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OPTIMAL CONTROL OF VARIATIONAL INEQUALITY WITH APPLICATIONS TO AXISYMMETRIC SHELLS

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Summary. The optimal centrol problem of variational inequality with applications to axisymmetric shells is discussed. First an existence result for the solution of the optimal control problem is given. Next is presented the formulation of first order necessary conditions of optimality for the control problem governed by a variational inequality with its coefficients as control variables.

Key words: elliptic variational inequalities, optimal control problems, shape of axisymmetric shells, second invariant of the stress deviator, first order necessary conditions of optimality, smooth regularized control problems.

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INTRODUCTION

We consider an optimal control problem in which the state variable of the system (which includes an elliptic, linear, symmetric operator, the coefficients of which are chosen as the design-control variables) is defined as the (unique) solution of a variational inequality. The existence result proved in Section 1 can be applied to the optimal design of the shape of axisymmetric shells (of Section 2; the theme of this section stems from the papers [5] and [6]). The meridian curves of their middle surfaces are taken for the design variable (the case of thickness is considered). Admissible functions are smooth curves of a given length, which are uniformly bounded together with their first and second derivatives, and such that the shell contains a given volume. The loading consists of the own weight, the hydrostatic pressure of a liquid and an external or internal pressure. As a cost functional, the integral of the second invariant of the stress deviator on both surfaces of the shell is chosen. Section 3 is concerned with the formulation of first order necessary conditions of optimality for the control problem governed by a variational inequality with its coefficients as control variables (nonsmooth and nonconvex infinite dimensional optimization problem.

We apply an idea of Barbu – Mignot – Tartar: the idea is to approximate the given optimal control problem by a family of "smooth" regularized control problems and then to pass to the limit in the approximating optimality equations.

1. EXISTENCE THEOREM

The proof of our existence theorem is based on results of [2], [4], [12].

Let $V(\Omega)$ be a real Hilbert space and $V^*(\Omega)$ its dual space, the pairing between $V(\Omega)$ and $V^*(\Omega)$ being denoted $\langle \cdot ; \cdot \rangle_{V(\Omega)}$. Next $H(\Omega)$ is a separable real Hilbert space such that $V(\Omega)$ is dense in $H(\Omega)$ and the injection of $V(\Omega)$ is completely continuous.

Let $U(\Omega)$ be a Hilbert space of controls, $U_{ad}(\Omega) \subset U(\Omega)$ a set of admissible controls ($U_{ad}(\Omega)$ is a compact set in $U(\Omega)$).

Let $A(e)$: $V(\Omega) \to V^*(\Omega)$ for every $e \in U_{ad}(\Omega)$ be a family of linear and symmetric operators with the following properties:

(HO)

1° For any $e \in U_{ad}(\Omega)$ the operator $A(e)$ belongs to $L(V(\Omega), V^*(\Omega))$ ($\{A(e)\}$ is uniformly bounded, i.e. $\|e\|_{U(\Omega)} \leqq c_1$, $\|v\|_{V(\Omega)} \leqq c_2 \Rightarrow \|A(e)\, v\|_{V^*(\Omega)} \leqq c$)

2° For any $e \in U_{ad}(\Omega)$ the operator $A(e)$ satisfies the $V(\Omega)$ – uniform coercivity condition (with respect to $U(\Omega)$)
$\langle A(e)\, v, v \rangle_{V(\Omega)} \geqq \alpha \|v\|^2_{V(\Omega)}$ for any $v \in V(\Omega)$, for any $e \in U(\Omega)$, and $\alpha > 0$ independent of e.

3° For every $v \in V(\Omega)$ the operator $A(\cdot)\, v$: $U_{ad}(\Omega) \to V^*(\Omega)$ is strongly – strongly continuous: $e_n \to e_0$ (strongly) in $U(\Omega)$ for $n \to \infty$, $A(e_n)\, v \to A(e_0)\, v$ in $V^*(\Omega)$ for all $v \in V(\Omega)$.

Consider the equation

$$A(e)\, u(e) + \partial\Phi(u(e)) \ni f + Be \tag{1.1}$$

where $\partial\Phi$: $V(\Omega) \to V^*(\Omega)$ is the subdifferential of Φ (Φ: $V(\Omega) \to \bar{R}$ is a lower semicontinuous convex function), B is a nonlinear continuous operator from $U(\Omega) \to H(\Omega)$ and f is a given element of $V^*(\Omega)$. In what follows we make use of the canonical injection $H(\Omega) \hookrightarrow V^*(\Omega)$. As seen earlier, (1.1) can be rewritten as the variational inequality

$$u(e) \in V(\Omega)\,, \tag{1.2}$$

$$\langle A(e)\, u(e), v - u(e) \rangle_{V(\Omega)} + \Phi(v) - \Phi(u(e)) \geqq \langle f + Be, v - u(e) \rangle_{V(\Omega)}$$

$$\text{for all} \quad v \in V(\Omega)\,, \quad e \in U_{ad}(\Omega)\,.$$

or

$$u(e) \in \mathfrak{K}(\Omega),$$

(1.2a) $$\langle A(e)\, u(e),\, v - u(e)\rangle_{V(\Omega)} + j(v) - j(u(e)) \geqq f + Be,\, v - u(e)\rangle_{V(\Omega)}$$

$$\text{for all} \quad v \in \mathfrak{K}(\Omega), \quad e \in U_{\mathrm{ad}}(\Omega)$$

where $\Phi = j + I_{\mathfrak{K}(\Omega)}$ ($I_{\mathfrak{K}(\Omega)}$ is the indicator of a convex, nonempty, closed subset of $V(\Omega)$, j is a convex lower semicontinuous proper functional on $V(\Omega)$). We shall assume in addition that $\mathfrak{K}(\Omega) \cap \operatorname{int} D(j) \neq \emptyset$ (int $C \equiv$ interior of the set C, $D(j) = \{v \in \mathfrak{K}(\Omega): j(v) < +\infty\}$). The parameter $e \in U_{\mathrm{ad}}(\Omega)$ is called the control, and the corresponding solution $u(e)$ is called the state of the system (1.2). Equation (1.1) itself will be referred to as the state system or control system. For every $f \in V^*(\Omega)$ and for every $e \in U_{\mathrm{ad}}(\Omega)$ the variational inequality (1.2) has a unique solution.

The optimal control problem can be set in the following general form:

Let a functional

$$\mathfrak{L}: (U(\Omega) \times V(\Omega)) \to R$$

be given, which satisfies the following condition

(1.3) $$\begin{cases} \text{if} \quad e_n,\ e \in U_{\mathrm{ad}}(\Omega),\ e_n \to e \text{ in } U(\Omega),\ v_n \rightharpoonup v \text{ (weakly)} \\ \text{in } V(\Omega) \Rightarrow \liminf\limits_{n \to \infty} \mathfrak{L}(e_n, v_n) \geqq \mathfrak{L}(e, v). \end{cases}$$

Defining the cost functional as

(1.4) $$\mathfrak{C}(e) = \mathfrak{L}(e, u(e))$$

where $u(e)$ denotes the solution of (1.2a), we may consider the optimal design problem $(\mathscr{B})$:

Minimize the function

$(\mathscr{B})$ $$\mathfrak{C}(e)$$

over all $u \in \mathfrak{K}(\Omega)$ and $e \in U_{\mathrm{ad}}(\Omega)$ subject to the state system (1.2a).

A pair $[e_0, u_0] \in U_{\mathrm{ad}}(\Omega) \times \mathfrak{K}(\Omega)$ for which the infimum in problem $(\mathscr{B})$ is attained is called the optimal pair, and the corresponding control e_0 is called the optimal control.

Theorem 1. *Under assumptions* (HO) *and* (1.3) , *problem* $(\mathscr{B})$ *has at least one optimal pair.*

Proof. Let $\{e_k\} \subset U_{\mathrm{ad}}(\Omega)$ be a minimizing sequence for $\mathfrak{C}(e)$, i.e.

(1.5) $$\lim_{k \to \infty} \mathfrak{C}(e_k) = \inf_{e \in U_{\mathrm{ad}}(\Omega)} \mathfrak{C}(e).$$

Since $U_{\mathrm{ad}}(\Omega)$ is compact, there exists a subsequence $\{e_n\} \subset \{e_k\}$ such that $e_n \to e_0$ (strongly) in $U(\Omega)$.

The means that $Be_n \to Be_0$ (strongly) in $V^*(\Omega)$. Setting $u_n = u(e_n)$ we can write

$$\langle A(e_n)(u_n - v), u_n - v\rangle_{V(\Omega)} \leqq j(v) - j(u_n) - \langle f + Be_n, v - u_n\rangle_{V(\Omega)} - $$
$$- \langle A(e_n) v, u_n - v\rangle_{V(\Omega)} \quad \text{for any} \quad v \in \mathfrak{K}(\Omega).$$

Then Proposition 1.7 ([3]) implies int $D(j) \subset D(\partial j)$, and if

(EO) $$\mathfrak{K}(\Omega) \cap \operatorname{int} D(j) \neq \emptyset,$$

there exists an element $v_0 \in \mathfrak{K}(\Omega) \cap D(\partial j)$ such that $j(v_0) - j(w) \leqq \langle p, v_0 - w\rangle_{V(\Omega)}$ where $p \in \partial j(v_0)$, $w \in \mathfrak{K}(\Omega)$. This means that the function $(\theta: \mathfrak{K}(\Omega) \to R)$ $\theta(w) = (j(v) - j(w))/\|v_0 - w\|_{V(\Omega)}$ is bounded. Further by assumption ((HO), 2°) we get $\alpha\|u_n - v_0\|_{V(\Omega)} \leqq \theta(u_n) + (\|f\|_{V^*(\Omega)} + \|Be_n\|_{V^*(\Omega)} + |\langle A(e_n) v_0, u_n - v_0\rangle_{V(\Omega)}| : \|u_n - v_0\|_{V(\Omega)}$.

