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DIFFERENTIAL STABILITY OF SOLUTIONS TO AIR QUALITY CONTROL PROBLEMS IN URBAN AREA

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Summary. The convex optimal control problem for a system described by the parabolic equation is considered. The form of the right derivative of an optimal solution with respect to the parameter is derived. The applications to an air quality control problem are discussed. Numerical result are provided.

Keywords: Differential stability, convex optimal control, air quality control problem.

1. INTRODUCTION

The paper concerns an optimal control problem for the atmospheric pollutant dispersion in the urban scale.

The computer forecasting model of the system [2] is used for the control purposes. The model is based on the two-dimensional advection – diffusion equation, which is numerically solved by a combined FE – characteristics method [3]. The convex, state and control constrained, optimal control problem is formulated.

The method of the sensitivity analysis of constrained optimization problems [8 – 12] is applied to the problem under consideration. We refer also to [5] for the results in the general convex case for the constraints depending on the parameter.

The Lipschitz continuity of an optimal control with respect to the coefficients of the state equation is obtained. The directional derivative of an optimal control with respect to the parameter is derived in the form of an optimal solution to an auxiliary optimal control problem. Numerical results for an example are presented. The results have been announced in [13].

2. FORECASTING MODEL

We consider the urban-scale forecasting model [2] intended for short-term prediction of air pollution in a large city. The horizontal scale of the simulated dispersion process is 20 – 40 km, while the time horizon of model's prediction is 1 ÷ 3 days.

The input data can be divided, in general, into the following three groups: i) the structural data independent of time, ii) the meteorological forecast, iii) characteristics of the emission sources.

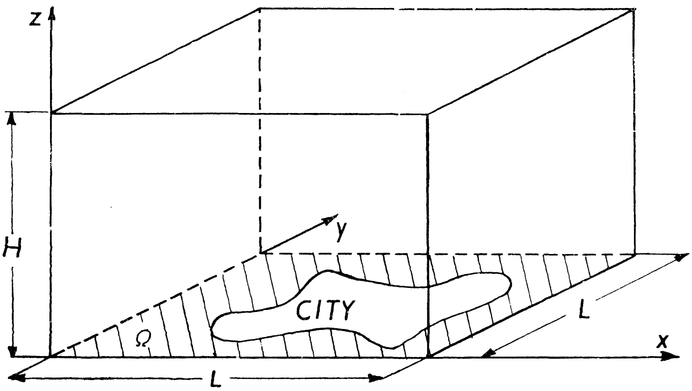


Fig. 1. Domain of simulation

The physical process of pollutant dispersion in the atmosphere is considered in a square domain $\Omega = (L \times L)$ (see Fig. 1). The process is described by two-dimensional, averaged over the mixing height H, advection–diffusion equation of the form [1, 6]

$$(2.1) \qquad \frac{\partial c}{\partial t} + \mathbf{w} \, . \, \nabla c - K_H \, \Delta c + \gamma c = \bar{Q} + (E - v_d c)/H \quad \text{in} \quad \Omega \times (0, T)$$

together with the boundary conditions

$$(2.2) \qquad \frac{\partial c}{\partial \mathbf{n}} = 0 \quad \text{on} \quad S^+ ,$$

$$c = 0 \quad \text{on} \quad S^- ,$$

and the initial condition

$$(2.3) \qquad c(0) = y^0 \quad \text{in} \quad \Omega ,$$

where

$$S^+ = \{(x, t) \in \partial\Omega \times (0, T) | \, \mathbf{w} \, . \, \mathbf{n} \geqq 0\} ,$$

$$S^- = \{(x, t) \in \partial\Omega \times (0, T) | \, \mathbf{w} \, . \, \mathbf{n} < 0\} .$$

Here we use the following notation:

c – pollutant concentration in [μg/m³],

$\mathbf{w} = \text{col}\,(u, v)$ – wind velocity vector in [m/s],

K_H	– horizontal diffusion coefficient in $[m^2/s]$,
$\bar{Q}$	– averaged over height H pointwise emission field in $[\mu g/m^3\ s]$,
E	– area emission field in $[\mu g/m^2\ s]$,
v_d	– dry deposition coefficient in $[m/s]$,
γ	– wet deposition factor depending on the precipitation intensity.

The values of the wind field vector $\mathbf{w}(x, y, t)$ in the domain $\Omega \times [0, T]$ are predicted by a special procedure which calculates successively:

i) the averaged over $\Omega \times [0, T]$ value $\mathbf{w}_0$ based on the meteorological forecast,

ii) topographical correction $\mathbf{w}_t$ depending on the ground topography and aerodynamical roughness,

iii) thermal correction $\mathbf{w}_\theta$ depending on a city "heat island" effect.

