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ON A SUPERCONVERGENT FINITE ELEMENT SCHEME FOR ELLIPTIC SYSTEMS

II. BOUNDARY CONDITIONS OF NEWTON'S OR NEUMANN'S TYPE

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Summary. A simple superconvergent scheme for the derivatives of finite element solution is presented, when linear triangular elements are employed to solve second order elliptic systems with boundary conditions of Newton's or Neumann's type. For bounded plane domains with smooth boundary the local $O(h^{3/2})$-superconvergence of the derivatives in the L^2-norm is proved. The paper is a direct continuation of [2], where an analogous problem with Dirichlet's boundary conditions is treated.

Keywords: finite elements, superconvergence, post-processing, averaged gradient, elliptic systems.

AMS Subject classification: 65 N 30, 73 C 99

1. INTRODUCTION

In [2] we have studied a simple superconvergent finite element scheme for second order elliptic systems with non-homogeneous Dirichlet boundary conditions. In this paper we analyze the same scheme in the case of boundary conditions of Newton's or Neumann's type. These types of boundary conditions, however, are very rarely investigated in connection with a superconvergence of the finite element method – see [5], p. 187 (and [4, 7]).

We recall [2] the main idea of the proposed scheme. The use of linear elements to elliptic systems yields a piecewise constant field of the first derivatives of the finite element solution. Thus employing a suitable post-processing (based on averaging at nodes), a new continuous piecewise linear field can be defined. We show that the latter field improves the approximation of the first derivatives of the solution, which are often more important than the original solution itself.

The paper is organized as follows. In Section 2, a variational formulation of a class of elliptic problems is given, and several lemmas for coercive case and numerical integration are presented. In Section 3, a local $O(h^{3/2})$ error estimate in the L^2-norm is proved for smooth domains. Finally, in Section 4 the same estimate is derived also for some non-coercive cases. To the authors' knowledge, there are no superconvergence results for non-coercive problems in the literature.

2. SOME LEMMAS FOR COERCIVE CASE OF ELLIPTIC SECOND ORDER SYSTEMS AND NUMERICAL INTEGRATION

Preserving the notation of [2], we assume that the functions

$$\boldsymbol{f} \in (L^2(\Omega))^M, \quad \boldsymbol{g} \in (L^2(\partial\Omega))^M$$

are given. As in [2], we introduce the operators $N_i(\boldsymbol{u})$, $i = 1, \ldots, \varkappa$, the bilinear form $a(\boldsymbol{u}, \boldsymbol{v})$ by means of a symmetric uniformly positive definite $\varkappa \times \varkappa$ matrix K with the entries $K_{ij} \in P_s(\bar{\Omega})$. Moreover we define another bilinear form

$$b(\boldsymbol{u}, \boldsymbol{v}) = \int_{\partial\Omega} \sum_{r,t=1}^{M} b_{rt} u_r v_t \, \mathrm{d}s ,$$

where b_{rt} are bounded measurable functions on $\partial\Omega$. For brevity, we shall use the notation

$$((\boldsymbol{u}, \boldsymbol{v})) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{v}) .$$

The following weak formulation of the boundary value problem will be considered:
Find $\boldsymbol{u} \in \boldsymbol{W}$ such that

$$((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} + (\boldsymbol{g}, \boldsymbol{v})_{0,\partial\Omega} \; \forall \boldsymbol{v} \in \boldsymbol{W} , \tag{2.1}$$

where $\boldsymbol{W} = (H^1(\Omega))^M$ and $(\cdot, \cdot)_{0,\partial\Omega}$ is the scalar product in $(L^2(\partial\Omega))^M$.
Assume that:

(H 1) the system of operators $\{N_i(\boldsymbol{u})\}_{i=1}^{\varkappa}$ is coercive on $\boldsymbol{W}$;

(H 2) $b(\boldsymbol{u}, \boldsymbol{u}) \geqq 0 \quad \forall \boldsymbol{u} \in \boldsymbol{W}$;

(H 3) an inequality of Korn's type holds: there exists a positive constant c_0 such that

$$((\boldsymbol{u}, \boldsymbol{u})) \geqq c_0 \|\boldsymbol{u}\|_{1,\Omega}^2 \quad \forall \boldsymbol{u} \in \boldsymbol{W} .$$

Then the problem (2.1) has a unique solution.

Remark 2.1. As follows from [3], Part I, the inequality of (H 3) can be proven from (H 1), (H 2) and the positive definiteness of $\boldsymbol{K}$, if

$$\boldsymbol{v} \in \boldsymbol{W}, \sum_{i=1}^{\varkappa} \|N_i(\boldsymbol{v})\|_{0,\Omega}^2 + b(\boldsymbol{v}, \boldsymbol{v}) = 0 \Rightarrow \boldsymbol{v} = 0 ,$$

There are two important possibilities when the latter condition holds:

(i) $b(\mathbf{v}, \mathbf{v}) \equiv 0$ and some of the operators $N_i(\mathbf{v})$ have "absolute" terms ($n_{im} \neq 0$),

(ii) $b(\mathbf{v}, \mathbf{v}) \neq 0$ and all $N_i(\mathbf{v})$ are without "absolute" terms ($n_{im} = 0$).

The first case corresponds with boundary conditions of Neumann's type on $\partial\Omega$, whereas in the second case the conditions of Newton's type are prescribed on a part $\Gamma \subset \partial\Omega$ of positive length, and conditions of Neumann's type on the remainder $\partial\Omega - \Gamma$.