Hence
$\|u_n\|_{V(\Omega)} \leqq c$ by (assumption ((HO), 1°), and there exists a subsequence (denoted again by $\{u_n\}$) such that $u_n \rightharpoonup u$ (weakly) in $V(\Omega)$, where $u \in \mathfrak{K}(\Omega)$ (the set $\mathfrak{K}(\Omega)$ is weakly closed).

For any $w \in V(\Omega)$ we have (by assumption ((HO), 3°))

$$\lim_{n\to\infty} \langle A(e_n) u(e_n); w\rangle_{V(\Omega)} = \lim_{n\to\infty} \langle A(e_n) w, u_n\rangle_{V(\Omega)} =$$
$$= \langle A(e_0) w, u\rangle_{V(\Omega)} = \langle A(e_0) u, w\rangle_{V(\Omega)}$$

and therefore

(1.6) $$A(e_n) u(e_n) \rightharpoonup A(e_0) u \quad \text{(weakly)} \quad \text{in } V^*(\Omega)$$

if $u_n \rightharpoonup u$ is weakly convergent in $V(\Omega)$. Furthermore, by assumption ((HO), 2°) we can write

$$\langle A(e_n)(u_n - u), u_n - u\rangle_{V(\Omega)} \geqq 0.$$

Hence we have (by passing to the limit)

$$\lim_{n\to\infty} 2\langle A(e_n), u, u_n\rangle_{V(\Omega)} = \liminf_{n\to\infty} \langle A(e_n) u_n, u_n\rangle_{V(\Omega)} + \lim_{n\to\infty} \langle A(e_n) u, u\rangle_{V(\Omega)}.$$

This yields

(1.7) $$\liminf_{n\to\infty} \langle A(e_n) u_n, u_n\rangle_{V(\Omega)} \geqq \langle A(e_0) u, u\rangle_{V(\Omega)}.$$

Now, letting n tend to $+\infty$ in the inequality

$$\langle A(e_n) u_n, u_n - v\rangle_{V(\Omega)} + j(u_n) \leqq j(v) + \langle f + Be_n, u_n - v\rangle_{V(\Omega)}$$

and taking lim inf, we conclude by (1.6) and (1.7) that

$$\langle A(e_0) u, u - v\rangle_{V(\Omega)} + j(u) \leqq j(v) + \langle f + Be_0, u - v\rangle_{V(\Omega)}$$

(because $j(v)$ is a convex, lower semicontinuous, proper functional on $V(\Omega)$).

We infer $u = u(e_0)$ as claimed and the whole sequence $\{u_n\}$ tends to $u(e_0)$ weakly in $V(\Omega)$ (since the variational inequality (1.2a) has a unique solution for any $e \in U_{ad}(\Omega)$). We can write

$$\langle A(e_n)\, u_n, v - u_n \rangle_{V(\Omega)} + j(v) - j(u_n) \geqq \langle f + Be_n, v - u_n \rangle_{V(\Omega)}$$
$$\text{for any} \quad v \in \mathfrak{K}(\Omega)\,, \quad e_n \in U_{ad}(\Omega)\,.$$

This yields

$$\langle A(e_n), u_n\, u_n \rangle_{V(\Omega)} - j(v) \leqq \langle A(e_n)\, u_n, v \rangle_{V(\Omega)} - \langle f + Be_n, v - u_n \rangle_{V(\Omega)} - j(u_n)\,.$$

Consequently,

$$\limsup_{n \to \infty} \langle A(e_n)\, u_n, u_n \rangle_{V(\Omega)} - j(v) \leqq \lim_{n \to \infty} (\langle A(e_n)\, u_n, v \rangle_{V(\Omega)} -$$
$$- \langle f + Be_n, v - u_n \rangle_{V(\Omega)}) - \liminf_{n \to \infty} j(u_n)\,.$$

Hence by (1.6) one has

$$\limsup_{n \to \infty} \langle A(e_n)\, u_n, u_n \rangle_{V(\Omega)} - j(v) \leqq \langle A(e_0)\, u, v \rangle_{V(\Omega)} - \tag{1.8}$$
$$- \langle f + Be_0, v - u \rangle_{V(\Omega)} - j(u) \quad \text{for any} \quad v \in \mathfrak{K}(\Omega)$$

and therefore (we take $v = u$)

$$\limsup_{n \to \infty} \langle A(e_n)\, u_n, u_n \rangle_{V(\Omega)} \leqq \langle A(e_0)\, u, u \rangle_{V(\Omega)}\,.$$

By (1.7) it follows that

$$\lim_{n \to \infty} \langle A(e_n)\, u_n, u_n \rangle_{V(\Omega)} = \langle A(e_0)\, u, u \rangle_{V(\Omega)}\,.$$

By virtue of ((HO), 2°, 3°) we have

$$\alpha \limsup_{n \to \infty} \|u_n - u\|^2_{V(\Omega)} \leqq \lim_{n \to \infty} \langle A(e_n)\,(u_n - u), u_n - u \rangle_{V(\Omega)} =$$
$$= \lim_{n \to \infty} \{\langle A(e_n)\, u_n, u_n \rangle_{V(\Omega)} + \langle A(e_n)\, u, u \rangle_{V(\Omega)} - 2\langle A(e_n)\, u, u_n \rangle_{V(\Omega)}\} = 0\,.$$

Hence
$\lim_{n \to \infty} u_n = u(e_0)\ (= u_0)$ holds in the strong topology of $V(\Omega)$.

Thus, we have shown that the map $e \to u(e)$ is strongly – strongly continuous from $U_{ad}(\Omega)$ to $V(\Omega)$.

By virtue of (1.3) and (1.5) we have

$$\inf_{e \in U_{ad}(\Omega)} \mathfrak{E}(e) = \liminf_{n \to \infty} \mathfrak{E}(e_n) = \liminf_{n \to \infty} \mathfrak{L}(e_n, u(e_n)) \geqq \mathfrak{L}(e_0, u(e_0)) = \mathfrak{E}(e_0)\,.$$

In other words, e_0 is an optimal control of problem $\mathscr{B}$.)

2. OPTIMIZATION OF THE SHAPE OF AXISYMMETRIC SHELLS WITH UNILATERAL CONSTRAINTS

Axisymmetric thin elastic shells of constant thickness with unilateral constraints are considered and the meridian curves of their middle surfaces taken for the design variable.

Admissible functions are smooth curves of a given length, which are uniformly bounded together with their first and second derivatives, and such that the shell contains a given volume. The loading consists of the own weight, the hydrostatic pressure of a liquid, and external or internal pressure.

As a cost functional, the integral of the second invariant of the stress deviator on both surfaces of the shell is chosen.

We shall apply the abstract Theorem 1 to the optimal shape design ($\mathscr{F} \equiv e$) in the case of axisymmetric problems for thin elastic shells.

Let z and r denote the axial radial coordinates, respectively. We describe the meridian curve by means of two functions $\mathscr{F}(s)$ and $\mathscr{Z}(s)$, as follows:

$$r = \mathscr{F}(s) \,; \quad z = \mathscr{Z}(s) \quad 0 \leqq s \leqq l$$

where s denotes the arc parameter and the length l is given.

We can write

$$\mathrm{d}\mathscr{Z}(s)/\mathrm{d}s = [1 - (\mathrm{d}\mathscr{F}(s)/\mathrm{d}s)^2]^{1/2} \,.$$

Let us choose $U(S) = C^{(1)}(\bar{S})$, $S = (0, l)$,

$$U_{\mathrm{ad}}(S) = \{\mathscr{F}(s) \in C^{(1),1}(\bar{S}) \colon r_0 \leqq \mathscr{F}(s) \leqq r_1 \,,$$

$$|\mathrm{d}\mathscr{F}(s)/\mathrm{d}s| \leqq C_1 < 1 \,, \quad |\mathrm{d}^2\mathscr{F}(s)/\mathrm{d}s^2| \leqq C_2 \,,$$

$$\int_0^e \mathscr{F}^2(s)\,(\mathrm{d}\mathscr{Z}(s)/\mathrm{d}s)\,\mathrm{d}s = C_3\}$$

where r_0, r_1, C_1, C_2, C_3 are given positive constants.

The integral condition means that the volume of the shell is prescribed. $C^{(1),1}(\bar{S})$ is the space of continuously differentiable functions in S, the derivatives of which are Lipschitzian.

Moreover, we define an auxiliary set

$$U^0(S) = \{\mathscr{F}(s) \in C^{(1)}(\bar{S}) \colon (1/2)\, r_0 \leqq \mathscr{F}(s) \leqq 2r_1$$

$$\mathrm{d}\mathscr{F}(s)/\mathrm{d}s \leqq (1/2)\,(1 + C_1) < 1\} \,.$$

We shall use the linear theory of shells ([13]) and formulate the equilibrium in terms of the displacement vector $\boldsymbol{u} = (u, w)$, where u is the meridional and w the normal displacement (see Fig. 1). Next, we set $\boldsymbol{v} = (\varphi, \theta)$. Let us define the following system of strains.