Finally, we set

$$\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_t + \mathbf{w}_\theta . \tag{2.4}$$

The model generates a sequence of forecasts of pollutant concentration within a period T, which is discretized with a time interval δT. Its length is determined by the frequency of the introducing meteorological data (in our case $\delta T = 6$ hrs). Each time interval is segmented with a discretization step τ of a numerical procedure solving the advection – diffusion equation (see Fig. 2).

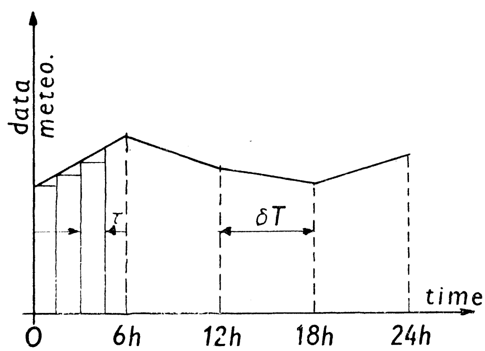


Fig. 2. Time discretization

All time-dependent data at the inner points of the interval δT are linearly interpolated. The initial-boundary value problem (2.1)–(2.3) is numerically solved by an effective combination of the method of characteristics with the finite element procedure [3].

The forecasting model was tested on real data for Warsaw and Krakow areas [2].

3. CONTROL PROBLEM

Basing on the forecasting model, the real-time emission control problem for the system of sources covering the area was formulated. The general idea of controlling consists in minimizing the environmental impacts by redistributing the production (emission) among the set of selected sources, according to the meteorological situation.

In order to define an optimal control problem, we introduce the state equation, the cost functional and the constraints in the following form:

State equation: find concentration $c = c(\mathbf{u}; \mathbf{x}; t)$, for a given vector function $\mathbf{u} \in L^2(0, T; R^N)$, $(\mathbf{x}, t) \in \Omega \times (0, T)$, which satisfies the parabolic equation

$$(3.1) \qquad \frac{\partial c}{\partial t} + \mathbf{w} \,.\, \nabla c - K_H \, \Delta c + \gamma c = Q + \sum_{i=1}^{n} \chi_i \, F_i(u_i) \,, \quad \text{in} \quad \Omega \times (0, T)$$

with the boundary conditions (2.2) and the initial conditions (2.3). Here $\mathbf{u} = \operatorname{col}(u_1, \ldots, u_N)$ denotes the control.

Cost functional:

$$(3.2) \quad J(\mathbf{u}) = \tfrac{1}{2}\alpha_{11} \int_0^T \int_\Omega r \max^2 (0, c(\mathbf{u}) - c_d) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \tfrac{1}{2}\alpha_{12} \int_0^T \sum_{i=1}^{N} \beta_i (u_i - \tilde{u}_i)^2 \, \mathrm{d}t \,,$$

where α_{11}, α_{12}, β_i, $i = 1, \ldots, N$ are given constants such that $\alpha_{11} \geqq 0$, $\alpha_{12} \geqq 0$, $\beta_i > 0$; $r \in L^\infty(\Omega \times (0, T))$ is a given function, $r(\mathbf{x}, t) \geqq 0$, $(\mathbf{x}, t) \in \Omega \times (0, T)$, and c_d is a given function.

Constraints:

state constraint:

$$(3.3) \qquad \alpha_{22} \int_0^T \int_\Omega r \max^2 (0, c(\mathbf{u}) - c_d) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leqq K_1 \,;$$

control constraints:

$$(3.4) \qquad {}_*u_i(t) \leqq u_i(t) \leqq {}^*u_i(t) \,,$$

$$(3.5) \qquad - D_i \leqq \frac{\mathrm{d}u_i}{\mathrm{d}t} \leqq D_i \,, \quad i = 1, \ldots, N \,,$$

$$(3.6) \qquad \sum_{i \in N_j} a_{ij} \, u_i(t) \geqq b_j \,, \quad j = 1, \ldots, M \,, \quad N_j \subset \{1, \ldots, N\}$$

almost everywhere on $(0, T)$.

The functions $\chi_i(\mathbf{x})$ in (3.1) describe the location of the controlled sources; $F_i(u_i)$ relate the emission to the production level. The factor $r(\mathbf{x}, t)$ in (3.2), (3.3) is a region weight function, $c_d(\mathbf{x})$ denotes the admissible level of pollution. The second term in (3.2) constitutes the cost of deviation of production levels u_i from the desired economic values $\tilde{u}_i$. The inequalities (3.4), (3.5) represent technological constraints, while (3.6) reflects demand requirements b_j imposed on homogeneous groups of plants N_j.

The parameters α_{ij} in (3.2), (3.3) make it possible to formulate a variety of optimization problems, ranging from the minimization of global costs (both environmental and control) for $\alpha_{11}, \alpha_{12} > 0$, $\alpha_{22} = 0$ to the minimization of outlays with environmental constraint, when $\alpha_{11} = 0$, $\alpha_{12}, \alpha_{22} > 0$.