Remark 2.2. In case of elastostatics (see [2], Example), the operators $N_i(\mathbf{v})$ have no absolute terms and we have the variant (ii). Then a sort of "elastic supports" is to be prescribed on a part of $\partial\Omega$ – cf. [3], Part II.

Let us consider the class $\mathscr{C}^3(d)$ of domains and a strongly regular family of triangulations $\mathfrak{M} = \{\mathscr{T}_h\}$, introduced in [2], Section 2. In contrast to the case of Dirichlet problem, however, we have to change slightly the definition of the interpolation operator $\mathbf{P}$.

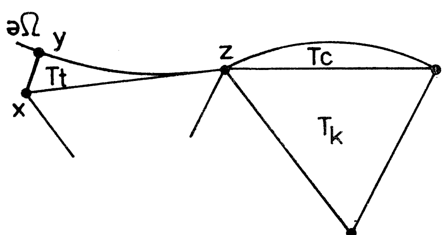

Fig. 1.

Denote by T_c, T_t the parts of $\Omega - \Omega_h$, i.e. the segments adjacent to the chords and to the tangents, respectively (see Fig. 1). For

$$W_h = \{v \in H^1(\Omega) \mid v|_T \in P_1(T)\ \forall T \in \mathscr{T}_h,\ v|_{T_c} \in P_1(T_c),\ v|_{T_t} \in P_1(T_t)\ \forall T_c, T_t \subset \Omega - \Omega_h\}$$

we put

$$\mathbf{W}_h \equiv (W_h)^M .$$

The interpolation operator P: $H^1(\Omega) \cap C(\bar{\Omega}) \to W_h$ will be defined as follows:

$$Pu = u$$

at all nodes $\mathbf{x} \in \bar{\Omega}_h$ and at the points $\mathbf{y} \in \partial\Omega$, where $\mathbf{y}$ is the point of $\partial\Omega$ nearest to the vertex $\mathbf{x}$ (see Fig. 1).

In T_c, the function Pu is defined by extension of the linear polynomial from the adjacent triangle T_k. In T_t, the function Pu is the linear interpolation of u with the nodes $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ (see Fig. 1). For $\mathbf{u} \in \mathbf{W} \cap (C(\bar{\Omega}))^M$ we define

$$\mathbf{P}\mathbf{u} = (Pu_1, Pu_2, \ldots, Pu_M) .$$

Lemma 2.1. *Let $\Omega \in \mathscr{C}^3(d)$, $\mathbf{u} \in (H^3(\Omega))^M$, $\mathbf{v} \in \mathbf{W}_h$. Then*

$$|((\mathbf{u} - \mathbf{P}\mathbf{u}, \mathbf{v}))| \leqq Ch^{3/2} \|\mathbf{u}\|_{3,\Omega} \|\mathbf{v}\|_{1,\Omega} . \tag{2.2}$$

Proof. We shall write

$$\mathbf{e} = \mathbf{u} - \mathbf{P}\mathbf{u} ,$$

$$a(\mathbf{e}, \mathbf{v}) = \int_{\Omega_h} \sum_{i,j=1}^{\varkappa} K_{ij} N_i(\mathbf{e}) N_j(\mathbf{v}) \, dx + \int_{\Omega - \Omega_h} \sum_{i,j=1}^{\varkappa} K_{ij} N_i(\mathbf{e}) N_j(\mathbf{v}) \, dx .$$

The first integral can be bounded by the right-hand side of (2.2), as follows by an argument which is parallel to that of Lemma 2.1 in [2].

We have to estimate the integral over $\Omega - \Omega_h$. For any $w \in H^n(\Omega)$ there exists Calderon's extension $Ew \in H^n(\mathbb{R}^2)$ such that (cf. [6], Chap. 2, § 3.7) $Ew|_\Omega = w$ and

$$\|Ew\|_{n,R^2} \leqq C(n) \|w\|_{n,\Omega} , \quad n \geqq 1 . \tag{2.3}$$

Let us consider an arbitrary segment T_c. We may write

$$\left| \int_{T_c} K_{ij} N_i(\mathbf{e}) N_j(\mathbf{v}) \, dx \right| \leqq C \|N_i(\mathbf{e})\|_{0,T_c} \|N_j(\mathbf{v})\|_{0,T_c} \leqq C_1 \|\mathbf{e}\|_{1,T_c} \|\mathbf{v}\|_{1,T_c} . \tag{2.4}$$

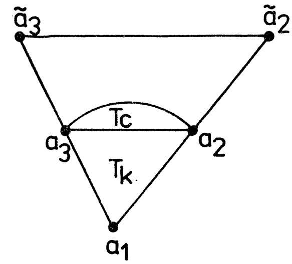

Fig. 2.