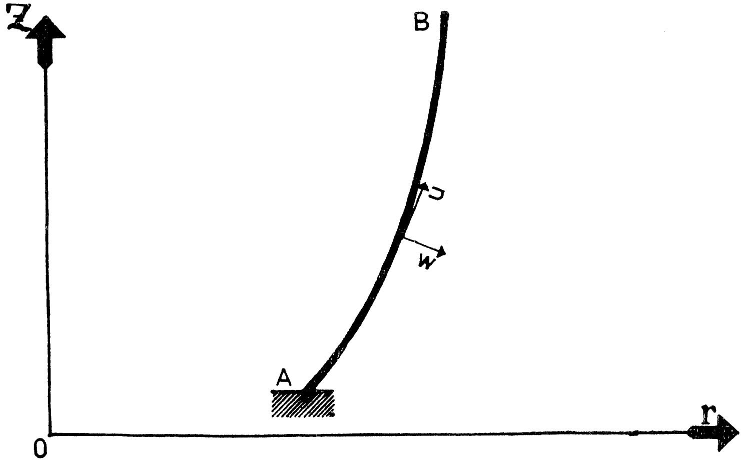

Fig. 1.

$$(2.1)\quad \begin{cases} \mathcal{N}_1(u) = \mathrm{d}u(s)/\mathrm{d}s, \ \mathcal{N}_2(u) = ((\mathrm{d}\mathcal{F}(s)/\mathrm{d}s)\, u + (\mathrm{d}\mathcal{Z}(s)/\mathrm{d}s)\, w)/\mathcal{F}(s)\,, \\ \mathcal{N}_3(u) = -\mathrm{d}^2 w(s)/\mathrm{d}s^2, \ \mathcal{N}_4(u) = (-\mathrm{d}\mathcal{F}(s)/\mathrm{d}s)\,(\mathrm{d}w(s)/\mathrm{d}s)/\mathcal{F}(s)\,, \end{cases}$$

and the matrix:

$$(2.2)\qquad \mathbf{K} = Eh/(1-\mu^2)\begin{bmatrix} 1 & \mu & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & 0 & (h^2/12) & (\mu h^2/12) \\ 0 & 0 & (\mu h^2/12) & (h^2/12) \end{bmatrix}_{(4,4)}$$

where E is the Young modulus, h the constant thickness of the shell and μ the Poisson ratio ($0 \leqq \mu < 1/2$). Henceforth $H^k(S)$, $k = 1, 2$ denote the usual Sobolev spaces with square – inegrable derivatives, and $\|\cdot\|_{H^k(S)}$their norms ($H^1_0(S)$ is a closed vector subspace of $H^1(S)$, the trace γv of $v \in H^1(S)$ equals zero on ∂S). Let us consider the space

$$W(S) = H^1(S) \times H^2(S)\,, \quad L(S) = L_2(S) \times L_2(S)$$

and write

$$\|\mathbf{u}\|_{W(S)} = (\|u\|^2_{H^1(S)} + \|w\|^2_{H^2(S)})^{1/2}\,.$$

We introduce the subspaces

$$(2.3)\quad \begin{cases} V(S) = \{\mathbf{v} = (\varphi, \theta) \in W(S) \colon \varphi(0) = \theta(0) = \mathrm{d}\theta(0)/\mathrm{d}s = 0\}\,, \\ \mathcal{R}(S) = \{\mathbf{v} \in V(S) \colon \mathcal{N}_i(v) = 0,\ i = 1, 2, 3, 4\}\,. \end{cases}$$

We define a bilinear form and an operator $A(\mathcal{F}) \colon V(S) \to V^*(S)$ by the equation

$$(2.4)\quad a(\mathcal{F}, \mathbf{u}, \mathbf{v}) = \langle A(\mathcal{F})\, \mathbf{u}, \mathbf{v}\rangle_{V(S)} = 2\pi \int_S \sum_{i,j=1}^{4} K_{ij}\, \mathcal{N}_i(\mathbf{u}, \mathcal{F})\, \mathcal{N}_j(\mathbf{v}, \mathcal{F})\, \mathcal{F}\, \mathrm{d}s\,.$$

Further, consider the form

$$(2.5)\qquad \langle \mathbf{f}(\mathcal{F}), \mathbf{v}\rangle_{V(S)} = 2\pi \int_S [k_0\theta(\mathcal{Z}(l) - \mathcal{Z}(s)) + k_1((\mathrm{d}\mathcal{F}(s)/\mathrm{d}s)\,\theta - (\mathrm{d}\mathcal{Z}(s)/\mathrm{d}s)\,\varphi) + k_3\theta]\,\mathcal{F}(s)\,\mathrm{d}s$$

where

$$\mathbf{f}(\mathcal{F}) = B\mathcal{F} \quad B: U(S) \to L(S)\,, \quad L(S) \subset V^*(S)\,,$$

k_0 and k_1 are non-negative constants denoting the specific weight of the liquid and of the shell, respectively. The first part of the loading corresponds to the volume of the shell full of the liquid. The constant k_3 denotes an internal or external pressure.

The boundary conditions in $V(S)$ correspond to the clamped edge $s = 0$. The subspace $\mathcal{R}(S)$ represents the virtual displacements of a rigid shell. It is easy to see that $\mathcal{R}(S) = \{0\}$.

In fact,

$$(2.6)\qquad \begin{cases} \mathcal{N}_1(\mathbf{u}) = 0 \to u = u_0 = \text{const}\,, \\ \mathcal{N}_3(\mathbf{u}) = 0 \to w = w_0 + w_1 s,\ w_0, w_1 = \text{const}\,. \end{cases}$$

Inserting the boundary conditions, we arrive at

$$u_0 = w_0 = w_1 = 0\,.$$

Further, we introduce the set of kinematically admissible displacement by

$$(2.7)\qquad \mathfrak{K}(S) = \{\mathbf{v} = [\varphi, \theta] \in V(S): \theta(s) \leqq 0 \text{ for } s \in S_0 \subset S\}\,.$$

Lemma 1. *The set $\mathfrak{K}(S)$ is non empty, convex and closed in $V(S)$.*

Proof. The convexity of $\mathfrak{K}(S)$ can be seen directly from definition (2.7). Let us now consider such a sequence $\mathbf{v}_n \in \mathfrak{K}(S)$, $n = 1, 2, 3, \ldots$, that $\mathbf{v}_n \to \mathbf{v}$ strongly in $V(S)$. If $\mathbf{v} = [\varphi, \theta]$, $\mathbf{v}_n = [\varphi, \theta_n]$, then $\theta_n \to \theta$ strongly in $H^2(S)$. Due to the imbeding theorem for the space $H^2(S)$ ([1]) we have $\lim_{n\to\infty} \theta_n(s) = \theta(s)$ for every point $s \in S_0$. As $\theta_n(s) \leqq 0$ in S_0 we obtain $\theta(s) \leqq 0$ in S_0 and hence $\mathbf{v} \in \mathfrak{K}(S)$, which concludes the proof.

If we define $a(\mathcal{F}, \mathbf{u}, \mathbf{v})$ and $\langle \mathbf{f}(\mathcal{F}), \mathbf{v}\rangle_{V(S)}$ by the formulas (2.4), (2.5) (cf. (2.1), (2.2)), and $V(S)$, $\mathfrak{K}(S)$ by (2.3) and (2.7), respectively, the state problem (1.2a) corresponds to a unilateral problem for a shell, the lower edge of which is clamped while on the upper edge we have the free boundary conditions under the combined effect of the own weight, of the weight of the liquid, and of a pressure.

Lemma 2. *The family $\{A(\mathcal{F}), \mathcal{F} \in U_{ad}(S)\}$ of operators defined by* (2.4) *satisfies the assumptions* ((HO), 1°, 2°, 3°).

Proof. By virtue of the definition of $U^0(S)$, we have

(2.8) $$\|A(\mathscr{F})\,\mathbf{v}\|_{V^*(S)} \leqq c\|\mathbf{v}\|_{V(S)}$$

where the positive constant c is independent of $(\mathscr{F}, \mathbf{v})$.

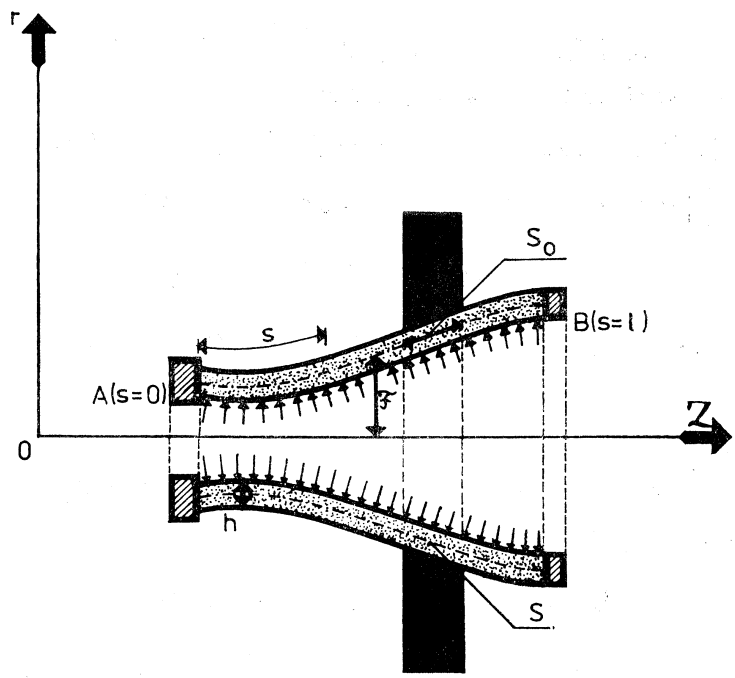

Fig. 2.

