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ON A SUPERCONVERGENT FINITE ELEMENT SCHEME FOR ELLIPTIC SYSTEMS III. OPTIMAL INTERIOR ESTIMATES

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Summary. Second order elliptic systems with boundary conditions of Dirichlet, Neumann's or Newton's type are solved by means of linear finite elements on regular uniform triangulations. Error estimates of the optimal order $O(h^2)$ are proved for the averaged gradient on any fixed interior subdomain, provided the problem under consideration is regular in a certain sense.

Keywords: finite elements, superconvergence, post-processing, averaged gradient, elliptic systems.
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1. INTRODUCTION

In Parts I and II of the present paper we proved the $O(h^{3/2})$ error estimate for averaged gradients of linear finite elements on the approximate domain Ω_h, assuming the $H^3(\Omega)$-regularity of the exact solution of an elliptic system. One can conjecture that an interior H^3-regularity could be sufficient to derive even better interior estimates for the averaged gradient on a subdomain Ω_0 of Ω if Ω_0 is independent of h. The present Part III is devoted to the verification of the latter conjecture.

In proving the main Theorem 4.1 of the paper [8] only an interior estimate for the norm

$$\|\boldsymbol{u}_h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega^*_h} \quad (\Omega_h^* \subset \Omega_h \subset \Omega)$$

was needed. Nevertheless, we employed the global estimate

$$\|\boldsymbol{u}_h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega} \leqq Ch^{3/2}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{f}\|_{2,\Omega}) \tag{1.1}$$

of Theorem 2.1. Thus a question arises, whether an interior $O(h^2)$ estimate can be deduced instead of (1.1) at least for some suitable "regular" boundary value problems. In what follows we shall give a positive answer to this question. As a consequence, we can easily derive an optimal $O(h^2)$-interior estimate for the averaged gradient. We employ a method of interior estimates, introduced by Nitsche, Schatz and Bramble in [14], [1] and some ideas of Zhu [16].

In Section 2 several auxiliary lemmas will be proven. In Section 3 we apply them to the proof of interior estimates of $\boldsymbol{u}_h - \boldsymbol{P}\boldsymbol{u}$ for the Dirichlet, Neumann and Newton type boundary value problems, assuming that the problem under consideration is regular in a certain sense. Finally, we derive $O(h^2)$-interior error estimates for the averaged gradient in Section 4. Similar results for convex domains were obtained by Westergren [15] who used a different approach.

Henceforth we shall write $\Omega_i \subset\subset \Omega$, whenever $\bar{\Omega}_i \subset \Omega$, $i = 1, 2, \ldots$. We keep the notations of Parts I, II and moreover define

$$B(\boldsymbol{w}, \boldsymbol{v}) = \int_{\Omega_1} \sum_{i,j=1}^{\varkappa} K_{ij} N_i(\boldsymbol{w}) N_j(\boldsymbol{v}) \, \mathrm{d}x ,$$

where $\Omega_1 \subset\subset \Omega$ is a fixed subdomain. For any Ω_i we set

$$\boldsymbol{W}_h(\Omega_i) = \{\boldsymbol{w}|_{\Omega_i} \mid \boldsymbol{w} \in \boldsymbol{W}_h\} ,$$
$$\boldsymbol{V}_h^0(\Omega_i) = \{\boldsymbol{v} \in \boldsymbol{W}_h \mid \operatorname{supp} \boldsymbol{v} \subset \Omega_i\} .$$

Definition 1.1. We say that the *Dirichlet problem* with homogeneous boundary conditions is *regular* if its weak solution for any right-hand side $\boldsymbol{f} \in [L^2(\Omega)]^M$ belongs to $[H^2(\Omega)]^M$.

We say that the *problem of Neumann's or Newton's type* $\mathscr{P}(\boldsymbol{f}, \boldsymbol{g})$ is *regular* if:

(i) it has a unique weak solution $\boldsymbol{u}$,

(ii) $\boldsymbol{u} \in [H^2(\Omega)]^M$,

(iii) the problem $\mathscr{P}(\boldsymbol{\varphi}, 0)$ has a unique solution $\boldsymbol{u}_0$ for any right-hand side $\boldsymbol{\varphi} \in \in [L^2(\Omega)]^M$; moreover $\boldsymbol{u}_0 \in [H^2(\Omega)]^M$.

Remark 1.1. For the homogeneous Dirichlet boundary conditions the regularity of the problem implies

$$\|\boldsymbol{u}\|_{2,\Omega} \leqq C\|\boldsymbol{f}\|_{0,\Omega} . \tag{1.2}$$

To see this let us recall the linear and continuous operator (see [8] for the classical formulation)

$$\boldsymbol{L}\boldsymbol{u} = \Big(\sum_{i,j=1}^{\varkappa} N_{jm}^*(K_{ij} N_i(\boldsymbol{u}))\Big)_{m=1}^M ,$$

which maps $[H_0^1(\Omega) \cap H^2(\Omega)]^M$ onto $[L^2(\Omega)]^M$ by the regularity of the problem. Since $\boldsymbol{L}$ is a one-to-one mapping ($\boldsymbol{L}$ maps onto $[L^2(\Omega)]^M$), we can employ the Theorem on Isomorphism (see e.g. [7], p. 216) to obtain that $\boldsymbol{L}^{-1}$ is continuous, i.e. (1.2) holds.

For the homogeneous boundary conditions of Neumann's or Newton's type we apply the same argument for the operator (cf. (4.4) in [9])

$$\boldsymbol{L}\colon \boldsymbol{V}_p \cap [H^2(\Omega)]^M \to \{\boldsymbol{f} \in [L^2(\Omega)]^M \mid (\boldsymbol{f}, \boldsymbol{q})_{0,\Omega} = 0 \;\; \forall \boldsymbol{q} \in \mathscr{P}\}$$

to verify (1.2).