Let us define the triangle $\tilde{T}_c = \Delta a_1 \tilde{a}_2 \tilde{a}_3$ (see Fig. 2), where $a_1 \tilde{a}_j = 2 a_1 a_j$, $j = 2, 3$. Finally let π denote the linear interpolation operator on $\tilde{T}_c$ such that

$$\pi \, w(a_i) = w(a_i) , \quad i = 1, 2, 3 .$$

Then $Pw = \pi w$ on $T_k \cup T_c$, so that

$$\|Pw - w\|_{1,T_c} \leqq \|\pi Ew - Ew\|_{1,\tilde{T}_k} \leqq Ch |Ew|_{2,\tilde{T}_c}$$

(see e.g. [1], p. 123) and consequently,

$$\|Pu_m - u_m\|_{1,T_c} \leqq Ch |Eu_m|_{2,\tilde{T}_c} , \quad m = 1, \ldots, M . \tag{2.5}$$

Next let us consider an arbitrary segment T_t. An analogue of (2.4) holds as well. Let us define a triangle $\tilde{T}_t$ with the vertices $\mathbf{x}$, $\mathbf{z}$, $\mathbf{r}$, where $\mathbf{xr} = \mathbf{xz}$ (see Fig. 3). Let π_1

be the interpolation operator on $\tilde{T}_t$ such that $\pi_1 w$ coincides with w at the points $\mathbf{x}$, $\mathbf{z}$, $\mathbf{y}$. By the same way as previously for T_c, we deduce

$$(2.6) \qquad \|Pu_m - u_m\|_{1,T_t} \leqq \|\pi_1 Eu_m - Eu_m\|_{1,\tilde{T}_t} \leqq Ch|Eu_m|_{2,\tilde{T}_t}\,.$$

(We can easily verify that the minimal angle of $\tilde{T}_t$ is bounded from below by a positive constant independent of h). We may write

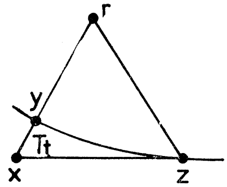

Fig. 3,

$$(2.7) \qquad \left|\int_{\Omega-\Omega_h} K_{ij}\, N_i(\mathbf{e})\, N_j(\mathbf{v})\, \mathrm{d}x\right| \leqq \sum_{T_c} \left|\int_{T_c} \dots \mathrm{d}x\right| + \sum_{T_t} \left|\int_{T_t} \dots \mathrm{d}x\right|.$$

The first sum can be estimated in the following way, using (2.5):

$$\sum_{T_c} \left|\int_{T_c} \dots \mathrm{d}x\right| \leqq \sum_{T_c} C_1 \|\mathbf{e}\|_{1,T_c} \|\mathbf{v}\|_{1,T_c} \leqq C_1 (\sum_{T_c} \|\mathbf{e}\|^2_{1,T_c})^{1/2} \|\mathbf{v}\|_{1,\Omega-\Omega_h} \leqq$$

$$\leqq C_2 \|\mathbf{v}\|_{1,\Omega} (\sum_{T_c} \sum_{m=1}^{M} h^2 |Eu_m|^2_{2,\tilde{T}_c})^{1/2} \leqq C_3\, h \|\mathbf{v}\|_{1,\Omega} (\sum_{m=1}^{M} 2 |Eu_m|^2_{2,\Omega_\delta})^{1/2}\,,$$

where Ω_δ is a "strip" containing all triangles $\tilde{T}_c$,

$$\Omega_\delta = \{\mathbf{x} \in \mathbb{R}^2 \mid \operatorname{dist}(\mathbf{x}, \partial\Omega) < Ch\}\,.$$

We employ twice the Iljin inequality with $\varepsilon = Ch$ (see [2], (2.31)) first to a domain $\tilde{\Omega} - \bar{\Omega}$, where $\tilde{\Omega} \supset \Omega$, and second to Ω. Thus we obtain, using also (2.3)

$$|Eu_m|^2_{2,\Omega_\delta} \leqq Ch \|Eu_m\|^2_{3,\tilde{\Omega}} \leqq C_1 h \|u_m\|^2_{3,\Omega}\,.$$

Inserting this into (2.5), we arrive at the upper bound

$$C_4 h^{3/2} \|\mathbf{u}\|_{3,\Omega} \|\mathbf{v}\|_{1,\Omega}\,.$$

The second sum in (2.7) can be estimated in a parallel way, using (2.6) instead of (2.5).

To deal with the boundary integral $b(\mathbf{e}, \mathbf{v})$, we first show that

$$(2.8) \qquad \|u_j - Pu_j\|_{0,\partial\Omega} \leqq Ch^{3/2} \|u_j\|_{2,\Omega}\,, \quad j = 1, \dots, M\,.$$

The proof of (2.8) is based on the following relation

$$\int_{\partial\Omega} w^2\, \mathrm{d}s \leqq C\left(\delta \int_{\omega_\delta} \|\nabla w\|^2\, \mathrm{d}x + \delta^{-1} \int_{\omega_\delta} w^2\, \mathrm{d}x\right).$$

This inequality holds for any function $w \in H^1(\Omega)$ and for an internal "boundary strip" $\omega_\delta \subset \Omega$, the width δ of which is small enough (see [5], p. 24).

If we enlarge ω_δ onto the whole domain Ω and substitute

$$\delta = h\,, \quad w = e_j\,,$$

we obtain

$$\|e_j\|_{0,\partial\Omega}^2 \leqq C(h|e_j|_{1,\Omega}^2 + h^{-1}\|e_j\|_{0,\Omega}^2)\,. \tag{2.9}$$

On the other hand, we have

$$\|e_j\|_{q,\Omega} \leqq Ch^{2-q}\|u_j\|_{2,\Omega}\,, \quad q = 0, 1\,. \tag{2.10}$$

This estimate is standard on Ω_h. We can verify it on $\Omega - \Omega_h$, using an argument parallel to the derivation of (2.5), (2.6), and the inequality (2.3) for $n = 2$.

