Ivan Hlaváček; Michal Křížek

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ON A SUPERCONVERGENT FINITE ELEMENT SCHEME FOR ELLIPTIC SYSTEMS

I. DIRICHLET BOUNDARY CONDITION

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

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Summary. Second order elliptic systems with Dirichlet boundary conditions are solved by means of affine finite elements on regular uniform triangulations. A simple averaging scheme is proposed, which implies a superconvergence of the gradient. For domains with enough smooth boundary, a global estimate $O(h^{3/2})$ is proved in the L^2-norm. For a class of polygonal domains the global estimate $O(h^2)$ can be proven.

Keywords: finite elements, superconvergence, post-processing, averaged gradient, elliptic systems

AMS Subject classification: 65 N 30, 73 C 99.

1. INTRODUCTION

In this article we deal with a system of linear second order elliptic equations with Dirichlet boundary conditions in a bounded plane domain. A simple averaging scheme guaranteeing a superconvergence of the derivatives of the Galerkin solution based on linear finite elements is presented. The article can be considered as a continuation of [9], where a local superconvergence for the Poisson equation has been analyzed. To the authors' knowledge, no superconvergence analysis has been published for systems of elliptic equations until now (except for [21] and a short note in [4]).

The elliptic systems considered here include Lamé's equations of linear anisotropic elasticity, the model of Cosserat continuum [7], and the standard Poisson equation with non-homogeneous boundary conditions.

In Section 2, we show that $\boldsymbol{u}_h - \boldsymbol{P}\boldsymbol{u}$ vanishes faster in the H^1-norm than $\boldsymbol{u} - \boldsymbol{u}_h$ or $\boldsymbol{u} - \boldsymbol{P}\boldsymbol{u}$, where $\boldsymbol{u}_h$ is the Galerkin approximation of the solution $\boldsymbol{u}$, and $\boldsymbol{P}\boldsymbol{u}$ is the linear interpolation of $\boldsymbol{u}$ over uniform triangular meshes. An analogous phenomenon has been observed [2, 4, 5, 11, 12, 14, 15, 16, 22] for the Poisson equation when employing linear triangular elements over uniform (quasiuniform or piecewise uniform) triangulations. In Section 3, an avaraged gradient (based on averaging at nodes) is introduced for linear elements, and its approximation properties in the L^2-norm are derived. In Section 4, we combine the results of Sections 2 and 3 to obtain a global superconvergence estimate for the derivatives of $\boldsymbol{u}$. Let us mention that the result of Section 2 can be used also for other averaging techniques at centroids or midpoints of sides, which were introduced in [4, 5, 11, 12]. Other important papers on post-processing with a superconvergence of the gradient include [3, 13, 20, 23] etc., see also the survey paper [10].

2. SOME LEMMAS FOR ELLIPTIC SECOND ORDER SYSTEMS AND NUMERICAL INTEGRATION

We shall consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. For the Euclidean norm in $\mathbb{R}^d$ we adopt the notation $\|\cdot\|$. Let us denote by $H^k(\Omega) = W^{k,2}(\Omega)$, $k = 0, 1, \ldots$, the standard Sobolev spaces with the norm $\|\cdot\|_{k,\Omega}$, and the seminorm $|\cdot|_{k,\Omega}$ of all the derivatives of k-th order. We also set

$$(\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x\,, \quad \boldsymbol{f}, \boldsymbol{v} \in (L^2(\Omega))^M\,,$$

where $M \geqq 1$ is a given integer, and we write for brevity

$$\boldsymbol{W} = (H^1(\Omega))^M\,.$$

Assume that the following functions are given:

$$\bar{\boldsymbol{u}} \in \boldsymbol{W}, \boldsymbol{f} \in (L^2(\Omega))^M\,,$$

a matrix $\boldsymbol{K}(\boldsymbol{x})$ of the type $\varkappa \times \varkappa$ with entries $K_{ij} \in P_s(\bar{\Omega})$ (i.e. polynomials of at most s-th order) for all $i, j \in \{1, \ldots, \varkappa\}$ and some integer $s \geqq 0$, $\boldsymbol{K}$ is symmetric and positive definite uniformly with respect to $x \in \Omega$, coefficients n_{im}, $n_{imt} \in \mathbb{R}$, $1 \leqq i \leqq \varkappa$, $1 \leqq m \leqq M$, $t = 1, 2$. We consider the system of operators

$$N_i(\boldsymbol{v}) = \sum_{m=1}^{M} \left(\sum_{t=1}^{2} n_{imt} \frac{\partial v_m}{\partial x_t} + n_{im} v_m \right), \quad \boldsymbol{v} \in \boldsymbol{W}, \quad 1 \leqq i \leqq \varkappa\,,$$

and the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sum_{i,j=1}^{\varkappa} K_{ij}\, N_i(\boldsymbol{u})\, N_j(\boldsymbol{v}) \,\mathrm{d}x\,.$$