Now ((HO), 1°) is an immediate consequence of (2.8). To prove the assumption ((HO), 2°), we first realize that $\mathbf{K}$ is positive definite; i.e. $\xi^T \mathbf{K} \xi \geqq c\xi^T\xi$ for any $\xi \in \mathrm{R}^4$; $c > 0$, and we may write

(2.9) $$\langle A(\mathscr{F})\,\mathbf{v}, \mathbf{v}\rangle_{V(S)} = a(\mathscr{F}, \mathbf{v}, \mathbf{v}) \geqq (c/2)\, r_0 \int_S [\mathscr{N}_1^2(\mathbf{v}) + \mathscr{N}_3^2(\mathbf{v})]\, \mathscr{F}\, \mathrm{d}s$$

$$\text{for any} \quad \mathscr{F} \in U_{\mathrm{ad}}(S)\,, \quad \mathbf{v} \in V(S)\,.$$

By virtue of (2.6) and the boundary conditions, we have

(2.10) $$\int_S [(d\varphi(s)/\mathrm{d}s)^2 + (\mathrm{d}^2\theta(s)/\mathrm{d}s^2)^2]\, \mathscr{F}\, \mathrm{d}s \geqq c\|\mathbf{v}\|^2_{V(S)}$$

for any $\mathbf{v} \in V(S)$ with $c > 0$ independent of $\mathbf{v}$ (see e.g. [8] – Chapt. 11, Lemma 3.2). Combining (2.9) and (2.10) we, obtain

(2.11) $$\langle A(\mathscr{F})\,\mathbf{v}, \mathbf{v}\rangle_{V(S)} \geqq \alpha\|\mathbf{v}\|^2_{V(S)} \quad \text{for any} \quad \mathbf{v} \in V(S)\,, \quad \mathscr{F} \in U^0(S)\,.$$

The positive constant α is independent of $(\mathcal{F}, \mathbf{v})$ and ((HO), 2°) is proved. Let $\mathcal{F}_n \to \mathcal{F}_0$ in $C^1(\bar{S})$ for $n \to \infty$. Then we may write

$$\|A(\mathcal{F}_n) - A(\mathcal{F}_0)\|_{L(V(S),V^*(S))} = \sup_{\mathbf{v}\in V(S), \|\mathbf{v}\|_{V(S)}=1} \|(A(\mathcal{F}_n) - A(\mathcal{F}_0))\, \mathbf{v}\|_{V^*(S)} =$$

$$= \sup_{\mathbf{v}\in V(S), \|\mathbf{v}\|_{V(S)}=1} \; \sup_{\boldsymbol{\omega}\in V(S), \|\boldsymbol{\omega}\|_{V(S)}=1} |\langle (A(\mathcal{F}_n) - A(\mathcal{F}_0))\, \mathbf{v}, \boldsymbol{\omega}\rangle_{V(S)}| \, .$$

Moreover, we obtain the following estimates:

$$(2.12) \quad |\langle A(\mathcal{F}_n)\, \mathbf{v}, \boldsymbol{\omega}\rangle_{V(S)} - \langle A(\mathcal{F}_0)\, \mathbf{v}, \boldsymbol{\omega}\rangle_{V(S)}| = |a(\mathcal{F}_n, \mathbf{v}, \boldsymbol{\omega}) - a(\mathcal{F}_0, \mathbf{v}, \boldsymbol{\omega})| \leqq$$

$$\leqq 2\pi \int_S |\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n)\, \mathcal{F}_n(s) - \mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_0)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0)\, \mathcal{F}_0(s)|\, \mathrm{d}s \leqq$$

$$\leqq 2\pi \int_S |\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n)\, \mathcal{F}_n(s) -$$

$$- \mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n)\, \mathcal{F}_0(s)|\, \mathrm{d}s + 2\pi \int_S |\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n)\, .$$

$$. \, \mathcal{F}_0(s) - \mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_0)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0)\, \mathcal{F}_0(s)|\, \mathrm{d}s$$

where $\boldsymbol{\omega} = (v, \omega)$,

$$\mathcal{N}(\mathbf{v}, \mathcal{F}) = [\mathcal{N}_1(\mathbf{v}, \mathcal{F}), \mathcal{N}_2(\mathbf{v}, \mathcal{F}), \mathcal{N}_3(\mathbf{v}, \mathcal{F}), \mathcal{N}_4(\mathbf{v}, \mathcal{F})] \, .$$

For the first integral we have

$$\int_S |\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n)| \, . \, |\mathcal{F}_n(s) - \mathcal{F}_0(s)|\, \mathrm{d}s \leqq$$

$$\leqq c\|\mathcal{F}_n(s) - \mathcal{F}_0(s)\|_{C(S)} \left[\int_S \sum_{j=1}^4 \mathcal{N}_j^2(\mathbf{v}, \mathcal{F}_n)\, \mathrm{d}s\right]^{1/2} \left[\int_S \sum_{j=1}^4 \mathcal{N}_j^2(\boldsymbol{\omega}, \mathcal{F}_n)\, \mathrm{d}s\right]^{1/2} \to 0 \, .$$

Next, for the second integral we have the upper bound

$$(2.13) \int_S \{|\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n)\, \mathcal{F}_0(s) - \mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0)\, \mathcal{F}_0(s)| +$$

$$+ |\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0)\, \mathcal{F}_0(s) - \mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_0)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0)\, \mathcal{F}_0(s)|\}\, \mathrm{d}s =$$

$$= \int_S \{|\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n)\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_n) - \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0))\, \mathcal{F}_0(s)| +$$

$$+ |(\mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_n) - \mathcal{N}^{\mathrm{T}}(\mathbf{v}, \mathcal{F}_0))\, \mathbf{K}\, \mathcal{N}(\boldsymbol{\omega}, \mathcal{F}_0)\, \mathcal{F}_0(s)|\}\, \mathrm{d}s \leqq$$

$$\leqq c\left[\int_S \sum_{j=1}^4 \mathcal{N}_j^2(\mathbf{v}, \mathcal{F}_n)\, \mathrm{d}s\right]^{1/2} \left[\int_S \sum_{j=1}^4 (\mathcal{N}_j(\boldsymbol{\omega}, \mathcal{F}_n) - \mathcal{N}_j(\boldsymbol{\omega}, \mathcal{F}_0))^2\right]^{1/2} +$$

$$+ c\left[\int_S \sum_{j=1}^4 (\mathcal{N}_j(\mathbf{v}, \mathcal{F}_n) - \mathcal{N}_j(\mathbf{v}, \mathcal{F}_0))^2\right]^{1/2} \left[\int_S \sum_{j=1}^4 \mathcal{N}_j^2(\boldsymbol{\omega}, \mathcal{F}_0)\right]^{1/2} .$$

Then (2.1) yields

$$(2.14)\qquad \int_S |\mathcal{N}_2(\mathbf{v}, \mathcal{F}_n) - \mathcal{N}_2(\mathbf{v}, \mathcal{F}_0)|^2 \, ds \leqq \int_S [|\varphi| \,.\, |(d\mathcal{F}_n(s)/ds)/\mathcal{F}_n - (d\mathcal{F}_0(s)/ds)/\mathcal{F}_0| + |\theta| \,.\, |(d\mathcal{Z}_n(s)/ds)/\mathcal{F}_n - (d\mathcal{Z}_0(s)/ds)/\mathcal{F}_0|^2 \, ds \to 0$$

since

$$\lim_{n\to\infty} \|(d\mathcal{F}_n(s)/ds)^2/\mathcal{F}_n - (d\mathcal{F}_0(s)/ds)/\mathcal{F}_0\|_{C(S)} = 0\,,$$

$$\lim_{n\to\infty} \|(d\mathcal{Z}_n(s)/ds/\mathcal{F}_n - (d\mathcal{Z}_0(s)/ds)/\mathcal{F}_0\|_{C(S)} = 0$$

holds if $\mathcal{F}_n \to \mathcal{F}_0$ in $C^{(1)}(\bar{S})$.

In a parallel way we obtain

$$(2.15)\qquad \int_S [\mathcal{N}_4(\mathbf{v}, \mathcal{F}_n) - \mathcal{N}_4(\mathbf{v}, \mathcal{F}_0)]^2 \, ds \leqq \leqq \int_S (d\theta/ds)^2 \, |d\mathcal{F}_n(s)/ds)/\mathcal{F}_n - (d\mathcal{F}_0(s)/ds)/\mathcal{F}_0|^2 \, ds \to 0\,.$$

Finally, inserting (2.14), (2.15) and the analogous relations with $\mathbf{v}$ replaced by ω into (2.13), we are led to the assertion that the second integral in (2.12) tends to zero.