Inserting (2.10) into (2.9), we are led to (2.8). On the basis of (2.8) we may write

$$\begin{aligned} |b(\mathbf{u} - \mathbf{P}\mathbf{u}, \mathbf{v})| &= \left|\int_{\partial\Omega} \sum_{i,j=1}^{M} b_{ji}(u_j - Pu_j)\, v_i \,\mathrm{d}s\right| \leqq \\ &\leqq C \sum_{i,j=1}^{M} \|u_j - Pu_j\|_{0,\partial\Omega}\, \|v_i\|_{0,\partial\Omega} \leqq C_1 h^{3/2} \sum_{i,j=1}^{M} \|u_j\|_{2,\Omega}\, \|v_i\|_{1,\Omega} \leqq \\ &\leqq C_2 h^{3/2}\|\mathbf{u}\|_{2,\Omega}\, \|\mathbf{v}\|_{1,\Omega}\,. \end{aligned} \tag{2.11}$$

Combining (2.11) with the previous estimates for $a(\mathbf{e}, \mathbf{v})$, we obtain the assertion (2.2) of the lemma. ∎

Lemma 2.2. *Let $\Omega \in \mathscr{C}^3(d)$, $f \in H^2(\Omega)$ and $v \in W_h$. Let us define the approximation*

$$(f, v)_{0,\Omega_h}^*$$

by the centroid rule on the triangulation $\mathscr{T}_h$ of Ω_h. Then

$$|(f, v)_{0,\Omega} - (f, v)_{0,\Omega_h}^*| \leqq Ch^2\|f\|_{2,\Omega}\, \|v\|_{1,\Omega} \tag{2.12}$$

holds for h small enough.

Proof. It is readily seen that

$$|(f, v)_{0,\Omega} - (f, v)_{0,\Omega_h}^*| \leqq \sum_{k=1}^{K} |E_k(fv)| + \left|\int_{\Omega - \Omega_h} fv \,\mathrm{d}x\right|, \tag{2.13}$$

where E_k denotes the local error on a triangle T_k. Lemma 2.2 of [2] yields the upper bound of (2.12) for the sum of local errors and it remains to estimate only the last term in (2.13). We have

$$I_h \equiv \left|\int_{\Omega - \Omega_h} fv \,\mathrm{d}x\right| \leqq \|f\|_{0,\Omega-\Omega_h}\, \|v\|_{0,\Omega-\Omega_h}\,.$$

Since $\Omega - \Omega_h$ is contained in a boundary strip Ω^ε of the width Ch^2, we may apply the Iljin inequality ([2], (2.31)) with $\varepsilon = Ch^2$ both to f and to v. Thus we obtain

$$I_h \leqq C_1 h^2 \|f\|_{1,\Omega} \|v\|_{1,\Omega}$$

for sufficiently small h. ∎

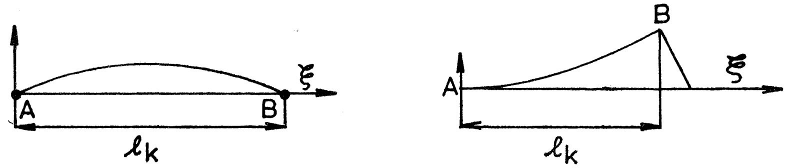

Fig. 4.

Lemma 2.3. *Let $\Omega \in \mathscr{C}^3(d)$. Assume that $g \in L^2(\partial\Omega)$ is piecewise from C^2, where the corresponding partition*

$$\partial\Omega = \bigcup_{i=1}^{I} \Gamma_i$$

is consistent with every $\mathscr{T}_h \in \mathfrak{M}$, and $v \in W_h$. Using local ξ-coordinate (see Fig. 4) *for the parametric representation of the arc segments, we define*

$$(2.14) \qquad (g, v)^*_{0,\partial\Omega} = \sum_{k=1}^{K_1} I_k(\tilde{g}v) + \sum_{k=1}^{K_2} \int_0^{l_k} \tilde{g}v \, d\xi ,$$

where

$$I_k(w) = l_k w(l_k/2) ,$$

$$\tilde{g}(\xi) = g(\xi)\,(1 + (\varphi_k'(\xi))^2)^{1/2} , \quad v = v|_{\partial\Omega} ,$$

φ_k: $[0, l_k] \to \mathbb{R}$ is a function, the graph of which coincides with the arc segment $\widehat{AB}$ (see Fig. 4). *The first and second sum in* (2.14) *corresponds to the arc segments adjacent to the chords and tangents, respectively. Then*

$$(2.15) \qquad |(g, v)_{0,\partial\Omega} - (g, v)^*_{0,\partial\Omega}| \leqq Ch^{3/2} \max_{1 \leqq i \leqq I} \|g\|_{C^2(\Gamma_i)} \|v\|_{1,\Omega} .$$

Proof. Let us consider the local error

$$E_k(w) = \int_0^{l_k} w(\xi) \, d\xi - I_k(w) .$$

Transforming the interval $S_k = [0, l_k]$ onto the unit interval $\sigma = [0, 1]$ by means of the mapping

$$\hat{x} = \xi/l_k ,$$

and introducing the functions

$$\hat{w}(\hat{x}) = w(l_k \hat{x}) ,$$

$$\hat{E}(\hat{w}) = \int_0^1 \hat{w} \, d\hat{x} - \hat{w}(1/2) ,$$

we derive easily

(2.16) $$|E_k(w)| = |l_k \hat{E}(\hat{w})| \leqq h|\hat{E}(\hat{w})| .$$

Using the Bramble-Hilbert lemma (see [1], Theorem 4.1.3), we arrive at the estimate

