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THE QUANTIZED JACOBI POLYNOMIALS

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Summary. The author studies a system of polynomials orthogonal at a finite set of points, its weight approximating that of the orthogonal system of classical Jacobi polynomials.

Keywords: Clebsch-Gordan coefficients, orthogonal polynomials, weight approximation, curve fitting.

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1. MOTIVATION

The use of orthogonal polynomials for the needs of computational mathematics is traditional, wide and ever growing. Particularly, some discrete argument analogues of the classical orthogonal polynomials may serve to process samples. From the viewpoint of a numerical analyst the Clebsch-Gordan coefficients (see [1], p. 181 in the Russian edition) are very close to some "very" sampled or quantized Jacobi polynomials. Peculiarities of quantum optics (see [2] for comments on its problems) suggest to attain this – more classical than quantum – ideal by varying the results of physicists according to the remarks in [1], pp. 25, 72.

We merely wish to call the attention of the computing community to this fruitful tool by our defining the quantized Jacobi polynomials and deriving their properties. They have been partly known since the past century when P. L. Čebyšev quantized the Legendre polynomials which may be held for a special case of the Jacobi polynomials (see [3], 10·23). The process of generalization ended apparently in 1952 when Weber and Erdélyi [4] commented on the polynomials due to W. Hahn.

2. THE SPECIAL CASE

The polynomials $P_{0,n}(n_1), P_{1,n}(n_1), \ldots, P_{n,n}(n_1)$ of the respective degrees $0, 1, \ldots, n$ orthogonal on the set $\{0, 1, \ldots, n\}$ are called the Čebyšev orthogonal polynomials.

Note that in the notation for the Čebyšev polynomial $P_{k,n}(n_1)$ the first subscript k stands for the degree of the polynomial and the second n for the size of the set less one.

The Čebyšev polynomials are explicitly written as follows [3]:

(1) $$P_{k,n}(n_1) = \sum_{s=0}^{n} (-1)^s \binom{k}{s} \binom{k+s}{s} \frac{n_1^{[s]}}{n^{[s]}},$$

where $n_1^{[s]} = n_1(n_1 - 1) \ldots (n_1 - s + 1)$ and $n^{[s]} = n(n-1) \ldots (n - s + 1)$ are the corresponding generalized powers.

The set of polynomials $\{P_{k,n}(n_1)\}$ is not normed. Nevertheless, it can be shown that

(2) $$\|P_{k,n}\|^2 = (n+1)^{-1} \sum_{n_1=0}^{n} [P_{k,n}(n_1)]^2 = \frac{(n+k+1)^{[k]}}{(2k+1)\, n^{[k]}}.$$

Dividing the polynomials $P_{k,n}(n_1)$ by their norms, we obtain the normed set of the Čebyšev orthogonal polynomials

(3) $$\tilde{P}_{k,n}(n_1) = \|P_{k,n}\|^{-1} P_{k,n}(n_1) \quad (k = 0, 1, 2, \ldots, n).$$

We shall show that the polynomials $P_{k,n}(n_1)$ satisfy the finite difference equation

(4) $$L(P_{k,n})(n_1) \equiv -n_1(n - n_1 + 1) P_{k,n}(n_1 - 1) + \\ + [(n_1 + 1)(n - n_1) + n_1(n - n_1 + 1)] P_{k,n}(n_1) - \\ - (n_1 + 1)(n - n_1) P_{k,n}(n_1 + 1) = k(k+1) P_{k,n}(n_1).$$

Proof. The left-hand side of (4) is equal to

$$n_1(n - n_1 + 1)[P_{k,n}(n_1) - P_{k,n}(n_1 - 1)] + \\ + (n_1 + 1)(n - n_1)[P_{k,n}(n_1 + 1) - P_{k,n}(n_1)].$$

By simple algebra

$$L n_1^{[s]} = s[(s+1) n_1^{[s]} - s(n - s + 1) n_1^{[s-1]}],$$

or

$$L(P_{k,n})(n_1) = \sum_{s=0}^{n} (-1)^s \binom{k}{s} \binom{k+s}{s} \frac{s(s+1) n_1^{[s]} - s^2(n - s + 1) n_1^{[s-1]}}{n^{[s]}}.$$

Again by simple algebra

$$L(P_{k,n})(n_1) = k(k+1) P_{k,n}(n_1),$$

Q.E.D.

