\tilde{u}_h)_{0,\Gamma} + \|\tilde{\lambda}^h\|^2 - 2\langle\tilde{\lambda}^h \cdot \mathbf{n}, g_1\rangle + 2(g_1, g_2)_{0,\Gamma} =$$
$$= \|\tilde{\lambda}^h - \operatorname{grad} \tilde{u}_h\|^2 + 2(\tilde{\lambda}^h, \operatorname{grad} \tilde{u}_h) - 2\langle\tilde{\lambda}^h \cdot \mathbf{n}, g_1\rangle -$$
$$- 2(f, \tilde{u}_h)_0 - 2(\gamma\tilde{u}_h - g_1, g_2)_{0,\Gamma} .$$

On the other hand, we may write

$$(\tilde{\lambda}^h, \operatorname{grad} \tilde{u}_h) - (f, \tilde{u}_h)_0 = (\tilde{\lambda}^h \cdot \mathbf{n}, \tilde{u}_h)_{0,\Gamma} .$$

Thus we arrive at

$$|\tilde{u}_h - u|_1^2 \leqq \|\tilde{\lambda}^h - \operatorname{grad} \tilde{u}_h\|^2 + 2\int_\Gamma (\tilde{\lambda}^h \cdot \mathbf{n} - g_2)(\gamma\tilde{u}_h - g_1) \, \mathrm{d}S ,$$

i.e., (4.3) holds.

We know that the solution (1.14) satisfies the condition

$$(\lambda^0, \lambda - \lambda^0) - \langle(\lambda - \lambda^0) \cdot \mathbf{n}, g_1\rangle \geqq 0 \quad \forall \lambda \in \mathscr{U} ,$$

therefore for any $\lambda \in \mathscr{U}$ we have

$$2[\mathscr{S}(\lambda) - \mathscr{S}(\lambda^0)] = \|\lambda\|^2 - \|\lambda^0\|^2 - 2\langle(\lambda - \lambda^0) \cdot \mathbf{n}, g_1\rangle =$$
$$= \|\lambda - \lambda^0\|^2 + 2(\lambda, \lambda^0) - 2\|\lambda^0\|^2 - 2\langle(\lambda - \lambda^0) \cdot \mathbf{n}, g_1\rangle \geqq \|\lambda - \lambda^0\|^2 .$$

Inserting $\lambda = \tilde{\lambda}^h$ and making use of (1.15), (1.16), we obtain

$$\|\tilde{\lambda}^h - \operatorname{grad} u\|^2 \leqq 2[\mathscr{S}(\tilde{\lambda}^h) + \mathscr{L}(u) + (g_1, g_2)_{0,\Gamma}] \leqq$$
$$\leqq 2[\mathscr{S}(\tilde{\lambda}^h) + \mathscr{L}(\tilde{u}_h) + (g_1, g_2)_{0,\Gamma}] \equiv E(\tilde{u}_h, \tilde{\lambda}^h) .$$

Theorem 4.2. *Let $\tilde{u}_h$ and $\tilde{\lambda}^h$ be as in Theorem 4.1. Then for $w^0 = u - G$ the following estimates hold:*

$$2[\mathscr{L}(G) - \mathscr{L}(\tilde{u}_h)] \leqq |w^0|_1^2 \leqq \|\lambda^f + {}^{\sim}\mathbf{q}^h - \operatorname{grad} G\| \equiv F({}^{\sim}\mathbf{q}^h) , \tag{4.5}$$

$$2[\mathscr{L}(G) - \mathscr{L}(\tilde{u}_h)] \leqq (f, w^0)_0 - (\operatorname{grad} G, \operatorname{grad} w^0) + (g_2, \gamma w^0)_{0,\Gamma} \leqq F({}^{\sim}\mathbf{q}^h), \tag{4.6}$$

where ${}^{\sim}\mathbf{q}^h = \tilde{\lambda}^h - \lambda^f$.