Thus we have proved that

$$(\text{NO})\qquad \lim_{n\to\infty} \|A(\mathcal{F}_n) - A(\mathcal{F}_0)\|_{L(V(S), V^*(S))} = 0\,.$$

On the other hand, (NO) implies ((HO), 3°). To prove the continuity of the operator $\boldsymbol{B}$ we first notice that

$$(2.16)\qquad \|d\mathcal{Z}_n(s)/ds - d\mathcal{Z}_0(s)/ds\|_{C(S)} \leqq c\|d\mathcal{F}_n(s)/ds - d\mathcal{F}_0(s)/ds\|_{C(S)} \to 0 \quad \text{holds if} \quad \mathcal{F}_n \in U^0(S), \mathcal{F}_n \to \mathcal{F}_0 \quad \text{in} \quad C^{(1)}(\bar{S})\,.$$

Then we have also

$$(2.17)\qquad |\mathcal{Z}_n(l) - \mathcal{Z}_0(s) - (\mathcal{Z}_0(l) - \mathcal{Z}_0(s))| \leqq \leqq \int_S |d\mathcal{Z}_n(s)/ds - d\mathcal{Z}(s)/ds| \, ds \leqq c\|d\mathcal{Z}_n(s)/ds - d\mathcal{Z}_0(s)/ds\|_{C(S)} \to 0\,.$$

For any $\mathbf{v} = (\varphi, \theta)$ we may write

$$|\langle \mathbf{f}(\mathcal{F}_n), \mathbf{v}\rangle_{V(S)} - \langle \mathbf{f}(\mathcal{F}_0), \mathbf{v}\rangle_{V(S)}| =$$

$$= 2\pi \left| \int_S \{k_0\theta[(\mathcal{Z}_n(l) - \mathcal{Z}_n(s))\,\mathcal{F}_n(s) - (\mathcal{Z}_0(l) - \mathcal{Z}_0(s))\,\mathcal{F}_0(s)] + k_1\theta((d\mathcal{F}_n(s)/ds)\,\mathcal{F}_n(s) - (d\mathcal{F}_0(s)/ds)\,\mathcal{F}_0(s) - k_1\varphi((d\mathcal{Z}_n(s)/ds)\,\mathcal{F}_n(s) - (d\mathcal{Z}_0(s)/ds)\,\mathcal{F}_0(s)) + k_3\theta(\mathcal{F}_n(s) - \mathcal{F}_0(s))\} \, ds \right|.$$

Thus, using (2.16), (2.17) and the convergence of $\mathcal{F}_n$ in $C^1(\bar{S})$, we establish the continuity of the operator B.

Lemma 3. *The set $U_{ad}(S)$ is compact in $C^1(\bar{S})$.*

Proof. Since the functions from $U_{ad}(S)$ are uniformly bounded and uniformly continuous, we apply Arzela's theorem. In every sequence there is a subsequence $\{\mathcal{F}_n\} \subset U_{ad}(S)$ such that $\mathcal{F}_n \to \mathcal{F}$ uniformly on $[0, l]$. It is easy to see that $\mathcal{F}$ fulfils the condition $|d\mathcal{F}/ds| \leqq C_1$. Since the derivatives $\{d\mathcal{F}_n/ds\}$ are uniformly bounded and uniformly continuous, there exist a function $\mathcal{H}$ and a subsequence $\{d\mathcal{F}_m/ds\}$ such that $d\mathcal{F}_m/ds \to \mathcal{H}$ uniformly on $[0, l]$. Using a classical theorem, we obtain $\mathcal{H} = d\mathcal{F}/ds$, hence $\mathcal{F}_m \to \mathcal{F}$ in $C^1(\bar{S})$.

Moreover, $|d^2\mathcal{F}/ds^2| \leqq C_2$, and

$$C_3 = \lim_{m\to\infty} \int_0^l \mathcal{F}_m^2(d\mathcal{Z}_m/ds)\, ds = \int_0^l \mathcal{F}^2(d\mathcal{Z}/ds)\, ds\,.$$

Now we define the cost functional. As in ([5]), let it be related to the second invariant of the stress tensor

$$(2.18) \qquad I_2(\sigma) = (2/3)\,(\sigma_s^2 + \sigma_\theta^2 - \sigma_s\sigma_\theta)$$

where σ_s and σ_θ denote the meridional and circumferential normal stresses, respectively. Thus we define

$$(2.19) \qquad \mathfrak{L}(\mathcal{F}, \mathbf{u}) = 2\pi \int_S \sigma^T(\mathbf{u})\, \mathbf{C}\sigma\,(\mathbf{u})\, \mathcal{F}\, ds\,,$$

where

$$\sigma(\mathbf{u}) = \langle \sigma_s^i, \sigma_s^e, \sigma_\theta^i, \sigma_\theta^e \rangle^T = \mathbf{HKN}(\mathbf{u})\,,$$

$$\mathbf{H} = \begin{bmatrix} 1/h & 0 & -6/h^2 & 0 \\ 1/h & 0 & 6/h^2 & 0 \\ 0 & 1/h & 0 & -6/h^2 \\ 0 & 1/h & 0 & 6/h^2 \end{bmatrix};$$

the superscripts i and e denote that the stress is considered on the internal and external surfaces of the shell, respectively.

$$\mathbf{C} = \begin{bmatrix} \beta_i(s) & 0 & (-1/2)\,\beta_i(s) & 0 \\ 0 & \beta_e(s) & 0 & (-1/2)\,\beta_e(s) \\ (-1/2)\,\beta_i(s) & 0 & \beta_i(s) & 0 \\ 0 & (-1/2)\,\beta_e(s) & 0 & \beta_e(s) \end{bmatrix}$$

where $\beta_i(s)$, $\beta_e(s)$ are positive, bounded weight functions.

Note that

$$(2.20) \qquad \mathfrak{L}(\mathcal{F}, \mathbf{u}) = (3/2)\,\pi \int_S (\beta_i I_2(\sigma^i(\mathbf{u})) + \beta_e I_2(\sigma^e(\mathbf{u}))\, \mathcal{F}(s)\, ds\,.$$

Lemma 4. *The cost functional* (2.20) *satisfies the condition* (1.3).

Proof. We write

$$(2.21)\qquad \mathfrak{L}(\mathscr{F}_n, \boldsymbol{u}_n) - \mathfrak{L}(\mathscr{F}_0, \boldsymbol{u}) = (\mathfrak{L}(\mathscr{F}_n, \boldsymbol{u}_n) - \mathfrak{L}(\mathscr{F}_0, \boldsymbol{u}_n)) + + (\mathfrak{L}(\mathscr{F}_0, \boldsymbol{u}_n) - \mathfrak{L}(\mathscr{F}_0, \boldsymbol{u})) .$$

For any fixed $\mathscr{F} \in U^0(S)$ the functional $\mathfrak{L}(\mathscr{F}, \mathbf{v})$ is weakly lower semicontinuous in $V(S)$.

Indeed, it is differentiable and convex, since

$$D^2\, \mathfrak{L}(\mathscr{F}, \boldsymbol{u}, \mathbf{v}, \mathbf{v}) = 4\pi \int_S \sigma^{\mathrm{T}}(\mathbf{v})\, \boldsymbol{C}\sigma(\mathbf{v})\, \mathscr{F}(s)\, \mathrm{d}s = 2\, \mathfrak{L}(\mathscr{F}, \mathbf{v}) .$$

Combining (2.20) with the positive definiteness of the form (2.18) we conclude that $\mathfrak{L}(\mathscr{F}, \mathbf{v})$ is non-negative. Consequently,

$$(2.22)\qquad \liminf_{n\to\infty} (\mathfrak{L}(\mathscr{F}, \boldsymbol{u}_n) - \mathfrak{L}(\mathscr{F}, \boldsymbol{u}) \geqq 0$$

provided $\boldsymbol{u}_n \to \boldsymbol{u}$.

Denoting

$$\boldsymbol{M} = \boldsymbol{K}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{H}\boldsymbol{K}$$

we may write

$$(2.23)\qquad |\mathfrak{L}(\mathscr{F}_n, \boldsymbol{u}_n) - \mathfrak{L}(\mathscr{F}_0, \boldsymbol{u}_n)| = |2\pi \int_S [\mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n)\, \mathscr{F}_n - - \mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_0)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_0)\, \mathscr{F}_0]\, \mathrm{d}s| \leqq$$

$$\leqq \int_S \{|\mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n)\, \mathscr{F}_n - \mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n)\, \mathscr{F}_0| + + |\mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n)\, \mathscr{F}_0 - \mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_0)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_0)\, \mathscr{F}_0|\}\, \mathrm{d}s \leqq$$

$$\leqq \int_S \{|\mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n)|\, |\mathscr{F}_n - \mathscr{F}_0| + + |\mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n) - \mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_0)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_0)| \,.\, |\mathscr{F}_0|\}\, \mathrm{d}s .$$

Since the entries of $\boldsymbol{M}$ are bounded function, we have

$$\int_S |\mathscr{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathscr{F}_n)\, \boldsymbol{M}\, \mathscr{N}(\boldsymbol{u}_n, \mathscr{F}_n)|\, |\mathscr{F}_n - \mathscr{F}_0|\, \mathrm{d}s \leqq \leqq \|\mathscr{F}_n - \mathscr{F}_0\|_{C(S)}\, C \sum_{j=1}^{4} \|\mathscr{N}_j(\boldsymbol{u}_n, \mathscr{F}_n)\|^2_{L_2(S)} \to 0 .$$

(Since

$$(2.24)\qquad \sum_{j=1}^{4} \|\mathscr{N}_j(\boldsymbol{u}_n, \mathscr{F}_n)\|^2_{L_2(S)} \leqq c\|\boldsymbol{u}_n\|^2_{V(S)} \leqq c \quad \text{for any} \quad n ,$$

$$\mathscr{F}_n \in U^0(S)$$

holds by virtue of the weak convergence of $\boldsymbol{u}_n$). The second part of the integral on the right-hand side of (2.23) has the upper bound

$$(2.25) \qquad 2r_1 \int_S \{|\mathcal{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathcal{F}_n)\, \boldsymbol{M}(\mathcal{N}(\boldsymbol{u}_n, \mathcal{F}_n) - \mathcal{N}(\boldsymbol{u}_n, \mathcal{F}_0))| +$$
$$+ |(\mathcal{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathcal{F}_n) - \mathcal{N}^{\mathrm{T}}(\boldsymbol{u}_n, \mathcal{F}_0))\, \boldsymbol{M}\, \mathcal{N}(\boldsymbol{u}_n, \mathcal{F}_0)|\}\, \mathrm{d}s \leqq$$
$$\leqq r_1 c [\sum_{j=1}^{4} \|\mathcal{N}_j(\boldsymbol{u}_n, \mathcal{F}_n)\|^2_{L_2(S)}]^{1/2} [\sum_{j=1}^{4} \|\mathcal{N}_j(\boldsymbol{u}_n, \mathcal{F}_n) -$$
$$- \mathcal{N}_j(\boldsymbol{u}_n, \mathcal{F}_0)\|^2_{L_2(S)}]^{1/2} + r_1 c [\sum_{j=1}^{4} \|\mathcal{N}_j(\boldsymbol{u}_n, \mathcal{F}_n) -$$
$$- \mathcal{N}_j(\boldsymbol{u}_n, \mathcal{F}_0)\|^2_{L_2(S)}]^{1/2} [\sum_{j=1}^{4} \|\mathcal{N}_j(\boldsymbol{u}_n, \mathcal{F}_0)\|^2_{L_2(S)}]^{1/2} \to 0$$

by virtue of (2.14), (2.15).