Remark 1.2. The regularity of an elliptic problem with a single equation has been established for bounded domains with piecewise continuously differentiable boundary, which has no "concave angles" (see e.g. [5], [6], [10], [11], [12]). If the boundary $\partial\Omega$ is infinitely differentiable, the Dirichlet problem with homogeneous boundary conditions is regular even for general elliptic systems ([13], p. 260, or [3]). Some sufficient conditions for the regularity of elliptic systems with boundary conditions of Neumann's or Newton's type can be found in [4].

2. AUXILIARY LEMMAS

First of all we shall prove four lemmas, which will be used in the proof of interior error estimates. Here we follow the ideas of the papers [14], [1] and [16].

Lemma 2.1. *Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$ be arbitrary subdomains and let $\omega \in C_0^\infty(\Omega_2)$. Then for any $\mathbf{v}, \mathbf{w} \in [H^1(\Omega_2)]^M$*

$$B(\omega\mathbf{w}, \mathbf{v}) = B(\mathbf{w}, \omega\mathbf{v}) + \int_{\Omega_2} \sum_{m=1}^{M} w_m \left(\sum_{p=1}^{M} \left(v_p \mu_0^{mp} + \sum_{q=1}^{2} \frac{\partial v_p}{\partial x_q} \mu_q^{mp} \right) \right) \mathrm{d}x \,,$$

where the coefficients $\mu_0^{mp}, \mu_1^{mp}, \mu_2^{mp} \in C_0^\infty(\Omega_2)$ are independent of $\mathbf{v}$ and $\mathbf{w}$.

Proof. According to [8], Sec. 2, we may write

$$B(\mathbf{w}, \mathbf{v}) = \int_{\Omega_2} \sum_{i,j=1}^{\varkappa} K_{ij} \left(\sum_{m=1}^{M} \left(\sum_{t=1}^{2} n_{imt} \frac{\partial w_m}{\partial x_t} + n_{im} w_m \right) \right) \left(\sum_{p=1}^{M} \left(\sum_{q=1}^{2} n_{jpq} \frac{\partial v_p}{\partial x_q} + \right.\right.$$

$$\left.\left. + n_{jp} v_p \right) \right) \mathrm{d}x = \int_{\Omega_2} \left(\sum_{m,p,t,q} \alpha_{tq}^{mp} \frac{\partial w_m}{\partial x_t} \frac{\partial v_p}{\partial x_q} + \sum_{m,p,q} \beta_q^{mp} w_m \frac{\partial v_p}{\partial x_q} + \right.$$

$$\left. + \sum_{m,p,t} \gamma_t^{mp} \frac{\partial w_m}{\partial x_t} v_p + \sum_{m,p} \delta^{mp} w_m v_p \right) \mathrm{d}x \,,$$

where the coefficients $\alpha_{tq}^{mp}, \ldots, \delta^{mp}$ are polynomials on Ω. Since $\omega \in C_0^\infty(\Omega_2)$, integration by parts yields

$$B(\omega\mathbf{w}, \mathbf{v}) - B(\mathbf{w}, \omega\mathbf{v}) = \int_{\Omega_2} \left(\sum_{m,p,t,q} \alpha_{tq}^{mp} \left(\frac{\partial(\omega w_m)}{\partial x_t} \frac{\partial v_p}{\partial x_q} - \frac{\partial w_m}{\partial x_t} \frac{\partial(\omega v_p)}{\partial x_q} \right) + \right.$$

$$\left. + \sum_{m,p,q} \beta_q^{mp} \left(\omega w_m \frac{\partial v_p}{\partial x_q} - w_m \frac{\partial(\omega v_p)}{\partial x_q} \right) + \sum_{m,p,t} \gamma_t^{mp} \left(\frac{\partial(\omega w_m)}{\partial x_t} v_p - \frac{\partial w_m}{\partial x_t} \omega v_p \right) \right) \mathrm{d}x =$$

$$= \int_{\Omega_2} \left(\sum_{m,p,t,q} w_m \left(\alpha_{tq}^{mp} \frac{\partial v_p}{\partial x_q} \frac{\partial \omega}{\partial x_t} + \frac{\partial}{\partial x_t} \left(\alpha_{tq}^{mp} v_p \frac{\partial \omega}{\partial x_q} \right) \right) - \right.$$

$$\left. - \sum_{m,p,q} w_m \beta_q^{mp} v_p \frac{\partial \omega}{\partial x_q} + \sum_{m,p,t} w_m \gamma_t^{mp} v_p \frac{\partial \omega}{\partial x_t} \right) \mathrm{d}x \,,$$

whence the result as required. ∎

Lemma 2.2. *Let $\Omega_0 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_1 \subset\subset \Omega$ and let $\mathbf{w}_h \in \mathbf{W}_h(\Omega_3)$ be such that*

$$B(\mathbf{w}_h, \boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in \mathbf{V}_h^0(\Omega_2)\,. \tag{2.1}$$

Then for sufficiently small h we have

$$\|\mathbf{w}_h\|_{1,\Omega_0} \leqq c\|\mathbf{w}_h\|_{0,\Omega_3}\,. \tag{2.2}$$

Proof. Following some ideas of [14] and [1], we assume that $\Omega_0 \subset\subset \Omega^* \subset\subset \Omega_2$ and let $\omega \in C_0^\infty(\Omega^*)$ be such that $\omega = 1$ in Ω_0. Setting

$$\mathbf{w}_h^* = \omega \mathbf{w}_h\,,$$

we get

$$\|\mathbf{w}_h\|_{1,\Omega_0} \leqq \|\mathbf{w}_h^*\|_{1,\Omega_2} \leqq \|\mathbf{w}_h^* - \mathbf{R}\mathbf{w}_h^*\|_{1,\Omega_2} + \|\mathbf{R}\mathbf{w}_h^*\|_{1,\Omega_2}\,, \tag{2.3}$$

where $\mathbf{R}\mathbf{w}_h^* \in \mathbf{V}_h^0(\Omega_2)$ is defined in the following way

$$B(\mathbf{w}_h^* - \mathbf{R}\mathbf{w}_h^*, \boldsymbol{\varphi}) = 0 \quad \forall \boldsymbol{\varphi} \in \mathbf{V}_h^0(\Omega_2)\,. \tag{2.4}$$