(2.17) $$|\hat{E}(\hat{w})| \leqq C|\hat{w}|_{2,\sigma} .$$

Since

(2.18) $$|\hat{w}|^2_{2,\sigma} = l_k^3|w|^2_{2,S_k} ,$$

combining (2.16)–(2.18), we obtain

$$|E_k(w)| \leqq Ch^{5/2}|w|_{2,S_k} \quad \forall w \in H^2(S_k) .$$

Let us consider an arc segment adjacent to a chord. We have

(2.19) $$|E_k(\tilde{g}v)| \leqq Ch^{5/2}|\tilde{g}v|_{2,S_k} .$$

Using $g \in C^2(\Gamma_i)$, $\varphi \in C^3(S_k)$, and the definition of v, we can prove that

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(\tilde{g}v)\right| \leqq C\|g\|_{C^2} \leqq (v^2 + \|\nabla v\|^2)^{1/2} ,$$

(where $\|g\|_{C^2} = \max\limits_{1 \leqq i \leqq I} \|g\|_{C^2(\Gamma_i)}$) so that

(2.20) $$|\tilde{g}v|^2_{2,S_k} \leqq C\|g\|^2_{C^2}\int_0^{l_k}(v^2 + \|\nabla v\|^2)\,\mathrm{d}\xi .$$

From (2.19)

(2.21) $$|(g, v)_{0,\partial\Omega} - (g, v)^*_{0,\partial\Omega}| = |\sum_{k=1}^{K_1} E_k(\tilde{g}v)| \leqq Ch^{5/2}\sum_{k=1}^{K_1}|\tilde{g}v|_{2,S_k}$$

follows. Since the number K_1 of all arc segments is bounded by Ch^{-1}, we have according to (2.20) that

(2.22) $$\sum_{k=1}^{K_1}|\tilde{g}v|_{2,S_k} \leqq Ch^{-1/2}(\sum_{k=1}^{K_1}|\tilde{g}v|^2_{2,S_k})^{1/2} \leqq$$
$$\leqq C_1h^{-1/2}\|g\|_{C^2}\left(\sum_{k=1}^{K_1}\int_0^{l_k}\left(v^2 + \left|\frac{\partial v}{\partial \xi}\right|^2\right)\mathrm{d}\xi\right)^{1/2} .$$

It is easy to find that

(2.23) $$\sum_{k=1}^{K_1}\int_0^{l_k} v^2\,\mathrm{d}\xi \leqq \int_{\partial\Omega} v^2\,\mathrm{d}s \leqq C\|v\|^2_{1,\Omega}$$

and

(2.24) $$\int_0^{l_k}\left|\frac{\partial v}{\partial \xi}\right|^2\mathrm{d}\xi = \left|\frac{\partial v}{\partial \xi}\right|^2 l_k \leqq |v|^2_{1,T_k}\, l_k(\mathrm{mes}\, T_k)^{-1} \leqq Ch^{-1}|v|^2_{1,T_k}$$

(where T_k is the adjacent triangle), using the strong regularity of the family $\mathfrak{M}$ of triangulations. Inserting (2.23), (2.24) into (2.22) and then in (2.21), we come to the following estimate

$$|(g, v)_{0,\partial\Omega} - (g, v)^*_{0,\partial\Omega}| \leqq Ch^2\|g\|_{C^2}(\|v\|^2_{1,\Omega} + h^{-1}|v|^2_{1,\Omega})^{1/2} \leqq$$
$$\leqq C_1 h^{3/2} \max_i \|g\|_{C^2(\Gamma_i)} \|v\|_{1,\Omega} . \qquad \blacksquare$$

The discrete problem will be defined as follows:

Find $\mathbf{u}_h \in \mathbf{W}_h^\sim$ such that

$$(2.25) \qquad ((\mathbf{u}_h, \mathbf{v}_h)) = (\mathbf{f}, \mathbf{v}_h)^*_{0,\Omega_h} + (\mathbf{g}, \mathbf{v}_h)^*_{0,\partial\Omega} \quad \forall \mathbf{v}_h \in \mathbf{W}_h^\sim ,$$

where the terms on the right-hand side are defined as the sums over $m = 1, \ldots, M$, of the approximations introduced in Lemma 2.2 and Lemma 2.3, respectively, $\mathbf{W}_h^\sim = \mathbf{P}(\mathbf{W} \cap (C(\bar{\Omega}))^M) \subset \mathbf{W}_h$.

Theorem 2.1. *Let Ω belong to the class $\mathscr{C}^3(d)$. Let $\mathbf{u} \in (H^3(\Omega))^M$ and $\mathbf{u}_h \in \mathbf{W}_h^\sim$ be the solution of* (2.1) *and* (2.25), *respectively, where $\mathbf{f} \in (H^2(\Omega))^M$, $\mathbf{g}$ is piecewise from C^2. Then*

$$\|\mathbf{u}_h - \mathbf{P}\mathbf{u}\|_{1,\Omega} \leqq Ch^{3/2}(\|\mathbf{u}\|_{3,\Omega} + \|\mathbf{f}\|_{2,\Omega} + \max_{1 \leqq i \leqq I} \|\mathbf{g}\|_{C^2(\Gamma_i)})$$

holds for h small enough.