Altogether, the right-hand side of (2.23) tends to zero. Combining this and (2.22) with (2.21), we obtain

$$\liminf_{n\to\infty} (\mathfrak{L}(\mathcal{F}_n, \boldsymbol{u}_n) - \mathfrak{L}(\mathcal{F}_0, \boldsymbol{u})) \geqq \liminf_{n\to\infty} (\mathfrak{L}(\mathcal{F}_0, \boldsymbol{u}_n) - \mathfrak{L}(\mathcal{F}_0, \boldsymbol{u})) \geqq 0\,.$$

Then Theorem 1 and Lemmas 1, 2, 3. 4 yield the following assertion (optimization of the shape of axisymmetric shells):

The optimal design problem $(\mathcal{B})$, where the data are defined as above, has at least one solution.

3. FIRST ORDER NECESSARY CONDITIONS OF OPTIMALITY

Since the control problems govergend by nonlinear equations are nonsmooth and nonconvex optimization problems, in order to derive necessary conditions of optimality, we approximate the given problem $(\mathcal{B}_1)$ by a family of smooth optimization problems $(\mathcal{B}_{1\varepsilon})$, and then pass to the limit in the corresponding optimality equations. Minimize the function

$$(\mathcal{B}_1) \qquad \mathfrak{L}(u) + \mathfrak{P}(e)$$

over all $\boldsymbol{u} \in \mathfrak{K}(\Omega)$ and $e \in U(\Omega)$ subject to the state system (1.2).

Here $\mathfrak{L}$: $H(\Omega) \to R$ and $\mathfrak{P}$: $U(\Omega) \to \bar{R}$ are given functions satisfying the following conditions (assumptions):

(E 1) 1° $\mathfrak{L}(u)$ is locally Lipschitz and non-negative on $H(\Omega)$;

$$2^\circ\ \mathfrak{P}(e) = \begin{cases} 0 \text{ if } e \in U_{\mathrm{ad}}(\Omega) \\ +\infty \text{ otherwise.} \end{cases}$$

With regard to the spaces $V(\Omega)$, $H(\Omega)$ and the operator $B \in L(U(\Omega), V^*(\Omega))$, we shall assume in addition that

(E 2) B is completely continuous from $U(\Omega)$ into $V^*(\Omega)$.

Lemma 5. *The map $e \to u(e)$ is weakly-strongly continuous from $U(\Omega)$ into $V(\Omega)$.*

Proof is based on an analogue of Theorem 1 and will be therefore omitted.

Theorem 2. *Under the assumptions* (E 1) *and* (E 2), *problem* $(\mathscr{B}_1)$ *has at least one optimal pair.*

Proof. Let $d = \inf\{\mathfrak{L}(u(e)),\ e \in U_{ad}(\Omega)\}$. By the assumptions (E 1) we see that $0 \leqq d < +\infty$. Now let $\{e_k\} \subset U_{ad}(\Omega)$ be such that $\mathfrak{L}(u(e_k)) \to d$. Since $U_{ad}(\Omega)$ is compact there exists subsequence $\{e_n\} \subset \{e_k\}$ such that $e_n \to e_0$ (strongly) in $U(\Omega)$. Moreover, by Lemma 5 we obtain $u(e_n) \to u(e_0) \equiv u_0$ (strongly) in $V(\Omega)$.

Since $\mathfrak{L}$ is continuous on $V(\Omega)$, we have ($\mathfrak{L}$ is locally Lipschitz and non-negative from $V(\Omega)$ into R) $\mathfrak{L}(u(e_0)) = d$. In other words, e_0 is an optimal control of problem $(\mathscr{B}_1)$.

Let $[e_0, u_0]$ be (any optimal pair of problem $(\mathscr{B}_1)$. For every $\varepsilon > 0$, consider the approximating control problem $(\mathscr{B}_{1\varepsilon})$:

Minimize

$$(\mathscr{B}_\varepsilon) \qquad \mathfrak{L}^\varepsilon(u) + (1/2)\,\|e - e_0\|^2_{U(\Omega)}$$

on all $[e, u] \in U_{ad}(\Omega) \times V(\Omega)$ subject to

$$(3.1) \qquad A(e)\,u(e) + \operatorname{grad} \Phi^\varepsilon(u(e)) = f + Be$$

(in the particular situation, grad Φ^ε is a penalty operator associated with the variational inequality (1.2a)), where

$\{\Phi^\varepsilon\}$, $\varepsilon \to 0^+$, is a family of convex functions, Φ^ε: $V(\Omega) \to R$, which are twice continuously differentiable and satisfy the following conditions:

$$(\text{H } 1) \quad \begin{cases} 1^\circ \ \Phi^\varepsilon(v) \geqq -c(\|v\|_{V(\Omega)} + 1) \text{ for all } v \in V(\Omega) \text{ and } \varepsilon > 0; \\ 2^\circ \ \lim\limits_{\varepsilon \to 0} \Phi^\varepsilon(v) = \Phi(v) \text{ for all } v \in V(\Omega); \\ 3^\circ \ \liminf\limits_{\varepsilon \to 0} \Phi^\varepsilon(v_\varepsilon) \geqq \Phi(v) \text{ for any weakly convergent sequence in } V(\Omega),\ v_\varepsilon \rightharpoonup v; \\ 4^\circ \ \langle \operatorname{grad} \Phi^\varepsilon(u) - \operatorname{grad} \Phi^\lambda(v), u - v\rangle_{V(\Omega)} \geqq \\ \quad \geqq -c(\varepsilon + \lambda)\,(\|\operatorname{grad} \Phi^\varepsilon(u)\|^2_{V^*(\Omega)} + \|\operatorname{grad} \Phi^\lambda(v)\|^2_{V^*(\Omega)} + 1) \\ \quad \text{for all } \varepsilon, \lambda > 0 \text{ and } u, v \in V(\Omega). \end{cases}$$

We define a function $\mathfrak{L}^\varepsilon$: $V(\Omega) \to R$ as follows:

(3.2) $\quad \mathfrak{L}^\varepsilon(v) = \displaystyle\int_{R^n} \mathfrak{L}(P_n v - \varepsilon \Lambda_n \sigma)\, \varrho_n(\sigma)\, d\sigma$ (using an obvious substitution we may write $\mathfrak{L}^\varepsilon(v)$ as

$$\mathfrak{L}^\varepsilon(v) = \varepsilon^{-n} \int_{R^n} \mathfrak{L}(\Lambda_n \vartheta)\, \varrho_n((\Lambda_n^{-1} P_n v - \vartheta)\, \varepsilon^{-1})\, d\vartheta)$$

where ϱ_n is a mollifier (a function which, for any choice of a real number $\varepsilon > 0$, has certain special properties ([1]) in R^n, $n = [\varepsilon^{-1}]$ (the integer part of ε^{-1}), and

P_n: $V(\Omega) \to X_n(\Omega)$ ($X_n(\Omega)$) is the finite dimensional subspace of $V(\Omega)$ generated by $\{\theta_i\}_{i=1}^n$) is the projection of $V(\Omega)$ into $X_n(\Omega)$, given by

$$P_n x = \sum_{i=1}^{n} x_i \theta_i \quad \text{where} \quad x = \sum_{i=1}^{\infty} x_i \theta_i .$$

Let Λ_n: $R^n \to X_n(\Omega)$ be the operator $\Lambda_n \boldsymbol{\sigma} = \sum_{i=1}^{n} \sigma_i \theta_i$, $\boldsymbol{\sigma} = [\sigma_1, \sigma_2 \dots \sigma_n]$.

If the function $\mathfrak{L}$ happens to be Fréchet differentiable, then we take $\mathfrak{L}^\varepsilon = \mathfrak{L}$.

Suppose that $\{A(e_\varepsilon), A(e_\varepsilon): V(\Omega) \to V^*(\Omega)\}$ is a family of linear operators which are uniformly convergent to $A(e)$, i.e.

(H 2) $$\lim_{\varepsilon \to 0} \|A(e_\varepsilon) - A(e)\|_{L(V(\Omega), V^*(\Omega))} = 0$$

whenever $e_\varepsilon \to e$ strongly in $U(\Omega)$ if $\varepsilon \to 0$.

Now we will give some results obtained by approximating the variational inequality (1.2a) by the penalized equation (3.1).

Theorem 3. *For any $\varepsilon > 0$ there exists at least one optimal pair $[e_\varepsilon, u_\varepsilon] \in U_{ad}(\Omega) \times$ $\times\ V(\Omega)$ of the problem $(\mathscr{B}_\varepsilon)$.*

Proof. (See the proof of Theorem 2.)