Note that $\mathbf{R}\mathbf{w}_h^*$ is uniquely determined, as

$$\|\mathbf{w}\|_{1,\Omega_2}^2 \leqq C\,B(\mathbf{w}, \mathbf{w}) \quad \forall \mathbf{w} \in [H_0^1(\Omega_2)]^M\,. \tag{2.5}$$

Therefore, the use of (2.5) for $\mathbf{w} = \mathbf{w}_h^* - \mathbf{R}\mathbf{w}_h^*$ and (2.4) yields

$$\|\mathbf{w}_h^* - \mathbf{R}\mathbf{w}_h^*\|_{1,\Omega_2}^2 \leqq C\|\mathbf{w}_h^* - \mathbf{R}\mathbf{w}_h^*\|_{1,\Omega_2}\|\mathbf{w}_h^* - \boldsymbol{\eta}\|_{1,\Omega_2} \quad \forall \boldsymbol{\eta} \in \mathbf{V}_h^0(\Omega_2)\,.$$

Thus applying the Leibniz rule on any $T \in \mathscr{T}_h$, $T \cap \Omega_2 \neq \emptyset$, we obtain

$$\|\mathbf{w}_h^* - \mathbf{R}\mathbf{w}_h^*\|_{1,\Omega_2}^2 \leqq Ch^2 \sum_{T} |\mathbf{w}_h^*|_{2,T}^2 \leqq C'h^2\|\mathbf{w}_h\|_{1,\tilde{\Omega}_2}^2\,, \tag{2.6}$$

where

$$\tilde{\Omega}_2 = \bigcup_{T \cap \Omega_2 \neq \emptyset} T\,,$$

as $\mathbf{w}_h$ is linear on every triangle T.

We estimate now the last term in (2.3). For $\mathbf{R}\mathbf{w}_h^* \neq 0$ we obtain by (2.5) and (2.4)

$$\|\mathbf{R}\mathbf{w}_h^*\|_{1,\Omega_2} \leqq CB(\mathbf{R}\mathbf{w}_h^*, \boldsymbol{\psi}) = CB(\mathbf{w}_h^*, \boldsymbol{\psi})\,, \tag{2.7}$$

where

$$\boldsymbol{\psi} = \frac{\mathbf{R}\mathbf{w}_h^*}{\|\mathbf{R}\mathbf{w}_h^*\|_{1,\tilde{\Omega}_2}}$$

so that

$$\|\boldsymbol{\psi}\|_{1,\tilde{\Omega}_2} = 1\,. \tag{2.8}$$

Using Lemma 2.1 we arrive at

$$B(\mathbf{w}_h^*, \psi) = B(\mathbf{w}_h, \psi^*) + \int_{\Omega_2} \sum_{m=1}^{M} w_{mh} \left(\sum_{p=1}^{M} \left(\psi_p \mu_0^{mp} + \frac{\partial \psi_p}{\partial x_1} \mu_1^{mp} + \frac{\partial \psi_p}{\partial x_2} \mu_2^{mp} \right) \right) \mathrm{d}x =$$

$$= \mathrm{B}(\mathbf{w}_h, \psi^*) + I\,,$$

where $\psi^* = \omega\psi$, and $\mu_i^{mp} \in C_0^\infty(\Omega_2)$ are independent of $\mathbf{w}_h$ and ψ. Then employing the assumption (2.1), we find that

$$B(\mathbf{w}_h^*, \psi) = B(\mathbf{w}_h, \psi^* - \boldsymbol{\eta}) + I \quad \forall \boldsymbol{\eta} \in \mathbf{V}_h^0(\Omega_2)\,.$$

From the definition of I and (2.8) it follows that

$$B(\mathbf{w}_h^*, \psi) \leqq C \|\mathbf{w}_h\|_{1,\Omega_2} \|\psi^* - \boldsymbol{\eta}\|_{1,\Omega_2} + C' \|\mathbf{w}\|_{0,\Omega_2}\,,$$

$$B(\mathbf{w}_h^*, \psi) \leqq C(h \|\mathbf{w}_h\|_{1,\Omega_2} + \|\mathbf{w}_h\|_{0,\Omega_2})\,.$$

Hence, by (2.7), (2.3) and (2.6) we get

$$\|\mathbf{w}_h\|_{1,\Omega_0} \leqq C(h \|\mathbf{w}_h\|_{1,\tilde{\Omega}_2} + \|\mathbf{w}_h\|_{0,\Omega_2}) \leqq C' \|\mathbf{w}_h\|_{0,\Omega_3}\,,$$

as follows from the Inverse Inequality (see [2], p. 142). ∎

In proving the next lemma, we cannot use the well-known result, following from Aubin-Nitsche trick (see e.g. [2]) directly, since we have a slightly different definition of the interpolation operator $\boldsymbol{P}$ in some triangles adjacent to the boundary. Let us introduce an "ideal" approximation of the homogeneous Dirichlet problem as follows:

$$\boldsymbol{u}_0^h \in \mathbf{V}_h\,, \quad a(\boldsymbol{u}_0^h, \mathbf{v}_h) = (\boldsymbol{f}, \mathbf{v}_h)_{0,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h\,.$$

(Note that in [8]–(2.49) the integral on the right-hand side is evaluated only approximately.)

Lemma 2.3. *Let $\Omega \in \mathscr{C}^3(d)$ be convex and let the Dirichlet problem be regular. Then if $\boldsymbol{f} \in [L^2(\Omega)]^M$ and $\bar{\boldsymbol{u}} \equiv 0$,*

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega} \leqq Ch^2 \|\boldsymbol{u}\|_{2,\Omega} \tag{2.9}$$

holds for sufficiently small h.