4. SENSITIVITY ANALYSIS

We denote by $K \subset H^1(0, T; R^N)$ the set of admissible controls of the form

(4.1) $$K = \{\boldsymbol{u} \in H^1(0, T; R^N) \mid \text{the element } \boldsymbol{u} \text{ satisfies (3.4), (3.5) and (3.6), the element } c(\boldsymbol{u}) \text{ satisfies (3.3)}\}.$$

The state equation (3.1) has a unique solution $c = c(\boldsymbol{u})$ determined for a given control $\boldsymbol{u} \in L^2(0, T; R^N)$ and for fixed constants K_H, γ; $K_H > 0$.

Let $\varepsilon \in [0, \delta)$ be a parameter; we denote

(4.2) $$d^\varepsilon = (K_H^\varepsilon, \gamma^\varepsilon),$$

where

(4.3) $$K_H^\varepsilon = K_H + \varepsilon K_H' + o(\varepsilon),$$

(4.4) $$\gamma^\varepsilon = \gamma + \varepsilon\gamma' + o(\varepsilon);$$

here

$$|o(\varepsilon)|/\varepsilon \to 0 \quad \text{with} \quad \varepsilon \downarrow 0.$$

Let us denote by $c_\varepsilon = c_\varepsilon(\boldsymbol{u})$ a unique solution of the state equation:

(3.1) $$\frac{\partial c_\varepsilon}{\partial t} + \boldsymbol{w} \,.\, \nabla c_\varepsilon - K_H^\varepsilon \,\Delta c_\varepsilon + \gamma^\varepsilon c_\varepsilon = Q + \sum_{i=1}^{N} \chi_i\, F_i(u_i) \quad \text{in} \quad \Omega \times (0, T)$$

with boundary conditions (2.2) and initial condition (2.3). Let us consider the following optimal control problem:

Problem (P_ε). Find an element $\boldsymbol{u}^\varepsilon = \boldsymbol{u}(d^\varepsilon) \in K$ which minimizes the cost functional

(4.5) $$J_\varepsilon(\boldsymbol{u}) = \frac{\alpha_{11}}{2} \int_0^T \int_\Omega r \max^2 (0, c_\varepsilon(\boldsymbol{u}) - c_d) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t + \frac{\delta}{2} \sum_{i=1}^{N} \int_0^T \left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)^2 \mathrm{d}t + \frac{\alpha_{12}}{2} \sum_{i=1}^{N} \int_0^T \beta_i (u_i - \tilde{u}_i)^2 \,\mathrm{d}t$$

over the set (4.1).

In order to ensure the uniqueness of an optimal control we assume that there exists a constant $\sigma > 0$ such that for $\varepsilon \in [0, \delta)$ we have

$$(4.6) \qquad \langle D\, J_\varepsilon(\boldsymbol{u}) - D\, J_\varepsilon(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v} \rangle > \sigma \|\boldsymbol{u} - \boldsymbol{v}\|^2_{H^1(0,T;R^N)} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in K\,.$$

Here $D\, J_\varepsilon(\boldsymbol{u})$ denotes the gradient of the functional (4.5).

The condition (4.6) is satisfied e.g. for $F_i(u_i) = u_i, \alpha_{11} > 0,\ \delta > 0, \alpha_{12} > 0, \beta_i > 0,\ i = 1, \ldots, N$.

It can be verified that there exists an optimal control $\boldsymbol{u}^\varepsilon \in K$ which is determined as the first component of the solution of the following, uniquely solvable [9], optimality system.

Optimality System for Problem (P_ε)

Find $(\boldsymbol{u}^\varepsilon, c^\varepsilon, p^\varepsilon)$ such that

$$(4.7) \qquad \frac{\partial c^\varepsilon}{\partial t} + \boldsymbol{w} \,.\, \nabla c^\varepsilon - K_H^\varepsilon \,\Delta c^\varepsilon + \gamma^\varepsilon c^\varepsilon = Q + \sum_{i=1}^{N} \chi_i\, F_i(u_i^\varepsilon) \quad \text{in} \quad \Omega \times (0, T)\,,$$

$$(4.8) \qquad \frac{\partial c^\varepsilon}{\partial n} = 0 \quad \text{on} \quad S^+\,,$$

$$(4.9) \qquad c^\varepsilon = 0 \quad \text{on} \quad S^-\,,$$

$$(4.10) \qquad c^\varepsilon(0) = y^0 \quad \text{on} \quad \Omega\,;$$

$$(4.11) \qquad -\frac{\partial p^\varepsilon}{\partial t} - \operatorname{div}(\boldsymbol{w} p^\varepsilon) - K_H^\varepsilon \,\Delta p^\varepsilon + \gamma^\varepsilon p^\varepsilon = \alpha_{11} r \max\{0, c^\varepsilon - c_d\} \quad \text{in} \quad \Omega \times (0, T)\,,$$