Theorem 4. *Let $\{[e_{\varepsilon_n}, u_{\varepsilon_n}]\}$, $\varepsilon_n \to 0$ be a sequence of solutions of the problem $(\mathscr{B}_{\varepsilon_n})$. Assume that the injection of $V(\Omega)$ into $H(\Omega)$ is compact and the functions Φ^{ε_n}: $V(\Omega) \to R$ satisfy conditions ((H 1), 1° to 3°). Then there is a subsequence $\varepsilon_n \to 0$ such that*

(3.3) $$\begin{cases} 1^\circ\ e_{\varepsilon_n} \to e_0 \ \textit{in}\ U(\Omega), \\ 2^\circ\ u_{\varepsilon_n} \rightharpoonup u(e_0) \equiv u_0 \ \textit{weakly in}\ V(\Omega)\ \textit{and strongly in}\ H(\Omega), \end{cases}$$

where $[e_0, u_0]$ is an optimal pair of problem $(\mathscr{B}_1)$.

In addition, if the functions Φ^ε satisfy condition ((H 1), 4°) and the operators $A(e)$ satisfy condition (H 2), then

(3.4) $$\begin{cases} 1^\circ\ u_{\varepsilon_n} \to u_0 \ \text{(strongly) in}\ V(\Omega), \\ 2^\circ\ A(e_{\varepsilon_n})\, u_{\varepsilon_n} \rightharpoonup A(e_0)\, u_0 \ \text{(weakly) in}\ V^*(\Omega), \\ 3^\circ\ \operatorname{grad} \Phi^{\varepsilon_n}(u_{\varepsilon_n}) \rightharpoonup f + Be_0 - A(e_0)\, u_0 \in \partial\Phi(u_0)\ (\textit{weakly})\ \textit{in}\ V^*(\Omega). \end{cases}$$

Proof. For every $\varepsilon > 0$ we have

(3.5) $$\mathfrak{L}^\varepsilon(u_\varepsilon) + (1/2)\, \|e_\varepsilon - e_0\|_{U(\Omega)}^2 \leqq \mathfrak{L}^\varepsilon(u_{0\varepsilon})$$

where $u_{0\varepsilon}$ is the solution to (3.1) with $e = e_0$. Let z be arbitrary but fixed in $D(\Phi)$. By (3.1) and the definition of gradient we have

(3.6) $$\langle A(e_0)\, u_{0\varepsilon}, u_{0\varepsilon} - z\rangle_{V(\Omega)} + \Phi^\varepsilon(u_{0\varepsilon}) - \Phi^\varepsilon(z) \leqq \langle f + Be_0, u_{0\varepsilon} - z\rangle_{V(\Omega)}$$
$$\text{for any} \quad z \in V(\Omega) \cap D(\Phi) .$$

Then by ((H0), 2°) and ((H 1) 1°, 2°) we see that $\{\|u_{0\varepsilon}\|_{V(\Omega)}\}$ is bounded for $\varepsilon \to 0$. Hence there exists $u_0^* \in V(\Omega)$ and a subsequence $\{u_{0\varepsilon_n}\}$ of $\{u_{0\varepsilon}\}$, $\varepsilon_n \to 0$ for $n \to \infty$ such that

$$u_{0\varepsilon_n} \to u_0^* \text{ (weakly) in } V(\Omega)\,. \tag{3.7}$$

Since the function $v \to \langle A(e)\, v, v \rangle_{V(\Omega)}$ is convex and continuous on $V(\Omega)$ for $e \in U_{ad}(\Omega)$ it is weakly lower semicontinuous.

Hence

$$\liminf_{n\to\infty} \langle A(e_0)\, u_{0\varepsilon_n}, u_{0\varepsilon_n} \rangle_{V(\Omega)} \geqq \langle A(e_0)\, u_0^*, u_0^* \rangle_{V(\Omega)}\,.$$

Together with ((H 1), 3°), (3.6) and (3.7), this yields

$$\langle A(e_0)\, u_0^*, u_0^* - z \rangle_{V(\Omega)} + \Phi(u_0^*) \leqq \Phi(z) + \langle f + Be_0, u_0^* - z \rangle_{V(\Omega)}$$

for all $z \in V(\Omega) \cap D(\Phi)$. Hence u_0^* is the solution of (1.2). Since the limit is unique we conclude that $u_{0\varepsilon} \to u_0^*$ in $V(\Omega)$, $u_0^* = u_0$ and $[e_0, u_0]$ is an optimal pair of $(\mathscr{B}_1)$. Then by Proposition (1.12) in ([3]) we can write ($u_{0\varepsilon} \to u_0$ strongly in $H(\Omega)$)

$$|\mathfrak{L}^\varepsilon(u_{0\varepsilon}) - \mathfrak{L}(u_0)| \leqq |\mathfrak{L}^\varepsilon(u_{0\varepsilon}) - \mathfrak{L}^\varepsilon(u_0)| + |\mathfrak{L}^\varepsilon(u_0) - \mathfrak{L}(u_0)| \to 0 \quad \text{for} \quad \varepsilon \to 0$$

because by (3.2) we see that $|\mathfrak{L}^\varepsilon(u_{0\varepsilon}) - \mathfrak{L}^\varepsilon(u_0)| \leqq L\|u_{0\varepsilon} - u_0\|_{H(\Omega)}$, where $L > 0$ is independent of ε.

Hence

$$\limsup_{\varepsilon\to 0} \left(\mathfrak{L}^\varepsilon(u_\varepsilon) + (1/2)\, \|e_\varepsilon - e_0\|^2_{U(\Omega)}\right) \leqq \mathfrak{L}(u_0) \tag{3.8}$$

follows from (3.5).

On the other hand, we have ($e_\varepsilon \in U_{ad}(\Omega)$ and $U_{ad}(\Omega)$ is compact)

$$e_{\varepsilon_n} \to e \quad \text{in } U(\Omega) \text{ for a subsequence } \{e_{\varepsilon_n}\} \text{ of } \{e_\varepsilon\}\,.$$

We conclude that

$$Be_{\varepsilon_n} \to Be \quad \text{strongly in} \quad V^*(\Omega)\,.$$

Let ω be arbitrary but fixed in $D(\Phi)$. By (1.9) and the definition of gradient we have

$$\langle A(e_{\varepsilon_n})\, u_{\varepsilon_n}, u_{\varepsilon_n} - \omega \rangle_{V(\Omega)} + \Phi^{\varepsilon_n}(u_{\varepsilon_n}) - \Phi^{\varepsilon_n}(\omega) \leqq \tag{3.9}$$
$$\leqq \langle f + Be_{\varepsilon_n}, u_{\varepsilon n} - \omega \rangle_{V(\Omega)} \quad \text{for any} \quad \omega \in V(\Omega)\,.$$

Then by ((H0), 1°, 2°) and ((H 1), 1°, 2°) we see that $\{\|u_{\varepsilon_n}\|_{V(\Omega)}\}$ is bounded for $\varepsilon_n \to 0$.

Hence there exists $u \in V(\Omega)$ and a subsequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to u \quad \text{(weakly) in} \quad V(\Omega)\,. \tag{3.10}$$

Next, by an analogue of (1.6) and (1.7), ((H 1), 3°) together with (3.9) yields

$$\langle A(e)\, u, u - \omega\rangle_{V(\Omega)} + \Phi(u) \leqq \Phi(\omega) + \langle f + Be, u - \omega\rangle_{V(\Omega)} \quad \text{for all} \quad \omega \in V(\Omega)\,.$$

Hence $u \equiv u(e)$ is the solution of (1.2). Since the limit is unique we conclude that $u_{\varepsilon_n} \rightharpoonup u(e)$ weakly in $V(\Omega)$ and strongly in $H(\Omega)$ and therefore

(3.11)
$$\lim_{\varepsilon_n \to 0} \mathfrak{L}^{\varepsilon_n}(u_{\varepsilon_n}) = \mathfrak{L}(u(e))\,.$$

We may write

$$\limsup_{\varepsilon_n \to 0} (1/2)\, \|e_{\varepsilon_n} - e_0\|^2_{U(\Omega)} \leqq \limsup_{\varepsilon_n \to 0} (\mathfrak{L}^{\varepsilon_n}(u_{\varepsilon_n}) + ((1/2)\, \|e_{\varepsilon_n} - e_0\|^2_{U(\Omega)}) + + \limsup_{\varepsilon_n \to 0} (-\mathfrak{L}^{\varepsilon_n}(u_{\varepsilon_n})) \leqq \mathfrak{L}(u_0) - \mathfrak{L}(u(e))$$

(which follows by virtue of (3.8), (3.11) and the definition of u_0).

We obviously have $\liminf_{\varepsilon_n \to 0} (1/2)\, \|e_{\varepsilon_n} - e_0\|^2_{U(\Omega)} \geqq 0$, and combining these two results, we arrive at $\lim_{\varepsilon_n \to 0} (1/2)\, \|e_{\varepsilon_n} - e_0\|^2_{U(\Omega)} = 0$.