Proof. We easily derive that

$$a(\boldsymbol{u} - \boldsymbol{u}_0^h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h\,, \tag{2.10}$$

$$\boldsymbol{u} - \boldsymbol{u}_0^h \in \mathbf{V}\,. \tag{2.11}$$

Define the adjoint problem:

find $\mathbf{v} \in \mathbf{V}$ such that

$$a(\mathbf{v}, \varphi) = (\boldsymbol{u} - \boldsymbol{u}_0^h, \varphi)_{0,\Omega} \quad \forall \varphi \in \mathbf{V}\,. \tag{2.12}$$

Using (2.10), (2.11) and (2.12), we come to

$$(2.13) \qquad \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega}^2 = a(\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{v}) = a(\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}) \leqq$$

$$\leqq C\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \|\boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}\|_{1,\Omega} .$$

Further we show that

$$(2.14) \qquad \|\boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}\|_{1,\Omega} \leqq Ch\|\boldsymbol{v}\|_{2,\Omega} \quad \forall \boldsymbol{v} \in [H_0^1(\Omega) \cap H^2(\Omega)]^M .$$

Obviously,

$$(2.15) \qquad \|\boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}\|_{1,\Omega}^2 = \|\boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}\|_{1,\Omega_h}^2 + \sum_{T_c} \|\boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}\|_{1,T_c}^2 ,$$

where T_c are segments adjacent to the curved parts of $\partial\Omega$. However,

$$(2.16) \qquad \sum_{T_c} \|v_m - Pv_m\|_{1,T_c}^2 = \sum_{T_c} \|v_m\|_{1,T_c}^2 \leqq \|v_m\|_{1,\Omega^\varepsilon}^2 ,$$

where Ω^ε is a boundary strip, the width of which is $\varepsilon \leqq Ch^2$ (see (Hl) in [8]). Applying Iljin's inequality (see (2.31) in [8]), we get

$$(2.17) \qquad \|v_m\|_{1,\Omega^\varepsilon}^2 \leqq C\varepsilon\|v_m\|_{2,\Omega}^2 \leqq C_1 h^2 \|v_m\|_{2,\Omega}^2 .$$

Now, the combination of (2.16), (2.17) and (2.15) together with the standard estimate on Ω_h leads to (2.14).

From the regularity of the problem we have $\boldsymbol{u} \in [H^2(\Omega)]^M$. Next, we derive the estimate

$$(2.18) \qquad \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \leqq Ch\|\boldsymbol{u}\|_{2,\Omega} .$$

In fact, the relations (2.10) and (2.11) give

$$C\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega}^2 \leqq a(\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{u} - \boldsymbol{u}_0^h) =$$

$$= a(\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{u} - \boldsymbol{P}\boldsymbol{u}) \leqq C_1\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \|\boldsymbol{u} - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega}$$

and thus referring to (2.14), we get

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \leqq C\|\boldsymbol{u} - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega} \leqq C_1 h\|\boldsymbol{u}\|_{2,\Omega} .$$

Substituting (2.18) and (2.14) into the inequality (2.13), we obtain

$$(2.19) \qquad \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega}^2 \leqq Ch^2\|\boldsymbol{u}\|_{2,\Omega}\|\boldsymbol{v}\|_{2,\Omega} .$$

As the problem is regular,

$$\|\boldsymbol{v}\|_{2,\Omega} \leqq C\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega} ,$$

which together with (2.19) implies (2.9). ∎

Note that in case of non-convex Ω, we are able to prove (2.14) only for $\boldsymbol{v} \in (H^3(\Omega))^M$.

In the next lemma the type of boundary conditions is not specified. A similar lemma has been proved also in [16].

Lemma 2.4. *Let Ω be a bounded domain with a Lipschitz boundary and let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, where Ω_1 is a convex domain with the boundary of the class $\mathscr{C}^\infty$. Assume that $\mathbf{u}|_{\Omega_1} \in [H^3(\Omega_1)]^M$ and that the following "interior Galerkin equation"*

(2.20)
$$B(\mathbf{u} - \mathbf{u}_0^h, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^0(\Omega_1)$$

is fulfiled. Then for sufficiently small h

$$\|\mathbf{u}_0^h - \mathbf{P}\mathbf{u}\|_{1,\Omega_0} \leqq C(h^2\|\mathbf{u}\|_{3,\Omega_1} + \|\mathbf{u} - \mathbf{u}_0^h\|_{0,\Omega_3}),$$

where Ω_3 is an arbitrary domain such that $\Omega_0 \subset\subset \Omega_3 \subset\subset \Omega_1$.

Proof. Let us choose $\Omega_0 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_1$ and a function $\omega \in C_0^\infty(\Omega_3)$ such that

$$\omega \equiv 1 \quad \text{in} \quad \Omega_2,$$

and let us put $\mathbf{u}^* = \omega \mathbf{u}$. Suppose that $\mathbf{u}^{*h} \in \mathbf{V}_h(\Omega_1)$ satisfies

(2.21)
$$a(\mathbf{u}^* - \mathbf{u}^{*h}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h(\Omega_1).$$

Here $\mathbf{V}_h(\Omega_1)$ is the space of finite elements on Ω_1, defined likewise $\mathbf{V}_h$ on Ω. Let $\mathbf{P}^1: [V(\Omega_1) \cap C(\bar{\Omega}_1)]^M \to \mathbf{V}_h(\Omega_1)$ be the corresponding interpolation mapping. From the $\mathbf{V}$-ellipticity of $a(\cdot, \cdot)$, from the fact that $\mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^* \in \mathbf{V}_h(\Omega_1)$ and since any function of $\mathbf{V}_h(\Omega_1)$ can be extended by zero to an element of $\mathbf{V}$, we obtain