$$(4.12) \qquad p^\varepsilon = 0 \quad \text{on} \quad S^-\,,$$

$$(4.13) \qquad K_H^\varepsilon \frac{\partial p^\varepsilon}{\partial n} + \boldsymbol{w} \,.\, \boldsymbol{n} p^\varepsilon = 0 \quad \text{on} \quad S^+\,,$$

$$(4.14) \qquad p^\varepsilon(x, T) = 0 \quad \text{on} \quad \Omega\,,$$

$$\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, \ldots, u_N^\varepsilon) \in K\,;$$

$$(4.15) \qquad \sum_{i=1}^{N} \left\{ \alpha_{11} \int_0^T \int_\Omega \chi_i\, F_i'(u_i^\varepsilon)\, p^\varepsilon (u_i - u_i^\varepsilon)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \delta \int_0^T \frac{\mathrm{d}u_i^\varepsilon}{\mathrm{d}t} \left(\frac{\mathrm{d}u_i}{\mathrm{d}t} - \frac{\mathrm{d}u_i^\varepsilon}{\mathrm{d}t} \right) \mathrm{d}t + \right.$$
$$\left. + \alpha_{12} \int_0^T \beta_i (u_i^\varepsilon - \tilde{u}_i)(u_i - u_i^\varepsilon)\, \mathrm{d}t \right\} \geqq 0 \quad \forall \boldsymbol{u} = (u_1, \ldots, u_N) \in K\,.$$

Theorem 1. *Assume that the condition (4.6) is satisfied. Then there exists a constant C such that for $\varepsilon > 0$, ε small enough,*

$$(4.16) \qquad \|\boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon\|_{H^1(0,T;R^N)} \leqq C\varepsilon\,.$$

Proof. We denote

$$J_\varepsilon(\mathbf{u}) = I(L_\varepsilon B(\mathbf{u})) + \omega(\mathbf{u}), \tag{4.17}$$

where

$$I(y) \stackrel{\text{def}}{=} \alpha_{11}/2 \int_0^T \int_\Omega r \max^2 \{0, y - c_d\} \, d\mathbf{x} \, dt, \quad \forall y \in L^2(\Omega \times [0, T]), \tag{4.18}$$

$$B(\mathbf{u}) \stackrel{\text{def}}{=} \sum_{i=1}^N \chi_i F_i(u)_i, \quad \forall \mathbf{u} = (u_1, \ldots, u_N) \in L^2(0, T; R^N), \tag{4.19}$$

$$\omega(\mathbf{u}) = \frac{\delta}{2} \sum_{i=1}^N \int_0^T \left(\frac{du_i}{dt}\right)^2 dt + \frac{\alpha_{12}}{2} \sum_{i=1}^N \int_0^T \beta_i(u_i - \tilde{u}_i)^2 \, dt, \quad \forall \mathbf{u} \in H^1(0, T; R^N) \tag{4.20}$$

and for any $\phi \in L^2(\Omega \times (0, T))$ the element $y = L_\varepsilon \phi$ is given by the unique solution of the parabolic equation

$$\frac{\partial y}{\partial t} + \mathbf{w} \,.\, \nabla y - K_H^\varepsilon \, \Delta y + \gamma^\varepsilon y = \phi \quad \text{in} \quad \Omega \times (0, T) \tag{4.21}$$

with the boundary conditions

$$y = 0 \quad \text{on} \quad S^-, \tag{4.22}$$

$$\frac{\partial y}{\partial n} = 0 \quad \text{on} \quad S^+, \tag{4.23}$$

and the initial condition

$$y(\mathbf{x}, 0) = y^0(\mathbf{x}) \quad \text{on} \quad \Omega. \tag{4.24}$$

It can be verified that

$$d\, J_\varepsilon(\mathbf{u}; v) \stackrel{\text{def}}{=} \lim_{\mu \downarrow 0} (J_\varepsilon(\mathbf{u} + \mu \mathbf{v}) - J_\varepsilon(\mathbf{u}))/\mu = \tag{4.25}$$

$$= (DI(L_\varepsilon B(\mathbf{u})), L_\varepsilon B'(\mathbf{u}) \, \mathbf{v})_{L^2(\Omega \times (0,T))} + (D\, \omega(\mathbf{u}), \mathbf{v})_{H^1(0,T;R^N)};$$

here

$$B'(\mathbf{u}) \, \mathbf{v} \stackrel{\text{def}}{=} \sum_{i=1}^N \chi_i F_i'(u_i) \, v_i, \quad \forall \mathbf{v} \in L^2(0, T; R^N); \tag{4.26}$$