The Legendre polynomials are defined by the relation

(5) $$P_k(x) = \frac{1}{2^k k!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (x^2 - 1)^k, \quad x \in \langle -1, 1 \rangle.$$

Denote

(6) $$G_k(x) = P_k(1 - 2x), \quad x \in \langle 0, 1 \rangle.$$

We obtain

$$(7)\qquad G_k(x) = \frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k [x^k(1-x)^k]\,.$$

Hence

$$(8)\qquad G_k(x) = \sum_{s=0}^{k} (-1)^s \binom{k}{s}\binom{k+s}{s} x^s\,.$$

Since

$$(9)\qquad \int_{-1}^{1} [P_k(x)]^2 \,\mathrm{d}x = \frac{2}{2k+1}\,,$$

we have

$$(10)\qquad \int_0^1 [G_k(x)]^2 \,\mathrm{d}x = \int_{-1}^{1} [P_k(x)]^2 \frac{\mathrm{d}x}{2} = \frac{1}{2k+1}\,.$$

Evidently

$$(11)\qquad P_{k,n}(n_1) \to G_k(x)$$

as $n \to \infty$, $n_1/n \to x$, $x \in \langle 0, 1\rangle$.

It can be easily seen that

$$(12)\qquad \sum_{n_1=0}^{n} P_{k,n}(n_1) \frac{n!}{n_1!\,(n-n_1)!} x^{n_1}(1-x)^{n-n_1} = G_k(x)\,.$$

This transformation leads from a function of a discrete argument to a function of a continuous argument.

We shall prove that the following "quantum relation" holds:

$$(13)\qquad \int_0^1 G_k(x) \frac{(n+1)!}{n_1!\,(n-n_1)!} x^{n_1}(1-x)^{n-n_1} \,\mathrm{d}x = \frac{n^{[k]}}{(n+k+1)^{[k]}} P_{k,n}(n_1)\,.$$

First of all, we have

$$(13')\qquad \int_0^1 G_k(x) \frac{(n+1)!}{n_1!\,(n-n_1)!} x^{n_1}(1-x)^{n-n_1} \,\mathrm{d}x = \sum_{s=0}^{k} (-1)^s \binom{k}{s}\binom{k+s}{s} \times$$
$$\times \frac{(n_1+s)^{[s]}}{(n+s+1)^{[s]}}\,.$$

By the composition of the transformations

$$(14)\qquad T(f)(n_1) \equiv \sum_{n_1=0}^{n} f(n_1) \frac{n!}{n_1!\,(n-n_1)!} x^{n_1}(1-x)^{n-n_1} = g(x)$$

and

$$(15)\qquad S(g)(x) \equiv \int_0^1 g(x) \frac{(n+1)!}{n_1!\,(n-n_1)!} x^{n_1}(1-x)^{n-n_1} \,\mathrm{d}x = h(n_1)$$

we obtain the transformation

$$(16)\qquad h(n_1) = \sum_{n_1''=0}^{n} \binom{2n+1}{n}^{-1} \binom{n_1 + n_1''}{n_1} \binom{2n - n_1 - n_1''}{n - n_1} f(n_1'')$$

It can be easily seen that if (a) the eigenfunctions of this transformation are polynomials of degree k then the eigenvalues are

$$(17)\qquad \frac{n^{[k]}}{(n+k+1)^{[k]}} \quad (k = 0, 1, \dots, n)\,.$$

Consequently, if we prove the assumption (a) then we shall complete the proof of the relation (13).