Assume that the system $\{N_i(\boldsymbol{v})\}_{i=1}^{\varkappa}$ is coercive on the space $\boldsymbol{W}$, i.e. a constant $c > 0$ exists such that

$$\sum_{i=1}^{\varkappa} \|N_i(\boldsymbol{v})\|_{0,\Omega}^2 + \|\boldsymbol{v}\|_{0,\Omega}^2 \geqq c\|\boldsymbol{v}\|_{1,\Omega}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{W}\,.$$

We define the space of test functions

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{W} \mid \gamma\mathbf{v} = 0 \text{ on } \partial\Omega\}$$

(where γ denotes the trace operator) and the problem to find $\mathbf{u} \in \bar{\mathbf{u}} + \mathbf{V}$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathbf{V} . \tag{2.1}$$

Let us derive the classical formulation of the boundary value problem (2.1). To this end we write

$$N_j(\mathbf{v}) = \sum_{m=1}^{M} N_{jm}(v_m) ,$$

where

$$N_{jm}(v_m) = \sum_{t=1}^{2} n_{jmt} \frac{\partial v_m}{\partial x_t} + n_{jm} v_m .$$

It is easy to deduce that (2.1) leads to the system

$$\sum_{i,j=1}^{\varkappa} N^*_{jm}(K_{ij} N_i(\mathbf{u})) = f_m , \quad m = 1, \ldots, M ,$$

in the domain Ω, where

$$N^*_{jm}(\mathbf{w}) = - \sum_{t=1}^{2} n_{jmt} \frac{\partial \mathbf{w}}{\partial x_t} + n_{jm} \mathbf{w}$$

is the operator formally adjoint to N_{jm}.

Remark 2.1. The following theorem holds (see [18], Th. 3.2). Let us put

$$\bar{N}_{im}\xi = \sum_{t=1}^{2} n_{imt} \xi_t , \quad 1 \leqq i \leqq \varkappa , \quad 1 \leqq m \leqq M .$$

The system $\{N_i(\mathbf{v})\}_{i=1}^{\varkappa}$ is coercive on $\mathbf{W}$ if and only if the rank of the matrix $(\bar{N}_{im}\xi)$ equals M for all non-zero vectors ξ from the complex two-dimensional space.

Example. Let us consider the two-dimensional theory of an elastic non-homogeneous anisotropic body. We define $\varkappa = 3$, $M = 2$,

$$N_1(\mathbf{v}) = \varepsilon_{11}(\mathbf{v}) = \frac{\partial v_1}{\partial x_1} , \quad N_2(\mathbf{v}) = \varepsilon_{22}(\mathbf{v}) = \frac{\partial v_2}{\partial x_2} ,$$

$$N_3(\mathbf{v}) = \sqrt{(2)}\, \varepsilon_{12}(\mathbf{v}) = \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) \Big/ \sqrt{2} ,$$

$$\mathbf{K} = \begin{bmatrix} c_{1111} & & \text{sym.} \\ c_{2211} & c_{2222} & \\ c_{1211}/\sqrt{2} & c_{1222}/\sqrt{2} & 2c_{1212} \end{bmatrix} ,$$

$$\sigma_{ij} = \sum_{m,k=1}^{2} c_{ijmk}\varepsilon_{mk}\,, \quad i, j = 1, 2\,,$$

holds for the stress components. Assume that

$$c_{mkij} = c_{jimk} = c_{ijmk} \in P_s(\bar{\Omega})\,, \quad s \geqq 0\,,$$

and a positive constant c_0 exists such that

$$\sum_{i,j,m,k=1}^{2} c_{ijmk}(x)\,\varepsilon_{ij}\varepsilon_{mk} \geqq c_0 \sum_{i,j=1}^{2} \varepsilon_{ij}^2 \tag{2.2}$$

holds for all symmetric matrices (ε_{ij}) and all $\mathbf{x} \in \bar{\Omega}$.