(2.22)
$$C\|\mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*\|_{1,\Omega}^2 \leqq a(\mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*, \mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*) =$$
$$= a(\mathbf{u}^* - \mathbf{P}^1\mathbf{u}^*, \mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*) = B(\mathbf{u}^* - \mathbf{P}^1\mathbf{u}^*, \mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*).$$

Moreover, for every $\mathbf{v} \in \mathbf{V}_h(\Omega_1)$ it is

(2.23)
$$|B(\mathbf{u}^* - \mathbf{P}^1\mathbf{u}^*, \mathbf{v})| \leqq Ch^2\|\mathbf{u}^*\|_{3,\Omega_3} \|\mathbf{v}\|_{1,\Omega}.$$

This inequality follows from some parts of the proof of Lemma 2.1 in [8], where the terms a_1, a_2 were estimated. The support $\operatorname{supp}(\mathbf{P}^1\mathbf{u})$ can be covered by a set, which is the union of the congruent parallelograms A_t, i.e.

$$\operatorname{supp}(\mathbf{P}^1\mathbf{u}^*) \subset \bigcup_t A_t \subset \Omega.$$

Then $S_m^* = 0$, since $\mathbf{u}^* - \mathbf{P}^1\mathbf{u}^* = 0$ outside the union, and thus (2.23) is valid.

Taking $\mathbf{v} = \mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*$ in (2.23), we find from (2.22) that

(2.24)
$$\|\mathbf{u}^{*h} - \mathbf{P}^1\mathbf{u}^*\|_{1,\Omega} \leqq Ch^2\|\mathbf{u}^*\|_{3,\Omega_3} \leqq C_1 h^2 \|\mathbf{u}\|_{3,\Omega_1}.$$

Further we prove that

$$B(\mathbf{u}_0^h - \mathbf{u}^{*h}, \varphi) = 0 \quad \forall \varphi \in \mathbf{V}_h^0(\Omega_2).$$

As $\mathbf{V}_h^0(\Omega_1) \subset \mathbf{V}_h(\Omega_1)$, we obtain by (2.21) that

$$B(\mathbf{u}^* - \mathbf{u}^{*h}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^0(\Omega_1).$$

Hence, setting

$$\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}^* = (1 - \omega)\,\boldsymbol{u}\,,$$

$$\boldsymbol{w}_h = \boldsymbol{u}_0^h - \boldsymbol{u}^{*h}\,,$$

and using the assumption (2.20), we find that

$$B(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{v}) = B((\boldsymbol{u} - \boldsymbol{u}^*) - (\boldsymbol{u}_0^h - \boldsymbol{u}^{*h}), \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h^0(\Omega_1)\,.$$

Consequently,

$$B(\boldsymbol{w}_h, \varphi) = B(\boldsymbol{w}, \varphi) = 0 \quad \forall \varphi \in \boldsymbol{V}_h^0(\Omega_2) \subset \boldsymbol{V}_h^0(\Omega_1)\,,$$

since $\operatorname{supp} \boldsymbol{w} \subset \Omega - \bar{\Omega}_2$. Now the assumptions of Lemma 2.2 are fulfilled and thus by (2.2)

$$(2.25) \quad \|\boldsymbol{u}_0^h - \boldsymbol{u}^{*h}\|_{1,\Omega_0} \leqq C\|\boldsymbol{u}_0^h - \boldsymbol{u}^{*h}\|_{0,\Omega_3} \leqq C(\|\boldsymbol{u}^* - \boldsymbol{u}_0^h\|_{0,\Omega_3} + \|\boldsymbol{u}^* - \boldsymbol{u}^{*h}\|_{0,\Omega_3})\,.$$

Using Lemma 2.3 for the convex domain Ω_1, we have by the definition (2.21)

$$(2.26) \qquad \|\boldsymbol{u}^* - \boldsymbol{u}^{*h}\|_{0,\Omega} \leqq Ch^2\|\boldsymbol{u}^*\|_{2,\Omega_1}\,.$$

Note that the Dirichlet problem is regular due to the smoothness of the boundary $\partial\Omega_1$ (see [3], p. 370; [13], p. 260). As

$$\|\boldsymbol{u}^*\|_{k,\Omega} \leqq C\|\boldsymbol{u}\|_{k,\Omega_1} \quad \text{for} \quad k = 0, 1, 2, 3\,,$$

the use of (2.26) and (2.25) yields

$$(2.27) \qquad \|\boldsymbol{u}_0^h - \boldsymbol{u}^{*h}\|_{1,\Omega_0} \leqq C(\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega_3} + h^2\|\boldsymbol{u}\|_{2,\Omega_1})\,.$$

Finally, the combination of (2.24) and (2.27) implies

$$\|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq \|\boldsymbol{u}_0^h - \boldsymbol{u}^{*h}\|_{1,\Omega_0} + \|\boldsymbol{u}^{*h} - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq$$
$$\leqq C(h^2\|\boldsymbol{u}\|_{3,\Omega_1} + \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega_3})\,,$$

since $\boldsymbol{P}^1\boldsymbol{u} \equiv \boldsymbol{P}\boldsymbol{u}$ on Ω_0. ■

3. INTERIOR ESTIMATES

In this section we apply the previous auxiliary lemmas to the proof of the interior error estimates for $\boldsymbol{u}^h - \boldsymbol{P}\boldsymbol{u}$. Again, we have to distinguish Dirichlet's problem and the problems of Neumann's or Newton's type.