$$(DI(y), \phi)_{L^2(Q)} = \alpha_{11} \int_0^T \int_\Omega r\phi \max \{0, y - c_d\} \, d\mathbf{x} \, dt, \tag{4.27}$$

$$\forall y, \phi \in L^2(\Omega \times (0, T)).$$

It can be shown that for $\varepsilon > 0$, ε small enough,

$$\|(L_\varepsilon - L_0) \, \phi\|_{L^2(\Omega \times (0,T))} \leqq C\varepsilon \|\phi\|_{L^2(\Omega \times (0,T))}, \quad \forall \phi \in L^2(\Omega \times (0, T)) \tag{4.28}$$

$$\|DI(L_\varepsilon \phi) - DI(L_0 \phi)\|_{L^2(\Omega \times (0,T))} \leqq C\varepsilon \|\phi\|_{L^2(\Omega \times (0,T))}, \tag{4.29}$$

$$\forall \phi \in L^2(\Omega \times (0, T)).$$

An optimal control $\boldsymbol{u}^\varepsilon$, $\varepsilon \in [0, \delta)$, is given by the unique solution of the variational inequality

$$\boldsymbol{u}^\varepsilon \in K: \langle D\, J_\varepsilon(\boldsymbol{u}^\varepsilon)\,,\ \boldsymbol{u} - \boldsymbol{u}^\varepsilon \rangle \geqq 0\,,\quad \forall \boldsymbol{u} \in K\,. \tag{4.30}$$

From (4.30), taking into account (4.6), (4.28), (4.29), by a standard argument we obtain

$$\begin{aligned}\sigma\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0\|^2_{H^1(0,T;R^N)} &\leqq \langle D\, J_\varepsilon(\boldsymbol{u}^\varepsilon) - D\, J_\varepsilon(\boldsymbol{u}^0), \boldsymbol{u}^\varepsilon - \boldsymbol{u}^0\rangle \leqq \\ &\leqq \langle D\, J_\varepsilon(\boldsymbol{u}^0) - D\, J(\boldsymbol{u}^0), \boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon\rangle = \\ &= \big(DI(L_\varepsilon\, B(\boldsymbol{u}^0), L_\varepsilon\, B'(\boldsymbol{u}^0)\,(\boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon)\big)_{L^2(\Omega\times(0,T))} - \\ &- \big(DI(L_0\, B(\boldsymbol{u}^0)), L_0\, B'(\boldsymbol{u}^0)\,(\boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon)\big)_{L^2(\Omega\times(0,T))} = \\ &= \big(DI(L_\varepsilon\, B(\boldsymbol{u}^0)), (L_\varepsilon - L_0)\, B'(\boldsymbol{u}^0)\,(\boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon)\big)_{L^2(\Omega\times(0,T))} + \\ &+ \big(DI(L_\varepsilon\, B(\boldsymbol{u}^0)) - DI(L_0\, B(\boldsymbol{u}^0)), L_0\, B'(\boldsymbol{u}^0)\,(\boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon)\big)_{L^2(\Omega\times(0,T))} \leqq \\ &\leqq C\varepsilon\|\boldsymbol{u}^0 - \boldsymbol{u}^\varepsilon\|_{H^1(0,T;R^N)}\,,\end{aligned}$$

which completes the proof.

5. DIFFERENTIAL STABILITY OF OPTIMAL CONTROLS

In this section we derive the form of the right-derivative of an optimal control $\boldsymbol{u}^\varepsilon$ with respect to the parameter ε at $\varepsilon = 0$. To this end we will define the control constrained optimal control problem (Π_ε), $\varepsilon \in [0, \delta)$.

We assume that the set of admissible controls U_{ad} is given by

$$U_{ad} = \{\boldsymbol{u} \in L^2(0, T; R^N) \mid \boldsymbol{u}(t) \text{ satisfies } (3.4), (3.6) \text{ for a.e. } t \in (0, T)\}\,. \tag{5.1}$$

Furthermore, we assume that

$$F_i(r) = r\,,\quad \forall r \in R\,,\quad i = 1, \ldots, N \tag{5.2}$$

and that the cost functional $J_\varepsilon(\boldsymbol{u})$ is defined by (4.5) for $\delta = 0$, $\boldsymbol{u} \in L^2(0, T; R^N)$, $\alpha_{12} > 0$, $\beta_i > 0$, $\alpha_{11} \geqq 0$, $i = 1, \ldots, N$.

Let us consider the following optimal control problem:

Problem (Π_ε). Find an element $\boldsymbol{u}^\varepsilon \in U_{ad}$ which minimizes the cost functional (4.5) over the set (5.1) of admissible controls.