It is readily seen that

$$\sum_{i,j=1}^{3} K_{ij}\, N_i(\mathbf{u})\, N_j(\mathbf{v}) = \sum_{i,j,m,k=1}^{2} c_{ijmk}\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{mk}(\mathbf{v})\,, \tag{2.3}$$

$$\sum_{i=1}^{3} N_i^2(\mathbf{v}) = \sum_{i,j=1}^{2} \varepsilon_{ij}^2(\mathbf{v})\,.$$

By virtue of (2.2), (2.3), the matrix $\mathbf{K}$ is uniformly positive definite. Using the theorem of Remark 2.1, one can prove that the system $\{N_i(\mathbf{v})\}_{i=1}^3$ is coercive on $\mathbf{W}$. ■

Moreover, we define the subspace

$$\mathscr{P} = \{v \in \mathbf{V} \mid \sum_{i=1}^{\varkappa} \|N_i(\mathbf{v})\|_{0,\Omega} = 0\}\,.$$

Let $\mathscr{P} = \{0\}$ (i.e. $\mathscr{P}$ reduces to the zero element). Then the following inequality (of Korn's type) holds for all $\mathbf{v} \in \mathbf{V}$:

$$a(\mathbf{v}, \mathbf{v}) \geqq c_0 \sum_{i=1}^{\varkappa} \|N_i(\mathbf{v})\|_{0,\Omega}^2 \geqq c_1 \|\mathbf{v}\|_{1,\Omega}^2\,. \tag{2.4}$$

(For the proof see e.e. [19], Lemma 11.3.2).

Henceforth we restrict ourselves to a certain subclass of domains with a Lipschitz boundary.

We say that Ω belongs to the class $\mathscr{C}^3(d)$ if:

(i) the boundary $\partial\Omega$ is 3-times continuously differentiable;

(ii) a positive constant d exists such that all circles, with radius d, which are tangential to $\partial\Omega$, have no other common points with $\partial\Omega$;

(iii) Ω is bounded.

Henceforth h will denote a positive small parameter, tending to zero and all constants C, C_i are positive, independent of h.

There exist polygonal approximations Ω_h of $\Omega \in \mathscr{C}^3(d)$ such that $\bar{\Omega}_h \subset \bar{\Omega}$,

$$\max_{x\in\partial\Omega_h} \operatorname{dist}(\mathbf{x}, \partial\Omega) \leqq Ch^2 \tag{H 1}$$

and the sides of $\partial\Omega_h$ are not longer than h. The sides of $\partial\Omega_h$ are chords or tangents of convex or of concave arcs, respectively. The points of inflexion of $\partial\Omega$ coincide with a vertex of $\partial\Omega_h$.

We consider triangulations $\mathscr{T}_h$ of Ω_h, consisting of

(a) a regular part $\mathscr{T}_h^*$ generated by uniform parallelograms and carving a domain Ω_h^* (so that the sides of $\partial\Omega_h^*$ are parallel with at most 3 different directions independent of h) – see Fig. 2.1, and such that

$$\operatorname{diam} T = h \quad \forall T \in \mathscr{T}_h^*$$

(T denotes any (closed) triangle of the triangulation). The boundary $\partial\Omega_h^*$ does not contain any tangent to $\partial\Omega$.

(b) an irregular part $\mathscr{T}_h^i$ carving the set $\Omega_h - \Omega_h^*$ and such that

$$\max_{T \in \mathscr{T}_h{}^i} \operatorname{diam} T \leqq h .$$

Assume that $\mathscr{T}_h^*$ is "maximal" in the following sense:

(H 2) $$\max_{x \in \partial\Omega_h} \operatorname{dist}(\mathbf{x}, \partial\Omega_h^*) \leqq 2h .$$

Finally, assume that the family $\mathfrak{M} = \{\mathscr{T}_h\}$, $h \to 0$, $\mathscr{T}_h = \mathscr{T}_h^* \cup \mathscr{T}_h^i$, is *strongly regular* in the standard sense, i.e. constants C_1, C_2 exist such that:

(i) all angles of all triangles in $\{\mathscr{T}_h\}$ are greater or equal to C_1,

(ii) the ratio of the lengths of any two sides in any $\mathscr{T}_h \in \mathfrak{M}$ is not less than C_2.