Theorem 3.1. *Let $\Omega \in \mathscr{C}^3(d)$ be a bounded convex domain and let $\Omega_0 \subset\subset \Omega$. Let a regular Dirichlet problem with $\boldsymbol{f} \in [H^2(\Omega)]^M$ and $\bar{\boldsymbol{u}} = 0$ be given on Ω. Then a subdomain Ω_1 exists such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and for sufficiently small h*

$$\|\boldsymbol{u}^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1})\,.$$

Proof. As $\boldsymbol{u}_0^h - \boldsymbol{u}^h \in \boldsymbol{V}_h$, we have by [8], Lemma 2.2, that

$$C_1\|\boldsymbol{u}_0^h - \boldsymbol{u}^h\|_{1,\Omega}^2 \leqq a(\boldsymbol{u}_0^h - \boldsymbol{u}^h, \boldsymbol{u}_0^h - \boldsymbol{u}^h) =$$
$$= (\boldsymbol{f}, \boldsymbol{u}_0^h - \boldsymbol{u}^h)_{0,\Omega} - (\boldsymbol{f}, \boldsymbol{u}_0^h - \boldsymbol{u}^h)_{0,\Omega}^* \leqq Ch^2\|\boldsymbol{f}\|_{2,\Omega}\,\|\boldsymbol{u}_0^h - \boldsymbol{u}^h\|_{1,\Omega}\,,$$

that is

$$\|\boldsymbol{u}_0^h - \boldsymbol{u}^h\|_{1,\Omega} \leqq Ch^2\|\boldsymbol{f}\|_{2,\Omega}\,. \tag{3.1}$$

Let us define

$$\Omega_1 = \bigcup_{i=1}^{I} K_i\,,$$

where K_i and $\tilde{K}_i$ are open circles such that $\tilde{K}_i \subset\subset K_i \subset\subset \Omega$ and

$$\bar{\Omega}_0 \subset \bigcup_{i=1}^{I} \tilde{K}_i\,.$$

Since $\boldsymbol{u} - \boldsymbol{u}_0^h$ satisfies the "interior equation" (2.20) on K_i and $\boldsymbol{u}|_{K_i} \in [H^3(K_i)]^M$, Lemma 2.4 holds on every K_i, i.e.,

$$\|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\tilde{K}_i} \leqq C(h^2\|\boldsymbol{u}\|_{3,K_i} + \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega}) \leqq Ch^2(\|\boldsymbol{u}\|_{3,K_i} + \|\boldsymbol{u}\|_{2,\Omega})\,,$$

where the last inequality follows from Lemma 2.3. Consequently,

$$\|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0}^2 \leqq \sum_i \|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\tilde{K}_i}^2 \leqq Ch^4(\sum_i \|\boldsymbol{u}\|_{3,K_i}^2 + \|\boldsymbol{u}\|_{2,\Omega}^2)\,. \tag{3.2}$$

Now the combination of (3.1) and (3.2) leads to

$$\|\boldsymbol{u}^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq \|\boldsymbol{u}^h - \boldsymbol{u}_0^h\|_{1,\Omega_0} + \|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq$$
$$\leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \sum_i \|\boldsymbol{u}\|_{3,K_i} + \|\boldsymbol{u}\|_{2,\Omega})\,.$$

Employing finally the regularity of the Dirichlet problem, we arrive at the estimate to be proved. ∎

Next we prove an analogue of Theorem 3.1 for regular Newton's or Neumann's boundary value problems even in case of non-convex domains. We introduce the definitions of the solutions $\boldsymbol{u}$, $\boldsymbol{u}_0^h$ and $\boldsymbol{u}^h$:

$$\boldsymbol{u} \in \boldsymbol{W}\,, \quad ((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} + (\boldsymbol{g}, \boldsymbol{v})_{0,\partial\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{W}\,, \tag{3.3}$$

$$\boldsymbol{u}_0^h \in \boldsymbol{W}_h\,, \quad ((\boldsymbol{u}_0^h, \boldsymbol{v}_h)) = (\boldsymbol{f}, \boldsymbol{v}_h)_{0,\Omega} + (\boldsymbol{g}, \boldsymbol{v}_h)_{0,\partial\Omega} \quad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h\,, \tag{3.4}$$
$$\boldsymbol{u}^h \in \boldsymbol{W}_h\,, \quad ((\boldsymbol{u}^h, \boldsymbol{v}_h)) = (\boldsymbol{f}, \boldsymbol{v}_h)_{0,\Omega_h}^* + (\boldsymbol{g}, \boldsymbol{v}_h)_{0,\partial\Omega} \quad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h\,,$$

where $((\boldsymbol{u}, \boldsymbol{v})) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{v})$ is introduced in [9], Sec. 2, and $(\boldsymbol{f}, \boldsymbol{v}_h)_{0,\Omega_h}^*$, W_h have been defined in [8], Sec. 2; $(\boldsymbol{W}_h = (W_h)^M)$. First we prove an auxiliary lemma which is similar to Lemma 2.3.

Lemma 3.1. *Let $\Omega \in \mathscr{C}^3(d)$ and let the problem* (3.3) *be regular. Then for sufficiently small h it is*

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega} \leqq Ch^2 \|\boldsymbol{u}\|_{2,\Omega} .$$

Proof. According to (3.3) and (3.4),

$$((\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{v}_h)) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h . \tag{3.5}$$

Define the adjoint problem with the right-hand side $\boldsymbol{u} - \boldsymbol{u}_0^h$ and with the homogeneous Newton or Neumann conditions on $\partial\Omega$, i.e., we look for $\boldsymbol{v} \in \boldsymbol{W}$ such that

$$((\boldsymbol{v}, \boldsymbol{z})) = (\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{z})_{0,\Omega} \quad \forall \boldsymbol{z} \in \boldsymbol{W} . \tag{3.6}$$

Hence, by (3.6) we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega}^2 = ((\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{v})) . \tag{3.7}$$

Referring to (3.5), we come to

$$((\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{v})) = ((\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{v} - \boldsymbol{P}\boldsymbol{v})) \leqq C\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \|\boldsymbol{v} - \boldsymbol{P}\boldsymbol{v}\|_{1,\Omega} . \tag{3.8}$$