It can be shown that the optimal control is uniquely determined by the following optimality system:

Find $(\boldsymbol{u}^\varepsilon, c^\varepsilon, p^\varepsilon)$ such that

$$\frac{\partial c^\varepsilon}{\partial t} + \boldsymbol{w}\,.\,\nabla c - K^\varepsilon_H\, \Delta c^\varepsilon + \gamma^\varepsilon c^\varepsilon = Q + \sum_{i=1}^N \chi_i u^\varepsilon_i \quad \text{in} \quad \Omega \times (0, T)\,, \tag{5.3}$$

c^ε satisfies conditions (4.8)–(4.10);

$$(5.4)\quad \frac{-\partial p^\varepsilon}{\partial t} - \operatorname{div}(\boldsymbol{w}p^\varepsilon) - K_H^\varepsilon\,\Delta p^\varepsilon + \gamma^\varepsilon p^\varepsilon = \alpha_{11} r \max\{0, c^\varepsilon - c_d\} \quad \text{in} \quad \Omega \times (0, T)$$

p^ε satisfies conditions (4.12)–(4.14);

$$\boldsymbol{u}^\varepsilon \in U_{\text{ad}},$$

$$(5.5)\quad \sum_{i=1}^{N} \left\{ \alpha_{11} \int_0^T \int_\Omega \chi_i\, p^\varepsilon (u_i - u_i^\varepsilon)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \right.$$

$$\left. + \alpha_{12} \int_0^T \beta_i (u_i^\varepsilon - \tilde{u}_i)(u_i - u_i^\varepsilon)\, \mathrm{d}t \right\} \geqq 0, \quad \forall \boldsymbol{u} \in U_{\text{ad}}.$$

Theorem 2. *For $\varepsilon > 0$, ε small enough,*

$$(5.6)\quad \boldsymbol{u}^\varepsilon = \boldsymbol{u}^0 + \varepsilon \boldsymbol{q} + o(\varepsilon) \quad in \quad L^2(0, T; R^N),$$

where

$$\|o(\varepsilon)\|_{L^2(0,T;R^N)}/\varepsilon \to 0 \quad with \quad \varepsilon \downarrow 0.$$

The element $\boldsymbol{q} \in L^2(0, T; R^N)$ is given by the unique solution of the following optimal control problem:

Problem (Q). Find an element $\boldsymbol{q} \in L^2(0, T; R^N)$ which minimizes the cost functional

$$I(\boldsymbol{u}) = \tfrac{1}{2}\alpha_{11} \int_0^T \int_\Omega r(\theta_0 \max^2\{0, z\} + \theta_1 z^2)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t -$$

$$- \int_{S^+} K_H' \frac{\partial p^0}{\partial \boldsymbol{n}} z\, \mathrm{d}\Sigma + \tfrac{1}{2}\alpha_{12} \int_0^T \beta_i u_i^2\, \mathrm{d}t$$

over the set of admissible controls (5.18), subject to the state equation (5.11)–(5.13) with $\boldsymbol{q} = \boldsymbol{u}$.

The elements θ_0, θ_1 are given by (5.22), (5.23), respectively.

Proof. It can be shown, by using the same argument as in the proof of Theorem 1, that

$$(5.7)\quad \|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0\|_{L^2(0,T;R^N)} \leqq C\varepsilon;$$

therefore (5.3), (5.5) imply [10, 11] for $\varepsilon > 0$, ε small enough:

$$(5.8)\quad \boldsymbol{u}^\varepsilon = \boldsymbol{u}^0 + \varepsilon \boldsymbol{q} + r_1(\varepsilon),$$

$$(5.9)\quad c^\varepsilon = c^0 + \varepsilon z + r_2(\varepsilon),$$

$$(5.10)\quad p^\varepsilon = p^0 + \varepsilon \omega + r_3(\varepsilon),$$

where the elements $\boldsymbol{q}$, z, ω are given by the unique solution of the following optimality system [10, 11]:

$$(5.11)\quad \frac{\partial z}{\partial t} + \mathbf{w}\,.\,\nabla z - K_H\,\Delta z - K'_H\,\Delta c^0 + \gamma^0 z + \gamma' c^0 = \sum_{i=1}^{N} \chi_i q_i \quad \text{in} \quad \Omega \times (0, T)\,,$$

$$(5.12)\quad \frac{\partial z}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad S^+\,, \quad z = 0 \quad \text{on} \quad S^-\,,$$

$$(5.13)\quad z(\mathbf{x}, 0) = 0 \quad \text{on} \quad \Omega\,;$$

$$(5.14)\quad \frac{-\partial \eta}{\partial t} - \operatorname{div}(\mathbf{w}\,.\,\nabla \eta) - K_H^0\,\Delta\eta - \alpha\,K_H\,\Delta p^0 + \gamma^0 \eta + \gamma' p^0 =$$

$$= \alpha_{11} r \theta_0 \max\{0, z\} + \alpha_{11} r \theta_1 z \quad \text{on} \quad \Omega \times (0, T)\,,$$