It can be easily proved that the commutation relation $AB = BA$ holds, where A is the symmetrical matrix of the transformation (16) and B is the symmetrical matrix of the transformation

$$(16')\qquad q(n_1'') = -n_1''(n - n_1'' + 1)\, p(n_1'' - 1) +$$
$$+ [(n_1'' + 1)(n - n_1'') + n_1''(n - n_1'' + 1)\, p(n_1'') - (n_1'' + 1)(n - n_1'')\, p(n_1'' + 1)\,.$$

Thus the transformation (16) has the same eigenfunctions as the transformation (16′) which is already known to have the eigenfunctions $P_{n,k}(n_1)$. Q.E.D.

3. THE GENERAL CASE

Let $R, S \geqq 0$ be integers. The particular Jacobi polynomials will be denoted by

$$(18)\quad G_k^{R,S}(x) = 1 + \sum_{s=1}^{k} (-1)^s \binom{k}{s} \frac{(S+R+k+1)\dots(S+R+k+s)}{(R+1)\dots(R+s)}\, x^s\,.$$

The quantized Jacobi polynomials will be denoted by

$$(19)\quad P_{k,n}^{R,S}(n_1) = 1 + \sum_{s=1}^{k} (-1)^s \binom{k}{s} \frac{(S+R+k+1)\dots(S+R+k+s)}{(R+1)\dots(R+s)} \frac{n_1^{[s]}}{n^{[s]}}\,.$$

Let us further denote

$$(20)\qquad \bar{P}_{k,n}^{R,S}(n_1) = \sqrt{((n+1)\, V_n^{R,S}(n_1))}\, P_{k,n}^{R,S}(n_1)\,,$$

where

$$(21)\qquad V_n^{R,S}(n_1) = \binom{n}{n_1} \frac{(R+n_1)\dots(R+1)(S+n-n_1)\dots(S+1)}{(R+S+2)\dots(R+S+n+1)}\,.$$

We are going to show that the functions $\bar{P}_{k,n}^{R,S}(n_1)$ satisfy the finite difference equation

$$\text{(22)}\quad -\sqrt{n_1}\sqrt{(R+n_1)}\sqrt{(n-n_1+1)}\sqrt{(n-n_1+S+1)}\,\bar{P}_{k,n}^{R,S}(n_1-1)+$$
$$+\tfrac{1}{2}[(n_1+1)(n-n_1)+(R+n_1+1)(n-n_1+S)+$$
$$+n_1(n-n_1+1)+(R+n_1)(n-n_1+S+1)]\,\bar{P}_{k,n}^{R,S}(n_1)-$$
$$-\sqrt{(n_1+1)}\sqrt{(R+n_1+1)}\sqrt{(n-n_1)}\sqrt{(n-n_1+S)}\,\bar{P}_{k,n}^{R,S}(n_1+1)=$$
$$=\left[l(l+1)-\left(\frac{S-R}{2}\right)^2\right]\bar{P}_{k,n}^{R,S}(n_1)\,,$$

where $l=(S+R)/2+k$.

Proof. Equivalently, we show that the polynomials $P_{k,n}^{R,S}(n_1)$ satisfy the finite difference equation

$$\text{(23)}\quad L(P_{k,n}^{R,S})(n_1)\equiv n_1(S+n-n_1+1)\,[P_{k,n}^{R,S}(n_1)-P_{k,n}^{R,S}(n_1-1)-$$
$$-(R+n_1+1)(n-n_1)\,[P_{k,n}^{R,S}(n_1+1)-P_{k,n}^{R,S}(n_1)]=$$
$$=k(k+R+S+1)\,P_{k,n}^{R,S}(n_1).$$

By simple algebra we obtain

$$Ln_1^{[s]}=s[(S+R+s+1)\,n_1^{[s]}-(R+s)(n-s+1)\,n_1^{[s-1]}]\,,$$

or

$$L(P_{k,n}^{R,S})(n_1)=\sum_{s=0}^{n}(-1)^s\binom{k}{s}\frac{(S+R+k+1)\dots(S+R+k+s)}{(R+1)\dots(R+s)}\times$$
$$\times\frac{s(S+R+s+1)\,n_1^{[s]}-s(R+s)(n-s+1)\,n_1^{[s-1]}}{n^{[s]}}\,,$$

as well as

$$L(P_{k,n}^{R,S})(n_1)=k(k+S+R+1)\,P_{k,n}^{R,S}(n_1)\,,$$

which proves (23) and thus also (22).