Proof. If we put

$$\mathbf{v}_h = \mathbf{u}_h - \mathbf{P}\mathbf{u} \in \mathbf{W}_h^\sim ,$$

and use the inequality of Korn's type (H 3), we may write

$$c_0\|\mathbf{v}_h\|^2_{1,\Omega} \leqq ((\mathbf{u}_h - \mathbf{P}\mathbf{u}, \mathbf{v}_h)) \leqq |((\mathbf{u} - \mathbf{P}\mathbf{u}, \mathbf{v}_h))| + |((\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h))| \leqq$$
$$\leqq |((\mathbf{u} - \mathbf{P}\mathbf{u}, \mathbf{v}_h))| + |(\mathbf{f}, \mathbf{v}_h)^*_{0,\Omega_h} - (\mathbf{f}, \mathbf{v}_h)_{0,\Omega}| + |(\mathbf{g}, \mathbf{v}_h)^*_{0,\partial\Omega} - (\mathbf{g}, \mathbf{v}_h)_{0,\partial\Omega}| .$$

Applying Lemma 2.i to the i-th term of the right-hand side, we obtain

$$c_0\|\mathbf{v}_h\|_{1,\Omega} \leqq Ch^{3/2}(\|\mathbf{u}\|_{3,\Omega} + \|\mathbf{f}\|_{2,\Omega} + \max_{1 \leqq i \leqq I} \|\mathbf{g}\|_{C^2(\Gamma_i)}) . \qquad \blacksquare$$

3. AVERAGED GRADIENT AND SUPERCONVERGENCE

Let us introduce the averaged gradient $G_h(v_{jh})$ for the j-th component of $\mathbf{v}_h \in \mathbf{W}_h$ according to (3.1) of [2]. Since the definition of $\mathbf{P}\mathbf{v}$ on Ω_h^* coincides with that of [2] (see Section 2), Theorem 3.1 of [2] holds again. Preserving the notation $\partial\mathbf{u}/\partial\mathbf{x}$ for the matrix of "exact" gradients and $\mathscr{G}(\mathbf{u}_h)$ for the matrix of averaged gradients, we are led to the main theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.1 be fulfilled. Then*

$$(3.1)\qquad \left\| \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}_h) \right\|_{0,\Omega_h^*} \leqq Ch^{3/2}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{f}\|_{2,\Omega} \max_{1 \leqq i \leqq I} \|\boldsymbol{g}\|_{C^2(\Gamma_i)})$$

holds for sufficiently small h.

Proof is parallel to that of Theorem 4.1 of [2].

Remark 3.1. The extension to the global $O(h^{3/2})$-superconvergence can be shown as in Corollary 4.1 of [2].

Remark 3.2. Let Ω have a polygonal boundary which consists of line segments parallel with one of three different directions and the ratio of the lengths of any two parallel sides is rational. Then we can put

$$\mathscr{T}_h = \mathscr{T}_h^*, \quad \Omega = \Omega_h = \Omega_h^*,$$

and prove (3.1), which represents also a global superconvergence estimate (i.e. up to the boundary). Note that the arguments of Remark 2.2 of [2], leading to the "improvement" of the rate $h^{3/2}$ to h^2 for polygonal domains and $\boldsymbol{u} \in (H^3(\Omega))^M$, cannot be employed here.

4. NON-COERCIVE CASES

In the present section we shall consider a class of boundary value problems, where the hypothesis (H 3) fails to hold. All the other assumptions will be preserved. Moreover, let $b(\boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{v}, \boldsymbol{u})$. We introduce the subspace

$$\mathscr{P} = \{\boldsymbol{v} \in \boldsymbol{W} \mid \sum_{i=1}^{\varkappa} \|N_i(\boldsymbol{v})\|_{0,\Omega}^2 + b(\boldsymbol{v}, \boldsymbol{v}) = 0\},$$

and assume that

$$\mathscr{P} \neq \{0\}, \quad \mathscr{P} \subset (P_1(\Omega))^M.$$

Let us choose a system of linear continuous functionals $p_i \in \boldsymbol{W}'$, $i = 1, \ldots, r$, such that

$$(4.1)\qquad \boldsymbol{v} \in \boldsymbol{W}, \quad \sum_{i=1}^{\varkappa} \|N_i(\boldsymbol{v})\|_{0,\Omega}^2 + b(\boldsymbol{v}, \boldsymbol{v}) + \sum_{i=1}^{r} p_i^2(\boldsymbol{v}) = 0 \Rightarrow \boldsymbol{v} = 0.$$

Example 4.1. In the case of the Poisson equation with Neumann's boundary conditions, we set $M = 1, \varkappa = 2$, $b(u, v) \equiv 0$, $N_i(v) = \partial v/\partial x_i$, $i = 1, 2$. Then $\mathscr{P} = P_0(\Omega)$ and we may choose

$$p_1(v) = \int_\Gamma v \, ds,$$

where $\Gamma \subset \bar{\Omega}$ is an arc segment of positive length.