$$(5.15)\quad \eta = 0 \quad \text{on} \quad S^-\,, \quad K_H \frac{\partial \eta}{\partial \boldsymbol{n}} + \mathbf{w}\boldsymbol{n}\eta = -K'_H \frac{\partial p^0}{\partial \boldsymbol{n}} \quad \text{on} \quad S^+\,,$$

$$(5.16)\quad \eta(\mathbf{x}, T) = 0 \quad \text{on} \quad \Omega\,;$$

$$(5.17)$$
$$\boldsymbol{q} \in C: \sum_{i=1}^{N} \{\alpha_{11} \int_0^T \int_\Omega \chi_i (u_i - q_i)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \alpha_{12} \int_0^T \beta_i q_i (u_i - q_i)\,\mathrm{d}t\} \geqq 0\,, \quad \forall \boldsymbol{u} \in C\,.$$

The set $C \subset L^2(0, T; R_N)$ is given by

$$(5.18)\quad C = \{\boldsymbol{u} \in L^2(0, T; R^N) \mid u_i(t) \geqq 0 \quad \text{a.e. on} \quad \Xi_1^i,\ u_i(t) \leqq 0 \quad \text{a.e. on} \quad \Xi_2^i\,,$$
$$\sum_{i \in N_j} a_{ij}\, u_i(t) \geqq 0 \quad \text{a.e. on} \quad \Xi_3^j, \quad j = 1, \ldots, M\,,$$
$$\sum_{i=1}^{N} (\alpha_{11} \int_\Omega \int_0^T X_i p^0 u_i\,\mathrm{d}x\,\mathrm{d}t + \alpha_{12} \int_0^T \beta_i (u_i^0 - u_i)\, u_i\,\mathrm{d}t) = 0\}\,.$$

Here

$$(5.19)\quad \Xi_1^i = \{t \in (0, T) \mid u_i^0(t) = {}_*u_i(t)\}\,, \qquad i = 1, \ldots, N\,,$$

$$(5.20)\quad \Xi_2^i = \{t \in (0, T) \mid u_i^0(t) = {}^*u_i(t)\}\,, \qquad i = 1, \ldots, N\,,$$

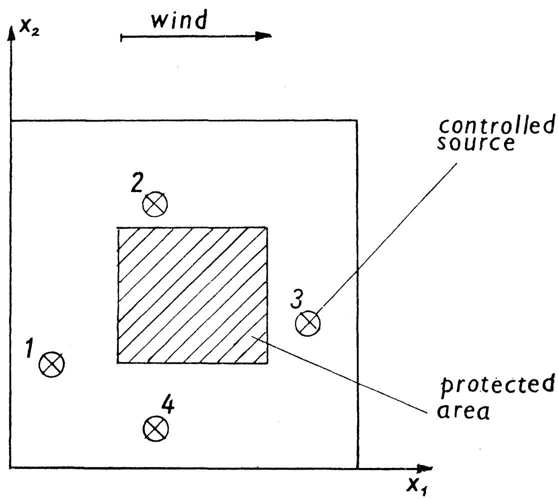
$$(5.21)\quad \Xi_3^j = \{t \in (0, T) \mid \sum_{i \in N_j} a_{ij}\, u_i^0(t) = b_j\}\,, \quad j = 1, \ldots, M\,,$$

$$(5.22)\quad \theta_0(\mathbf{x}, t) = \begin{cases} 1, & c^0(\mathbf{x}, t) = c_d(\mathbf{x}, t)\,, \\ 0, & c^0(\mathbf{x}, t) \neq c_d(\mathbf{x}, t)\,; \end{cases}$$

$$(5.23)\quad \theta_1(\mathbf{x}, t) = \begin{cases} 1, & c^0(\mathbf{x}, t) > c_d(\mathbf{x}, t)\,, \\ 0, & c^0(\mathbf{x}, t) \leqq c_d(\mathbf{x}, t)\,. \end{cases}$$

From (5.14)–(5.18) it follows that the element $\boldsymbol{q} \in C$ is given by the unique solution of the control problem (Q).

In order to illustrate the results of the previous sections, a somewhat simplified control problem was studied. The most important simplification concerned the horizon of control, which was taken as equal to δT, see Fig. 2. In each of the intervals



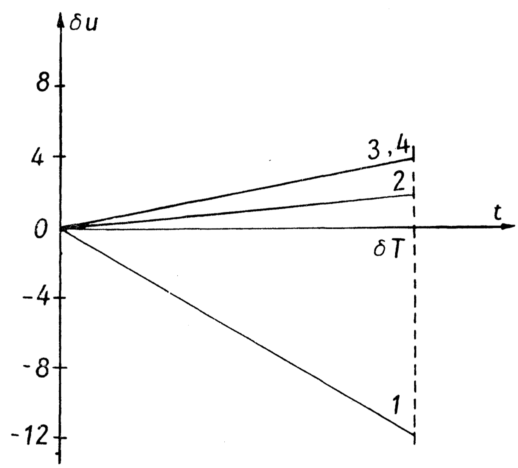


Fig. 3. The control for $K_H = 300$, $\alpha_{12} = 2$.

of the length δT the controls were approximated by linear functions, analogously to the meteorological data. Thus the (discrete) decision variables were reduced to the increments of u_i, $i = 1, \ldots, N$, over the period δT, denoted by δu_i.