(Recall that the operator $\boldsymbol{P}$ has been defined in [9], Sec. 2). However,

$$\|\boldsymbol{w} - \boldsymbol{P}\boldsymbol{w}\|_{1,\Omega} \leqq Ch\|\boldsymbol{w}\|_{2,\Omega} \quad \forall \boldsymbol{w} \in [H^2(\Omega)]^M , \tag{3.9}$$

since for the single components w_m we have (we drop the subscript m)

$$\|w - Pw\|_{1,\Omega}^2 \leqq \sum_{T_c} \|w - Pw\|_{1,T_c \cup T_k}^2 + \sum_{T_t} \|w - Pw\|_{1,T_t}^2 + \sum_{T_k} \|w - Pw\|_{1,T_k}^2 .$$

The first and second sum can be estimated as in the proof of Lemma 2.1 of [9] (the relations (2.5) and (2.6)). Thus e.g. for the first sum we get the bound

$$Ch^2 \sum_{\tilde{T}_c} |Ew|_{2,\tilde{T}_c}^2 \leqq C_1 h^2 |Ew|_{2,\tilde{\Omega}}^2 \leqq C_2 h^2 \|w\|_{2,\Omega}^2 ,$$

where $\tilde{T}_c$ is a triangle of Fig. 2 in [9] and Ew is the Calderon extension of w to $\tilde{\Omega}$, $\Omega \subset\subset \tilde{\Omega}$. Similarly we estimate the second sum with the help of (2.6) in [9] and the third sum has the standard estimation. Hence, (3.9) holds.

From the coercitivity of the bilinear form $((\cdot, \cdot))$ and (3.5) we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega}^2 \leqq C((\boldsymbol{u} - \boldsymbol{u}_0^h, \boldsymbol{u} - \boldsymbol{P}\boldsymbol{u})) \leqq C_1 \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \|\boldsymbol{u} - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega} .$$

Utilizing further (3.9) and the regularity of the problem (3.3), we get

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{1,\Omega} \leqq Ch\|\boldsymbol{u}\|_{2,\Omega} . \tag{3.10}$$

On the basis of (3.9), (3.10) and (3.8), we find by (3.7) that

$$\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega}^2 \leqq Ch^2 \|\boldsymbol{u}\|_{2,\Omega} \|\boldsymbol{v}\|_{2,\Omega} . \tag{3.11}$$

Using the regularity of the adjoint problem and Remark 1.1, we deduce

$$\|\boldsymbol{v}\|_{2,\Omega} \leqq C\|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega} .$$

Consequently, the lemma follows immediately from (3.11). ∎

Theorem 3.2. *Let $\Omega \in \mathscr{C}^3(d)$ be a bounded domain and let the problem* (3.3) *be regular. If $\boldsymbol{f} \in [H^2(\Omega)]^M$ and $\Omega_0 \subset\subset \Omega$, then a subdomain Ω_1 exists such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and for sufficiently small h*

$$\|\boldsymbol{u}^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1}).$$

Proof. By Lemma 2.2 in [9] we have

$$C\|\boldsymbol{u}_0^h - \boldsymbol{u}^h\|_{1,\Omega}^2 \leqq ((\boldsymbol{u}_0^h - \boldsymbol{u}^h, \boldsymbol{u}_0^h - \boldsymbol{u}^h)) =$$
$$= (\boldsymbol{f}, \boldsymbol{u}_0^h - \boldsymbol{u}^h)_{0,\Omega} - (\boldsymbol{f}, \boldsymbol{u}_0^h - \boldsymbol{u}^h)_{0,\Omega_h}^* \leqq Ch^2\|\boldsymbol{f}\|_{2,\Omega}\,\|\boldsymbol{u}_0^h - \boldsymbol{u}^h\|_{1,\Omega},$$

and thus

$$\|\boldsymbol{u}_0^h - \boldsymbol{u}^h\|_{1,\Omega} \leqq Ch^2\|\boldsymbol{f}\|_{2,\Omega}. \tag{3.12}$$

Consider the same set of circles K_i, $\tilde{K}_i$ as in the proof of Theorem 3.1. Since $\boldsymbol{u}$ and $\boldsymbol{u}_0^h$ satisfy the "interior Galerkin equation" (2.20), Lemma 2.4 holds for every $\tilde{K}_i$, i.e.,

$$\|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\tilde{K}_i} \leqq C(h^2\|\boldsymbol{u}\|_{3,K_i} + \|\boldsymbol{u} - \boldsymbol{u}_0^h\|_{0,\Omega}) \leqq Ch^2(\|\boldsymbol{u}\|_{3,K_i} + \|\boldsymbol{u}\|_{2,\Omega}),$$

where the last inequality follows from Lemma 3.2. Consequently,

$$\|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0}^2 \leqq \sum_i \|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\tilde{K}_i}^2 \leqq Ch^4(\sum_i \|\boldsymbol{u}\|_{3,K_i}^2 + \|\boldsymbol{u}\|_{2,\Omega}^2). \tag{3.13}$$

Gathering (3.12) and (3.13), we derive

$$\|\boldsymbol{u}^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq \|\boldsymbol{u}^h - \boldsymbol{u}_0^h\|_{1,\Omega_0} + \|\boldsymbol{u}_0^h - \boldsymbol{P}\boldsymbol{u}\|_{1,\Omega_0} \leqq$$
$$\leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \sum_{i=1}^{I} \|\boldsymbol{u}\|_{3,K_i} + \|\boldsymbol{u}\|_{2,\Omega}). \quad \blacksquare$$

4. INTERIOR SUPERCONVERGENCE

By a slight modification of the argument used in Section 4 of [8], we can prove an optimal $O(h^2)$-interior estimate for the averaged gradient. We use the same notation as in [8].