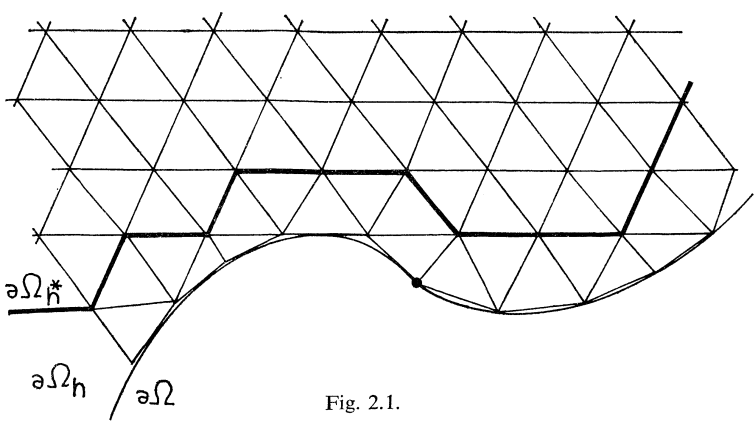

Fig. 2.1.

A constant regular 2×2 matrix exists such that

$$\mathscr{T}_h = F(\mathscr{T}_h^0), \quad \mathscr{T}_h^* = F(\mathscr{T}_h^{*0}), \quad \mathscr{T}_h^i = F(\mathscr{T}_h^{i0}),$$

where the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by the matrix $\mathscr{F}$,

$$\mathbf{x} = \mathscr{F}\mathbf{X} + \mathbf{x}_h^0, \quad \mathbf{x} = (x_1, x_2), \quad \mathbf{X} = (X_1, X_2),$$

and $\mathscr{T}_h^{*0}$ is generated by a uniform square mesh of step-size h. Here $\mathbf{x}_h^0 \in \mathbb{R}^2$ may depend on h whereas $\mathscr{F}$ is independent of h. We shall write

$$\Omega^0 = F^{-1}(\Omega), \quad \Omega_h^0 = F^{-1}(\Omega_h), \quad \Omega_h^{*0} = F^{-1}(\Omega_h^*).$$

Denote by T_c, $T_t \subset \Omega - \Omega_h$ the segments adjacent to the chords and tangents, respectively. We define the space

$$W_h = \{v \in H^1(\Omega) \mid v|_T \in P_1(T) \quad \forall T \in \mathscr{T}_h,$$

$$v|_{T_c} \in P_1(T_c), v|_{T_t} \in P_1(T_t) \ \forall T_c, T_t \subset \Omega - \Omega_h\}$$

and the interpolation mapping $P: H^1(\Omega) \cap C(\bar{\Omega}) \to W_h$ by the relations:

$$Pu(\mathbf{x}) = u(\mathbf{x}) \quad \text{at all nodes} \quad \mathbf{x} \in \Omega_h,$$

$$Pu(\mathbf{x}) = u(\mathbf{y}) \quad \text{at all nodes of} \quad \partial\Omega_h,$$

where $\mathbf{y}$ is the point of $\partial\Omega$ nearest to $\mathbf{x}$, $\partial Pu/\partial t = 0$ in $\Omega - \Omega_h$, where the direction t is parallel with $\mathbf{xy}$ in parts adjacent to tangent segments and perpendicular to chords in parts adjacent to the latter. Let $\mathbf{Pu} = (Pu_1, \ldots, Pu_M)$ for $\mathbf{u} \in \mathbf{W} \cap (C(\bar{\Omega}))^M$.

Finally let us define

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} \mid \mathbf{v}|_T \in (P_1(T))^M \ \forall T \in \mathscr{T}_h, \ \mathbf{v} = 0 \text{ in } \Omega - \Omega_h\}.$$

Note that $\mathbf{V}_h = \mathbf{P}(\mathbf{V} \cap (C(\bar{\Omega}))^M)$.

To analyze the approximate solution (cf. (2.49)), we shall need the three following lemmas.

Lemma 2.1. *Let Ω belong to the class $\mathscr{C}^3(d)$, $\mathbf{u} \in (H^3(\Omega))^M$, $\mathbf{v} \in \mathbf{V}_h$. Then*

$$|a(\mathbf{Pu} - \mathbf{u}, \mathbf{v})| \leqq Ch^{3/2} \|\mathbf{u}\|_{3,\Omega} \|\mathbf{v}\|_{1,\Omega} \tag{2.5}$$

holds for sufficiently small h.