The emission field and the topographical data correspond to the real neighbourhood of Warsaw. Four artificial controlled sources with emmission 35 each (1 unit $\approx$ 57·6 kg/sec) were added. The numerical values of other parameters were $\alpha_{11} = 1$, $\alpha_{12} = 2$, $\alpha_{22} = 0$, $T = \delta T = 6h$; ${}_{*}u_i = 20$, ${}^{*}u_i = 50$, $D_i = 2$ for $i = 1, \ldots, 4$. Only the city center was considered as a protected area, which means that $r(\mathbf{x}, t)$ was zero outside the shaded region in Fig. 3. The value of K_H was 300. Only one demand constraint was imposed, namely,

$$\sum_{i=1}^{4} u_i \geqq 140 .$$

It means that only a transfer of power from one source to another was permissible.

The results of computation for $F_i(u_i) \equiv u_i$ and $\beta_i = 1$ $i = 1, \ldots, 4$, are shown in Fig. 3.

In the next step the directional derivatives of u_i, denoted by q_i (5.8), corresponding to the change of K_H in the direction K'_H were calculated.

Two cases were considered: one, when the dual variable corresponding to the active constraint (3.4) imposed on δu_1 was positive (strictly complementary situation) and

Table 1.

Source number		1	2	3	4
complementary case $K_H = 300$					
$K_H = 100$, sensitivity		0	−0·27	0·12	0·15
$\delta u(K_H = 200)$	approx.	−12·00	2·79	4·61	4·61
	exact	−12·00	2·76	4·60	4·64
$\delta u(K_H = 100)$	approx.	−12·00	3·06	4·49	4·46
	exact	−12·00	3·08	4·44	4·48
noncomplementary case $K_H = 300$					
$K'_H = 200$, sensitivity		1·70	−0·82	−0·46	−0·42
$\delta u(K_H = 500)$	approx.	−10·30	1·95	4·14	4·21
	exact	−10·13	2·10	4·01	4·02
$K'_H = -200$, sensitivity		0·0	0·66	−0·30	−0·36
$\delta u(K_H = 100)$	approx.	−12·00	3·43	4·30	4·27
	exact	−12·00	3·25	4·36	4·39

the other, when the strict complementarity condition was not satisfied. In order to achieve the approximately noncomplementary situation the value of α_{12} was changed to $\alpha_{12} = 2{\cdot}55$.

The difference between both cases consists in the fact, that in the noncomplementary case the sensitivity had to be calculated twice, separately for $K'_H > 0$ and $K'_H < 0$.

The results are summarized in Table 1. The approximated values of controls δu_i are obtained by using formula (5.8), while exact numbers correspond to the solution of the original problem with new K_H.

All optimization problems involved in this work have been solved by the quickly convergent version [3] of the linearisation algorithm [7].

7. CONCLUDING REMARKS

In the paper some theoretical and numerical results for the sensitivity analysis of an optimal control problem arising in air pollution control are presented.

We have proved that the optimal control is locally stable (Lipschitz continuous) with respect to perturbations of the coefficients of the state equation.

Our numerical results confirm that the formulae for the right-derivatives of the optimal control can be used in order to approximate the increments of the optimal control with respect to the perturbations of data.

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Souhrn

DIFERENCIÁLNÍ STABILITA ŘEŠENÍ PROBLÉMU REGULACE KVALITY OVZDUŠÍ V MĚSTSKÉ OBLASTI

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Autoři studují konvexní problém optimální regulace popsaný parabolickou rovnicí. Odvozují tvar derivace zprava optimálního řešení podle parametru, a vyšetřují aplikace na problém regulace kvality ovzduší. V článku jsou uvedeny také numerické výsledky.

Резюме

ДИФФЕРЕНЦИАЛЬНАЯ УСТОЙЧИВОСТЬ РЕШЕНИЯ ПРОБЛЕМЫ РЕГУЛЯЦИИ КАЧЕСТВА АТМОСФЕРЫ В ГОРОДСКОЙ ОБЛАСТИ

PIOTR HOLNICKI, JAN SOKOŁOWSKI, ANTONI ŻOCHOWSKI

Авторы изучают выпуклую проблему оптимального управления, описанную параболическим уравнением. Выводят форму производной справа оптимального решения по параметру и исследуют приложения к проблеме регуляции качества атмосферы. Приводят также численные результаты.

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