Example 4.2. Let $M = 2$, $\varkappa = 3$, $b(\boldsymbol{u}, \boldsymbol{v}) \equiv 0$, $N_i(\boldsymbol{v})$ be identified with the strain components of two-dimensional elasticity — cf. Example of [2], Section 2. Here

$$\mathscr{P} = \{(q_1, q_2) \mid q_1 = a_1 - bx_2, q_2 = a_2 + bx_1, a_1, a_2, b \in R\} .$$

We can choose $r = 3$,

$$p_i(\boldsymbol{v}) = \int_\Gamma v_i \,\mathrm{d}s , \quad i = 1, 2 ,$$

$$p_3(\boldsymbol{v}) = \int_\Gamma (x_1 v_2 - x_2 v_1) \,\mathrm{d}s ,$$

(cf. [3], Part II, the traction boundary value problem). ■

Henceforth let us restrict ourselves to the set of functionals $p_i \in ((L^2(\Gamma))^M)' \subset \boldsymbol{W}'$, and let us introduce the following subspace

$$\boldsymbol{V}_p = \{\boldsymbol{v} \in \boldsymbol{W} \mid \sum_{i=1}^r p_i^2(\boldsymbol{v}) = 0\} .$$

Then an inequality of Korn's type holds in $\boldsymbol{V}_p$, i.e. a constant $c_0 > 0$ exists such that

$$((\boldsymbol{u}, \boldsymbol{u})) \geqq c_0 \|\boldsymbol{u}\|_{1,\Omega}^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V}_p \tag{4.2}$$

(see [3], Part I, Remark 3). The subspace $\boldsymbol{V}_p$ coincides with the orthogonal complement of $\mathscr{P}$ in $\boldsymbol{W}$, equipped with the following scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_W \equiv \sum_{i=1}^{\varkappa} (N_i(\boldsymbol{u}), N_i(\boldsymbol{v}))_{0,\Omega} + b(\boldsymbol{u}, \boldsymbol{v}) + \sum_{i=1}^{r} p_i(\boldsymbol{u})\, p_i(\boldsymbol{v}) .$$

We define the problem:

Find $\boldsymbol{u} \in \boldsymbol{V}_p$ such that

$$((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} + (\boldsymbol{g}, \boldsymbol{v})_{0,\partial\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{V}_p . \tag{4.3}$$

The solution exists if and only if

$$(\boldsymbol{f}, \boldsymbol{q})_{0,\Omega} + (\boldsymbol{g}, \boldsymbol{q})_{0,\partial\Omega} = 0 \quad \forall \boldsymbol{q} \in \mathscr{P} \tag{4.4}$$

(see [3], Part I, Theorem 2.2 and Remark 4). Note that (4.3) is equivalent with the same definition, where the test functions are taken from the whole space $\boldsymbol{W}$ instead of $\boldsymbol{V}_p$, provided the condition (4.4) is fulfilled.

By virtue of (4.2), there exists a unique solution of the problem (4.3).

Passing to the finite element approximations, we shall suppose that Γ is a fixed straight line segment consistent with every $\mathscr{T}_h \in \mathfrak{M}$. Let us define a modified interpolation

$$\boldsymbol{P}_p : \boldsymbol{V}_p \cap (C(\bar{\Omega}))^M \to \boldsymbol{W}_h \cap \boldsymbol{V}_p$$

by means of the relation

$$\boldsymbol{P}_p \boldsymbol{u} = \boldsymbol{P}\boldsymbol{u} + \boldsymbol{Q}\boldsymbol{u} ,$$

where $\mathbf{u} \in \mathbf{V}_p \cap (C(\bar{\Omega}))^M$, $\mathbf{P}$ denotes the interpolation operator introduced in Section 2 and $\mathbf{Qu}$ is a suitable "correction". On the basis of the orthogonal decomposition

$$\mathbf{W} = \mathbf{V}_p \oplus \mathscr{P},$$

we conclude that a unique $\mathbf{Qu} \in \mathscr{P}$ exists such that $\mathbf{P}_p\mathbf{u} \in \mathbf{V}_p$. In fact, $-\mathbf{Qu}$ equals to the orthogonal projection of $\mathbf{Pu}$ onto $\mathscr{P}$.

Lemma 4.1. *Let* $\mathbf{u} \in (H^3(\Omega))^M \cap \mathbf{V}_p$. *Then*

$$\|\mathbf{Qu}\|_{1,\Omega} \leqq Ch^2\|\mathbf{u}\|_{3,\Omega}. \tag{4.5}$$

Proof. By definition, we have for any $i = 1, \ldots, r$,

$$0 = p_i(\mathbf{P}_p\mathbf{u}) = p_i(\mathbf{Pu}) + p_i(\mathbf{Qu})$$

and

$$p_i(\mathbf{u}) = 0.$$

Consequently, we may write

$$|p_i(\mathbf{Qu})| = |p_i(\mathbf{Pu})| = |p_i(\mathbf{Pu} - \mathbf{u})| \leqq C\|\mathbf{Pu} - \mathbf{u}\|_{0,\Gamma}. \tag{4.6}$$

The trace theorem yields that $\mathbf{u}|_\Gamma \in (H^2(\Gamma))^M$ and it is well-known that

$$\|Pu_j - u_j\|_{0,\Gamma} \leqq Ch^2|u_j|_{2,\Gamma}, \quad j = 1, \ldots, M. \tag{4.7}$$