Hence $e = e_0$ and $u(e) = u_0$ as claimed. Let us show (3.4, 1°). To this end we use the conditions ((H0), 2°), ((H 1), 4°). We can write

(3.12)
$$\begin{aligned}
\alpha\|u_\varepsilon - u_\lambda\|^2_{V(\Omega)} &\leqq \langle A(e_\varepsilon)\, u_\varepsilon - A(e_\varepsilon)\, u_\lambda, u_\varepsilon - u_\lambda\rangle_{V(\Omega)} = \\
&= \langle A(e_\lambda)\, u_\lambda - A(e_\varepsilon)\, u_\lambda, u_\varepsilon - u_\lambda\rangle_{V(\Omega)} - \langle \operatorname{grad} \Phi^\varepsilon(u_\varepsilon) - \\
&- \operatorname{grad} \Phi^\lambda(u_\lambda), u_\varepsilon - u_\lambda\rangle_{V(\Omega)} + \langle B(e_\varepsilon - e_\lambda), u_\varepsilon - u_\lambda\rangle_{V(\Omega)} \leqq \\
&\leqq c(\varepsilon + \lambda)\, (1 + \|\operatorname{grad} \Phi^\varepsilon(u_\varepsilon)\|^2_{V^*(\Omega)} + \|\operatorname{grad} \Phi^\lambda(u_\lambda)\|^2_{V^*(\Omega)} + \\
&+ \|B(e_\varepsilon - e_\lambda)\|_{V^*(\Omega)}\, \|u_\varepsilon - u_\lambda\|_{V(\Omega)} + c\|A(e_\lambda) - A(e_\varepsilon)\|_{L(V(\Omega), V^*(\Omega))}\, \|u_\lambda\|_{V(\Omega)} \times \\
&\times \|u_\varepsilon - u_\lambda\|_{V(\Omega)} \leqq c(\varepsilon + \lambda)\, (1 + \|\operatorname{grad} \Phi^\varepsilon(u_\varepsilon)\|^2_{V^*(\Omega)} + \|\operatorname{grad} \Phi^\lambda(u_\lambda)\|^2_{V^*(\Omega)}) + \\
&+ \|B(e_\varepsilon - e_\lambda)\|_{V^*(\Omega)}\, (\|u_\varepsilon\|_{V(\Omega)} + \|u_\lambda\|_{V(\Omega)}) + \\
&+ c\|A(e_\lambda) - A(e_\varepsilon)\,\|_{L(V(\Omega), V^*(\Omega))}\, (\|u_\lambda\|^2_{V(\Omega)} + \|u_\lambda\|_{V(\Omega)}\, \|u_\varepsilon\|_{V(\Omega)})\,.
\end{aligned}$$

Let $\varepsilon \to 0$, $\lambda \to 0$ be subsequences of $\{\varepsilon_n\}$. Since the operator B is continuous and $e_n \to e_0$ in $U(\Omega)$, $u_{\varepsilon_n} \rightharpoonup u_0$ (weakly) in $V(\Omega)$, we have $\lim_{\varepsilon, \lambda \to 0} \|B(e_\varepsilon - e_\lambda)\|_{V^*(\Omega)} = 0$ and $\|u_\lambda\|_{V(\Omega)} \leqq c$, $\|u_\varepsilon\|_{V(\Omega)} \leqq c$. On the other hand, by (1.6) we see that

$$\|\operatorname{grad} \Phi^\varepsilon\, u(e_\varepsilon)\|_{V^*(\Omega)} \leqq c\,,$$

$$\|\operatorname{grad} \Phi^\lambda\, u(e_\lambda)\|_{V^*(\Omega)} \leqq c \quad \text{for any} \quad \varepsilon \to 0,\ \lambda \to 0\,.$$

Then taking into account conditions (H 2) and passing to the limit in (3.12) for $\varepsilon, \lambda \to 0$ we obtain $\lim_{\varepsilon,\lambda\to 0} \|u_\varepsilon - u_\lambda\|_{V(\Omega)} = 0$ ($\{u_\varepsilon\}$ is a Cauchy sequence).

This means that $u_\varepsilon \to u$ (strongly) in $V(\Omega)$ for $\varepsilon \to 0$. In virtue of ((3.3), 2°) we get $u_\varepsilon \to u = u(e_0)$.

Hence for $\varepsilon \to 0$ (due to (1.6)) grad $\Phi^\varepsilon(u_\varepsilon) \to \xi = (f + Be_0) - A(e_0)\, u_0$ (weakly) in $V^*(\Omega)$. Then if ε_n tends to zero in the inequality

$$\Phi^{\varepsilon_n}(u_{\varepsilon_n}) - \Phi^{\varepsilon_n}(v) \leqq \langle \text{grad}\, \Phi^{\varepsilon_n}(u_{\varepsilon_n}), u_{\varepsilon_n} - v\rangle_{V(\Omega)}$$

for any $v \in V(\Omega)$, it follows by ((H 1), 2°, 3°) that $\Phi(u_0) \leqq \Phi(v) + \langle \xi, u_0 - v\rangle_{V(\Omega)}$ for any $v \in V(\Omega)$ as claimed.

Lemma 6. *There exists $p(e_\varepsilon) = p_\varepsilon \in V(\Omega)$ satisfying together with u_ε and e_ε the system*

$$(3.13)\qquad \begin{cases} 1^\circ\ A(e_\varepsilon)\, u_\varepsilon + \text{grad}\, \Phi^\varepsilon(u_\varepsilon) = f + Be_\varepsilon, \\ 2^\circ\ -A^*(e_\varepsilon)\, p_\varepsilon - \text{grad}\,(\text{grad}\, \Phi^\varepsilon(u_\varepsilon))\, p_\varepsilon = \text{grad}\, \mathfrak{L}^\varepsilon(u_\varepsilon)\,, \\ 3^\circ\ B^* p_\varepsilon \in \partial \mathfrak{P}(e_\varepsilon) + e_\varepsilon - e_0\,. \end{cases}$$

Proof is given in ([3]).

Now take the scalar product of ((3.13), 2°) with p_ε and use the coercivity condition ((H0), 2°) and the positivity of the operator grad (grad $\Phi^\varepsilon(u_\varepsilon)$) to get

$$\alpha \|p_\varepsilon\|_{V(\Omega)} \leqq \|\text{grad}\, \mathfrak{L}^\varepsilon(u_\varepsilon)\|_{H(\Omega)} \quad \text{for all} \quad \varepsilon > 0\,.$$

Since $\mathfrak{L}$ is locally Lipschitzian, the map $u \to \text{grad}\, \mathfrak{L}^\varepsilon(u)$ is uniformly bounded on bounded subsets.

Hence

$$\|p_\varepsilon\|_{V(\Omega)} \leqq c \quad \text{for all} \quad \varepsilon > 0\,.$$

Therefore, we may conclude that there exists a sequence $\varepsilon_n \to 0$ and $p_0 \in V(\Omega)$, $\eta \in$ $\in H(\Omega)$ such that

$$(3.14)\qquad \begin{cases} 1^\circ\ p_{\varepsilon_n} \to p_0 \quad \text{(weakly) in} \quad V(\Omega), \\ 2^\circ\ \text{grad}\, \mathfrak{L}^{\varepsilon_n}(u_{\varepsilon_n}) \to \eta \quad \text{(weakly) in} \quad H(\Omega)\,. \end{cases}$$

By Theorem 4 and ((3.14), 2°) it follows via Proposition (1.12), ([3°]) that $\eta \in \partial\mathfrak{L}(u_0)$, where $\partial\mathfrak{L}$ is the generalized gradient of $\mathfrak{L}$([3°]). Now, if ε tends to zero in ((3.13), 3°) then Lemma 3.2 and Theorem 1.2 ([3°]) imply that $B^* p_0^* \in \partial\mathfrak{P}(e_0)$.

We may view p_0 as a dual extremal element of problem ($\mathscr{B}_1$) and

$$(3.15)\qquad \begin{cases} A(e_0)\, u_0 + \Phi(u_0) = f + Be_0\,, \\ -A^*(e_0)\, p_0 - D^2\, \Phi(u_0)\, p_0 = \partial\mathfrak{L}(u_0)\,, \\ B^* p_0 \in \partial\mathfrak{P}(e_0) \end{cases}$$

as generalized first order necessary conditions of optimality. (The element $D^2\, \Phi(u_0)\, p_0 \in V^*(\Omega)$ is defined by $D^2\, \Phi(u_0)\, p_0 = \text{weak} - \lim_{n\to\infty} \text{grad}\,(\text{grad}\, \Phi^{\varepsilon_n}(u_{\varepsilon_n})\, p_{\varepsilon n})$.)

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Súhrn

OPTIMÁLNE RIADENIE VARIAČNEJ NEROVNICE S APLIKÁCIOU NA ROTAČNE SYMETRICKÚ ŠKRUPINU

JÁN LOVÍŠEK

Je študovaná úloha optimálneho riadenia variačnou nerovnicou s riadeniami v koeficientoch operátora nerovnice, a v pravej strane nerovnice. Dokazuje sa existencia optimálneho riadenia. V aplikácii na pružnú rotačne symetrickú škrupinu konštantnej hrúbky, meridianova krivka sa berie za návrhovú premennú. Je predpísaná jej dĺžka a objem, ktorý jej zodpovedá, derivácie do 2 rádu sú v daných hraniciach. Zaťaženie predstavuje hydrostatický tlak, vlastná tiaž a pretlak. Účelový funkcionál je integrál druhého invariantu napätia pri okrajových povrchoch škrupiny.

Резюме

ТЕОРИЯ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ ВАРИАЦИОННОГО НЕРАВЕНСТВА С ПРИМЕНЕНИЕМ К ОСЕСИММЕТРИЧНОЙ ОБОЛОЧКЕ

Ján Lovíšek

Цель этой работы заключается в следующем: Формулировать и решить задачу теории оптимального управления для вариационного неравенства с управлением в операторе и в правой части. Доказывается существование задачи оптимального управления, когда функция стоимости имеет квадратичный вид. Кроме того формулируется условие первого рода для задачи оптимального управления.

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