Proof. It is readily seen that the problem (1.3) is equivalent to the problem: find $w^0 \in K_0$ such that

$$\mathscr{L}_1(w^0) \leqq \mathscr{L}_1(w) \quad \forall w \in K_0 . \tag{4.7}$$

Then w^0 satisfies the relation

$$(\text{grad}\, w^0, \text{grad}\,(w - w^0)) + (\text{grad}\, G, \text{grad}\,(w - w^0)) - (f, w - w^0)_0 - \\ - (g_2, \gamma w - \gamma w^0)_{0,\Gamma} \geqq 0 \quad \forall w \in K_0 . \tag{4.8}$$

Inserting $w = 0$ and $w = 2w^0$ into (4.8), we arrive at

$$|w^0|_1^2 + (\text{grad}\, G, \text{grad}\, w^0) = (f, w^0)_0 + (g_2, \gamma w^0)_{0,\Gamma} . \tag{4.9}$$

Therefore we have

$$\mathscr{L}(u) = \mathscr{L}_1(w^0) + \mathscr{L}(G) = -\tfrac{1}{2}|w^0|_1^2 + \mathscr{L}(G) , \\ |w^0|_1^2 = 2[\mathscr{L}(G) - \mathscr{L}(u)] \geqq 2[\mathscr{L}(G) - \mathscr{L}(\tilde{u}_h)] . \tag{4.10}$$

On the other hand, we also have

$$|w^0|_1^2 = 2[\mathscr{L}(G) - \mathscr{L}(u)] = 2[\mathscr{L}(G) + \mathscr{S}(\lambda^0) + (g_1, g_2)_{0,\Gamma}] \leqq \\ \leqq 2[\mathscr{L}(G) + \mathscr{S}(\tilde{\lambda}^h) + (g_1, g_2)_{0,\Gamma}] = \\ = |G|_1^2 - 2(f, G)_0 - 2(g_2, g_1)_{0,\Gamma} + \|\tilde{\lambda}^h\|^2 - 2\langle \tilde{\lambda}^h . \mathbf{n}, g_1 \rangle + 2(g_2, g_1)_{0,\Gamma} = \\ = \|\tilde{\lambda}^h - \text{grad}\, G\|^2 + 2(\text{grad}\, G, \tilde{\lambda}^h) - 2(f, G)_0 - 2\langle \tilde{\lambda}^h . \mathbf{n}, g_1 \rangle = \\ = \|\tilde{\lambda}^h - \text{grad}\, G\|^2 = \|\lambda^f + {}^{\sim}\mathbf{q}^h - \text{grad}\, G\|^2 \equiv F({}^{\sim}\mathbf{q}^h) . \tag{4.11}$$

Combining this with (4.10), we arrive at (4.5). Then (4.6) follows from (4.5) and (4.9).

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Souhrn

ANALÝZA PRIMÁRNÍ A DUÁLNÍ VARIAČNÍ FORMULACE SEMI-KOERCIVNÍCH ELIPTICKÝCH ÚLOH S NEHOMOGENNÍMI PŘEKÁŽKAMI NA HRANICI METODOU KONEČNÝCH PRVKŮ

TRAN VAN BON

Práce se zabývá aproximací eliptického problému druhého řádu s nehomogenními jednostrannými okrajovými podmínkami na hranici metodou konečných prvků. Primární variační problém je aproximován po částech lineárními funkcemi na trojúhelnících. Je dokázána $O(h)$-konvergence za předpokladu dostatečné regularity řešení na polygonální nebo na konvexní oblasti s hranicí dostatečně hladkou. Studuje se i konvergence aproximací bez předpokladu regularity. Pomocí principu minima doplňkové energie je definována duální variační formulace. Přípustná konvexní množina napětí se aproximuje po částech lineárními vektorovými funkcemi s nulovou divergencí na celé oblasti a je dokázána $O(h^{3/2})$-konvergence aproximací. Na základě primární a duální variační formulace jsou odvozeny aposteriorní odhady chyb a oboustranné odhady energie řešení.

Резюме

АНАЛИЗ ПРИМАРНОЙ И ДВОЙСТВЕННОЙ ВАРИАЦИОННОЙ ФОРМУЛИРОВКИ СЕМИ-КОЭРЦИТИВНЫХ ЭЛЛИПТИЧЕСКИХ ЗАДАЧ С НЕОДНОРОДНЫМИ ПРЕПЯТСТВИЯМИ НА ГРАНИЦЕ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

TRAN VAN BON

Уравнение Пуасона с неоднородным односторонним краевым условием решено посредством конечных элементов. Прямая задача приближается при помощи линейных треугольных элементов и доказывается $O(h)$-сходимость при предположении, что точное решение достаточно регулярно. Для изучения двойственной задачи используются кусочно линейные приближения с нулевой дивергенцией и доказывается $O(h^{3/2})$-сходимость для регулярного решения. Приведены тоже некоторые апостериорные оценки.

Author's address: Dr. *Tran Van Bon,* CSc., Khoa Có bán-Ngoai ngú 2C-453 Nha trang-Phú Khánh, Vietnam.