Combining (4.6), (4.7) and the inequality

$$|u_j|_{2,\Gamma} \leqq C\|u_j\|_{3,\Omega},$$

we get

$$|p_i(\mathbf{Qu})| \leqq Ch^2\|\mathbf{u}\|_{3,\Omega}. \tag{4.8}$$

On the other hand, the norm

$$\|\mathbf{v}\|_W = (\mathbf{v}, \mathbf{v})_W^{1/2}$$

is equivalent with the norm of $(H^1(\Omega))^M$ (see [3], Part I, Theorem 2.3). Thus we have

$$c\|\mathbf{Qu}\|_{1,\Omega} \leqq \|\mathbf{Qu}\|_W = \Big(\sum_{i=1}^r p_i^2(\mathbf{Qu})\Big)^{1/2} \leqq Ch^2\|\mathbf{u}\|_{3,\Omega}. \qquad \blacksquare$$

Next we shall verify an analogue of Lemma 2.1 for the modified interpolation operator $\mathbf{P}_p$. In fact,

$$\mathbf{e} = \mathbf{u} - \mathbf{P}_p\mathbf{u} = \mathbf{u} - \mathbf{Pu} - \mathbf{Qu},$$

$$((\mathbf{e}, \mathbf{v})) = ((\mathbf{u} - \mathbf{Pu}, \mathbf{v})) + ((\mathbf{Qu}, \mathbf{v})).$$

The term with $\mathbf{u} - \mathbf{Pu}$ has been estimated in Lemma 2.1. Using Lemma 4.1, we may write

$$|((\mathbf{Qu}, \mathbf{v}))| \leqq C\Big(\|\mathbf{Qu}\|_{1,\Omega}\,\|\mathbf{v}\|_{1,\Omega} + \sum_{i,j=1}^{M}\|Qu_i\|_{0,\partial\Omega}\,\|v_j\|_{0,\partial\Omega}\Big) \leqq$$

$$\leqq C_1\|\mathbf{Qu}\|_{1,\Omega}\,\|\mathbf{v}\|_{1,\Omega} \leqq C_2h^2\|\mathbf{u}\|_{3,\Omega}\,\|\mathbf{v}\|_{1,\Omega}.$$

Altogether, Lemma 2.1 holds for the modified operator $\mathbf{P}_p$, too.

Let us define the discrete problem:

Find $\mathbf{u}_h \in \mathbf{W}_h^{\sim} \cap \mathbf{V}_p$ such that

$$((\mathbf{u}_h, \mathbf{v}_h)) = (\mathbf{f}, \mathbf{v}_h)^*_{0,\Omega_h} + (\mathbf{g}, \mathbf{v}_h)^*_{0,\partial\Omega} \quad \forall \mathbf{v}_h \in \mathbf{W}_h^{\sim} \cap \mathbf{V}_p . \tag{4.9}$$

On the basis of the inequality (4.2), we can prove easily an analogue of Theorem 2.1 for the modified operator $\mathbf{P}_p$.

Next we apply the same averaging technique as before. The main Theorem 3.1 can be proven with a slight change in the argument, as follows.

Since G_h is linear, we have

$$G_h(P_p u_j) = G_h(P u_j) + G_h(Q u_j) , \quad j = 1, \ldots, M ,$$

$$\|\operatorname{grad} u_j - G_h(P_p u_j)\|_{0,\Omega_h^*} \leqq \|\operatorname{grad} u_j - G_h(P u_j)\|_{0,\Omega_h^*} + \|G_h(Q u_j)\|_{0,\Omega_h^*} . \tag{4.10}$$

On the other hand, the definition (3.1) of [2] yields

$$G_h(Q u_j) = \operatorname{grad} Q u_j \quad \text{in} \quad \Omega_h^* ,$$

since $Q u_j \in P_1(\Omega)$. Using Lemma 4.1, we obtain

$$\|G_h(Q u_j)\|_{0,\Omega_h^*} \leqq \|Q u_j\|_{1,\Omega} \leqq C h^2 \|\mathbf{u}\|_{3,\Omega} . \tag{4.11}$$

Combining (4.10), (4.11) and the argument used in the proof of Theorem 4.1 of [2], we are led to the assertion (3.1) of Theorem 3.1.

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Souhrn

O JEDNOM SUPERKONVERGENTNÍM SCHÉMATU V METODĚ KONEČNÝCH PRVKŮ PRO ELIPTICKÉ SYSTÉMY

II. OKRAJOVÉ PODMÍNKY NEWTONOVA NEBO NEUMANNOVA TYPU

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

V článku se předkládá jednoduché superkonvergentní schéma pro derivace lineárních trojúhelníkových elementů, které jsou použity k řešení eliptických systémů 2. řádu s okrajovými podmínkami Newtonova nebo Neumannova typu. Pro omezené rovinné oblasti s hladkou hranicí je dokázána lokální superkonvergence derivací v L^2 — normě řádu $O(h^{3/2})$. Článek je přímým pokračováním práce [2], která pojednává o podobném problému s Dirichletovými okrajovými podmínkami.

Резюме

ОБ ОДНОЙ СУПЕРСХОДЯЩЕЙСЯ СХЕМЕ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ

II. ГРАНИЧНЫЕ УСЛОВИЯ ТИПА НЬЮТОНА И НЕЙМАНА

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

В статье предлагается простая схема с суперсходимостью для производных конечноэлементного решения построенного с помощью линейных треугольных элементов, используемых для решения эллиптических систем 2-го порядка с граничными условиями типа Ньютона или Неймана. Для ограниченных плоских областей с гладкой границей доказана локальная суперсходимость производных в L^2 — норме порядка $O(h^{3/2})$. Статья является прямым продолжением работы [2], которая посвящена подобной проблеме с граничными условиями Дирихле.

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