Theorem 4.1. *Let $\Omega \in \mathscr{C}^3(d)$ be a bounded convex domain. Let a regular Dirichlet problem with $\boldsymbol{f} \in [H^2(\Omega)]^M$ and $\bar{\boldsymbol{u}} = 0$ be given on Ω. Let $\Omega_0 \subset\subset \Omega$ be an arbitrary subdomain. Then a subdomain Ω_1 exists such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and*

$$\left\|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}_h)\right\|_{0,\Omega_0} \leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1})$$

holds for h small enough.

Proof. We may write

$$
(4.1) \qquad \left\| \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}_h) \right\|_{0,\Omega_0} \leqq \sum_{j=1}^{M} \|\text{grad}\, u_j - G_h(u_{jh})\|_{0,\Omega_0} \leqq
$$

$$
\leqq \sum_{j=1}^{M} \left(\|\text{grad}\, u_j - G_h(Pu_j)\|_{0,\Omega_0} + \|G_h(Pu_j - u_{jh})\|_{0,\Omega_0} \right).
$$

Let Ω' be a subdomain such that

$$
\Omega_0 \subset\subset \Omega' \subset\subset \Omega,
$$

and let Ω^*_{h0} be the smallest union of triangles $T \in \mathscr{T}^*_h$ such that Ω_0 is contained in it, i.e.,

$$
\Omega_0 \subset \Omega^*_{h0}.
$$

Obviously, we have

$$
\Omega^*_{h0} \subset \Omega'
$$

for sufficiently small h.

Following the proof of Theorem 3.1 of [8], we obtain

$$
(4.2) \qquad \|\text{grad}\, u_j - G_h(Pu_j)\|_{0,\Omega_0} \leqq \|\text{grad}\, u_j - G_h(Pu_j)\|_{0,\Omega^*_{h0}} \leqq
$$

$$
\leqq Ch^2 |u_j|_{3,\Omega^*_{h1}} \leqq Ch^2 |u_j|_{3,\Omega'},
$$

where

$$
\Omega^*_{h1} = \bigcup_{T \subset \Omega^*_{h0}} D(T)
$$

is the union of all "neighbourhoods" $D(T)$ (cf. (3.2) in [8]) and therefore

$$
\Omega^*_{h1} \subset \Omega'
$$

holds for sufficiently small h.

Arguing as in the proof of Theorem 4.1 of [8], we deduce

$$
(4.3) \qquad \|G_h(Pu_j - u_{jh})\|_{0,\Omega_0} \leqq \|G_h(Pu_j - u_{jh})\|_{0,\Omega^*_{h0}} \leqq
$$

$$
\leqq 3\sqrt{13}\, \|Pu_j - u_{jh}\|_{1,\Omega^*_{h1}} \leqq 3\sqrt{13}\, \|Pu_j - u_{jh}\|_{1,\Omega'}.
$$

Let us apply Theorem 3.1 to get the estimate

$$
(4.4) \qquad \|\boldsymbol{Pu} - \boldsymbol{u}_h\|_{1,\Omega'} \leqq Ch^2 (\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1}),
$$

where $\Omega_1 \supset\supset \Omega'$ is the finite union of circles, constructed as in the proof of Theorem 3.1.

Combining (4.3) and (4.4), we obtain

$$
(4.5) \qquad \sum_{j=1}^{M} \|G_h(Pu_j - u_{jh})\|_{0,\Omega_0} \leqq C_1 \|\boldsymbol{Pu} - \boldsymbol{u}_h\|_{1,\Omega'} \leqq Ch^2 (\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1}).
$$

Inserting (4.2) and (4.5) into (4.1), we arrive at

$$\left\| \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}_h) \right\|_{0,\Omega_0} \leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1}) . \quad \blacksquare$$

Theorem 4.2. *Let $\Omega \in \mathscr{C}^3(d)$ and let a regular Neumann or Newton boundary value problem with $\boldsymbol{f} \in [H^2(\Omega)]^M$ be given on Ω. Let $\Omega_0 \subset\subset \Omega$ be an arbitrary subdomain. Then a subdomain Ω_1 exists such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and*

$$\left\| \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}^h) \right\|_{0,\Omega_0} \leqq Ch^2(\|\boldsymbol{f}\|_{2,\Omega} + \|\boldsymbol{u}\|_{3,\Omega_1} + \|\boldsymbol{u}\|_{2,\Omega})$$

holds for h small enough.

Proof. The proof is the same as that of Theorem 4.1 with the only change: instead of Theorem 3.1 we apply Theorem 3.2. $\blacksquare$

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Souhrn

O JEDNOM SUPERKONVERGENTNÍM SCHÉMATU V METODĚ KONEČNÝCH PRVKŮ PRO ELIPTICKÉ SYSTÉMY III. OPTIMÁLNÍ VNITŘNÍ ODHADY

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

Systémy eliptických rovnic 2. řádu s okrajovými podmínkami Dirichletova, Neumannova nebo Newtonova typu se řeší pomocí lineárních konečných prvků na regulárních a pravidelných triangulacích. Odvozují se optimální odhady chyby řádu $O(h^2)$ pro tzv. zprůměrovaný gradient na každé pevné vnitřní podoblasti za předpokladu, že uvažovaný problém je v jistém smyslu regulární. Článek je přímým pokračováním prací [8, 9], kde jsou odvozeny globální odhady chyby řádu $O(h^{3/2})$.

Резюме

ОБ ОДНОЙ СУПЕРСХОДЯЩЕЙСЯ СХЕМЕ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ III. ОПТИМАЛЬНЫЕ ВНУТРЕННИЕ ОЦЕНКИ

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

Системы эллиптических уравнений 2-го порядка с граничными условиями типа Дирихле, Неймана или Ньютона решаются при помощи линейных конечных элементов на регулярных однородных триангуляциях. Получаются оптимальные оценки ошибок порядка $O(h^2)$ для так называемого осредненного градиента на каждой фиксированной внутренней подобласти при предположении, что рассматриваемая проблема является регулярной в некотором смысле.

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