It can be easily seen that

$$\text{(24)}\quad \sum_{n_1=0}^{n}\sqrt{(V_n^{R,S}(n_1))}\,P_{k,n}^{R,S}(n_1)\,\frac{\sqrt{n!}\sqrt{(R+S+n+1)!}}{\sqrt{n_1!}\sqrt{(n_1+R)!}\sqrt{(n-n_1)!}\sqrt{(n-n_1+S)!}}\times$$
$$\times\,x^{n_1+R/2}(1-x)^{n-n_1+S/2}=\sqrt{(v^{R,S}(x))}\,G_k^{R,S}(x)\,,$$

where

$$v^{R,S}(x)=\frac{(R+S+1)!}{R!\,S!}\,x^R(1-x)^S\,.$$

This transformation leads from a function of a discrete argument to a function of a continuous argument.

We shall prove that the following "quantum relation" holds:

$$(25) \quad \int_0^1 \sqrt{(v^{R,S}(x))}\, G_k^{R,S}(x) \frac{\sqrt{(n+1)!}\sqrt{(R+S+n+1)!}}{\sqrt{n_1!}\sqrt{(n_1+R)!}\sqrt{(n-n_1)!}\sqrt{(n-n_1+S)!}} \times$$

$$\times\, x^{n_1+R/2}(1-x)^{n-n_1+S/2}\, dx = \frac{n^{[k]}}{(n+R+S+1+k)^{[k]}} \times$$

$$\times \sqrt{((n+1)\, V_n^{R,S}(n_1))}\, P_{k,n}^{R,S}(n_1) \equiv \sqrt{((n+1)\, V_n^{R,S}(n_1))}\, Q_{k,n}^{R,S}(n_1)\,,$$

where

$$Q_{k,n}^{R,S}(n_1) = \sum_{s=0}^{k} (-1)^s \binom{k}{s} \frac{(S+R+k+1)\dots(S+R+k+s)}{(R+1)\dots(R+s)} \times$$

$$\times \frac{(n_1+R+s)^{[s]}}{(R+S+n+s+1)^{[s]}}\,.$$

By the composition of the transformations

$$(26) \quad T(f)(n_1) \equiv$$

$$\equiv \sum_{n_1=0}^{n} f(n_1) \frac{(n+1)^{-1}\sqrt{(n+1)!}\sqrt{(R+S+n+1)!}}{\sqrt{n_1!}\sqrt{(n_1+R)!}\sqrt{(n-n_1)!}\sqrt{(n-n_1+S)!}}\, x^{n_1+R/2}(1-x)^{n-n_1+S/2} =$$

$$= g(x)$$

and

$$(27) \quad S(g)(x) \equiv \int_0^1 g(x) \frac{\sqrt{(n+1)!}\sqrt{(R+S+n+1)!}}{\sqrt{n_1!}\sqrt{(n_1+R)!}\sqrt{(n-n_1)!}\sqrt{(n-n_1+S)!}} \times$$

$$\times\, x^{n_1+R/2}(1-x)^{n-n_1+S/2}\, dx = h(n_1)\,,$$

we obtain the transformation

$$(28) \quad h(n_1) = \sum_{n_1''=0}^{n} \binom{R+S+2n+1}{n}^{-1} \binom{n_1+n_1''+R}{n_1}^{1/2} \binom{n_1+n_1''+R}{n_1+R}^{1/2} \times$$

$$\times \binom{2n-n_1-n_1''+S}{n-n_1}^{1/2} \binom{2n-n_1-n_1''+S}{n-n_1+S}^{1/2} f(n_1'')\,.$$

It can be easily seen that if (a) the eigenfunctions of this transformation are of the form $\sqrt{((n+1)\, V_n^{R,S}(n_1))}\, U(n_1)$, where U is a polynomial of degree k, then the eigenvalues are

$$(29) \quad \lambda_k = \frac{n^{[k]}}{(R+S+n+k+1)^{[k]}} \; (k = 0, 1, 2, \dots, n)\,.$$

Consequently, if we prove the assumption (a) then we shall complete the proof of the relation (25).

It can be easily proved that the commutation relation

$$(30) \quad AB = BA$$

holds, where A is the symmetrical matrix of the transformation (28) and B is the symmetrical matrix of the transformation

(31)

$$q(n_1'') = -\sqrt{n_1''}\sqrt{(R+n_1'')}\sqrt{(n-n_1''+1)}\sqrt{(n-n_1''+S+1)}\,p(n_1''-1) +$$
$$+\tfrac{1}{2}[(n_1''+1)(n-n_1'') + (R+n_1''+1)(n-n_1''+S) +$$
$$+ n_1''(n-n_1''+1) + (R+n_1'')(n-n_1''+S+1)]\,p(n_1'') -$$
$$-\sqrt{(n_1''+1)}\sqrt{(R+n_1''+1)}\sqrt{(n-n_1'')}\sqrt{(n-n_1''+S)}\,p(n_1''+1)\,.$$

Thus the transformation (28) has the same eigenfunctions as the transformation (31) which is already known to have the eigenfunctions $\sqrt{((n+1)\,V_n^{R,S}(n_1))}\,P_{k,n}^{R,S}(n_1)$.

Q. E. D.

4. THE NORM OF THE QUANTIZED JACOBI POLYNOMIALS

We shall prove that

(32)
$$\|P_{k,n}^{R,S}\|^2 = \sum_{n_1=0}^{n} V_n^{R,S}(n_1)\,[P_{k,n}^{R,S}(n_1)]^2 =$$
$$= \frac{k!}{S+R+1+2k}\,\frac{(S+1)\ldots(S+k)}{(R+1)\ldots(R+k)}\,\frac{(R+S+1)!}{(R+S+k)!}\,\frac{(n+R+S+1+k)^{[k]}}{n^{[k]}},$$

which represents a certain generalization of the relation

(32′)
$$\|G_k^{R,S}\|^2 = \int_0^1 \frac{(R+S+1)!}{R!\,S!}\,x^R(1-x)^S\,[G_k^{R,S}(x)]^2\,\mathrm{d}x =$$
$$= \frac{k!}{S+R+1+2k}\,\frac{(S+1)\ldots(S+k)}{(R+1)\ldots(R+k)}\,\frac{(R+S+1)!}{(R+S+k)!}\,.$$

Proof. We are going to apply the method of singular values and singular functions. Let us consider the transformation T as an operator from the space $L^2(\mu)$ to the space $L^2(\mu_n)$ [and the transformation S as an operator from the space $L^2(\mu_n)$ to the space $L^2(\mu)$, where μ denotes the Lebesgue measure on the interval $\langle 0, 1\rangle$, and μ_n the measure ascribing to each point of the set $\{0, \ldots, n\}$ the weight $(n+1)^{-1}$].

[Evidently

(33)
$$T = S^{\mathrm{T}}\,,$$

i.e., T is the adjoint operator to S.]

The operator T has the orthonormal system

(34)
$$\|P_{k,n}^{R,S}\|^{-1}\sqrt{((n+1)\,V_n^{R,S}(n_1))}\,P_{k,n}^{R,S}(n_1)$$

in its domain and the orthonormal system

(35) $$\|G_k^{R,S}\|^{-1} \sqrt{(v^{R,S}(x))}\, G_k^{R,S}(x)$$

in its codomain, and it has the singular values

(36) $$v_k = \sqrt{\lambda_k} = \sqrt{\left(\frac{n^{[k]}}{(R+S+n+k+1)^{[k]}}\right)}.$$

So it replaces the function (34) by the function

(34′) $$\sqrt{(\lambda_k)}\, \|G_k^{R,S}\|^{-1} \sqrt{(v^{R,S}(x))}\, G_k^{R,S}(x).$$

The relation (24) implies that this operator replaces the function (34) by the function

(34″) $$\|P_{k,n}^{R,S}\|^{-1} \sqrt{(v^{R,S}(x))}\, G_k^{R,S}(x).$$

By comparison it follows that

$$\|P_{k,n}^{R,S}\|^2 = \lambda_k^{-1} \|G_k^{R,S}\|^2,$$

which is (32).

Dividing the functions $\bar{P}_{k,n}^{R,S}(n_1)$ by the norms of the polynomials $P_{k,n}^{R,S}(n_1)$, we obtain the set of normed and orthogonal functions

$$\tilde{\bar{P}}_{k,n}^{R,S}(n_1) = \|P_{k,n}^{R,S}\|^{-1} \bar{P}_{k,n}^{R,S}(n_1) \quad (k = 0, 1, 2, \ldots, n).$$

5. NOTES

(i) Similarly as the decomposition

$$T_{l_1}(u) \otimes T_{l_2}(u) = \sum_{l=|l_1-l_2|}^{l_1+l_2} T_l(u)$$

leads to the Clebsch-Gordan coefficients $C(l_1, l_2, l; j, k, j+k)$ (see [1], Eq. (5) at p. 182). the related decomposition

$$T_{l_1}(u) \otimes \overline{T_{l_2}(u)} = \overline{\sum_{l=|l_1-l_2|}^{l_1+l_2} T_l(u)}$$

where the bar stands for the complex conjugation helps to introduce analogous coefficients

$$P(l_1, l_2, l; j, k, k-j) = (-1)^{l_1-j}\, C(l_1, l_2, l; j, -k, j-k).$$

Putting $n_1 + R = m_1$, $n - n_1 = n_2$, $n - n_1 + S = m_2$ and expressing n_1, m_1, n_2, m_2 as $l_1 - j$, $l_2 - k$, $l_1 + j$, $l_2 + k$, respectively, we obtain that

$$\tilde{\bar{P}}_{k,n}^{R,S}(n_1) = \sqrt{(n+1)} \times$$
$$\times P\left(\frac{n}{2}, \frac{n+R+S}{2}, \frac{R+S}{2} + k; \frac{n}{2} - n_1, \frac{n+S-R}{2} - n_1, \frac{S-R}{2}\right).$$

(ii) Replacing the factorials by the gamma functions with arguments increased by one wherever necessary we see that the parameters R, S may take on all real values greater than -1.

(iii) Our considerations concerning $A = ST = SS^T$ partly imitated the proof of Theorem (v) in [5], 1c.3. Our implicit use of the Hilbert-Schmidt theorem is legitimate since

$$\sum_{n_1=0}^{n} \int_0^1 \left[\frac{\sqrt{n!}\,\sqrt{\Gamma(R+S+n+2)}}{\sqrt{n_1!}\,\sqrt{\Gamma(n_1+R+1)}\,\sqrt{(n-n_1)!}\,\sqrt{\Gamma(n-n_1+S+1)}} \times \right.$$
$$\left. \times\; x^{n_1+R/2}(1-x)^{n-n_1+S/2} \right]^2 \mathrm{d}x < \infty$$

for $R, S > -1$.

6. APPLICATION

Taking into account the fact that we have studied a system of polynomials orthogonal at a finite set of points with a weight approximating that of the orthogonal system of the Jacobi polynomials, the methods of application are standard (see [6], Ch. 6, [7], p. 34).

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Souhrn

KVANTOVANÉ JACOBIHO POLYNOMY

ANTONÍN LUKŠ

V práci se studuje systém polynomů ortogonálních na konečné množině bodů s vahou aproximující váhu ortogonálního systému Jacobiho polynomů.

Резюме

КВАНТОВАННЫЕ МНОГОЧЛЕНЫ ЯКОБИ

ANTONÍN LUKŠ

В работе исследуется конечная последовательность ортогональных многочленов дискретного переменного, для которых вес является приближением веса, соответствующего классическим многочленам Якоби.

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