Proof. Given any function $\varphi(\mathbf{x})$ in Ω, we define

$$\varphi^0(\mathbf{X}) = \varphi(\mathscr{F}\mathbf{X} + \mathbf{x}_h^0)$$

Then we may write for all $j = 1, \ldots, \varkappa$,

$$N_j(\mathbf{v}) = \sum_{m=1}^{M} \left(\sum_{t=1}^{2} n_{jmt} \sum_{i=1}^{2} \frac{\partial v_m^0}{\partial X_i} \mathscr{F}_{it}^{-1} + n_{jm} v_m^0 \right) = N_j^0(\mathbf{v}^0) =$$

$$= \sum_{m=1}^{M} \left(b_{jm}^{(1)} \frac{\partial v_m^0}{\partial X_1} + b_{jm}^{(2)} \frac{\partial v_m^0}{\partial X_2} + n_{jm} v_m^0 \right),$$

where $b_{jm}^{(r)} = \text{const.}$, $r = 1, 2, m = 1, \ldots, M$. Defining

$$\mathbf{e} = \mathbf{u} - \mathbf{Pu},$$

we have

$$a(\mathbf{e}, \mathbf{v}) = \int_{\Omega^0} \sum_{i,j=1}^{\varkappa} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, N_j^0(\mathbf{v}^0) \, |\det \mathscr{F}| \, \mathrm{d}X =$$

$$= \sum_{m=1}^{M} \int_{\Omega^0} \sum_{i,j=1}^{\varkappa} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, b_{jm}^{(1)} \, |\det \mathscr{F}| \, \frac{\partial v_m^0}{\partial X_1} \, \mathrm{d}X +$$

$$+ \sum_{m=1}^{M} \int_{\Omega^0} \sum_{i,j=1}^{\varkappa} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, b_{jm}^{(2)} \, |\det \mathscr{F}| \, \frac{\partial v_m^0}{\partial X_2} \, \mathrm{d}X +$$

$$+ \sum_{m=1}^{M} \int_{\Omega^0} \sum_{i,j=1}^{\varkappa} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, n_{jm} v_m^0 |\det \mathscr{F}| \, \mathrm{d}X = a_1 + a_2 + a_3 \, .$$

Let us consider the term a_1. Since v_m is linear in any triangle $T \in \mathscr{T}_h$, $\partial v_m^0 / \partial X_1 \in$ $\in P_0(T^0)$. We may write

$$a_1 = \sum_{m=1}^{M} (S_m + S_m^*) \, , \tag{2.6}$$

where

$$S_m = \sum_{t=1}^{K_A} \frac{\partial v_m^0}{\partial X_1} \bigg|_{A_t{}^0} \sum_{i,j=1}^{\varkappa} b_{jm}^{(1)} \, |\det \mathscr{F}| \int_{A_t{}^0} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, \mathrm{d}X \, , \tag{2.7}$$

$$S_m^* = \sum_{k=1}^{K_B} \frac{\partial v_m^0}{\partial X_1} \bigg|_{B_k{}^0} \sum_{i,j=1}^{\varkappa} b_{jm}^{(1)} |\det \mathscr{F}| \int_{B_k{}^0} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, \mathrm{d}X \, , \tag{2.8}$$

A_t^0 are parallelograms with a diagonal parallel with the X_1-axis (see Fig. 2.2),

$$A_t^0 = T_k^0 \cup T_{k'}^0 \, ,$$

T_k^0, $T_{k'}^0 \in \mathscr{T}_h^*$ and $B_k^0 \in \mathscr{T}_h$ are the remaining triangles.

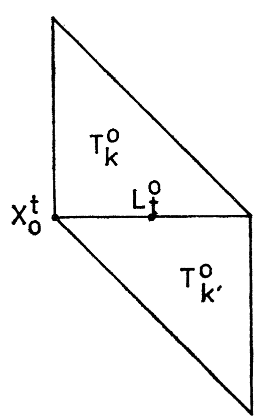

Fig. 2.2.

Note that the continuity of v_m^0 implies that $\partial v_m^0/\partial X_1$ is constant in every parallelogram A_t^0. Let G_k^0 denote the centroid of T_k^0, and L_t^0 the centroid of A_t^0.

We decompose S_m as follows:

$$S_m = S_{m1} + S_{m2} + S_{m3}\,, \tag{2.9}$$

where

$$S_{m1} = \sum_{t=1}^{K_A} \frac{\partial v_m^0}{\partial X_1}\bigg|_{A_t{}^0} \sum_{i,j=1}^{\varkappa} b_{jm}^{(1)} |\det \mathscr{F}| \frac{h^2}{2} K_{ij}^0(L_t^0) \left(N_i^0(\mathbf{e}^0)|_{G_k{}^0} + N_i^0(\mathbf{e}^0)|_{G_k{}'^0}\right),$$

$$S_{m2} = \sum_{k=1}^{K_A} \frac{\partial v_m^0}{\partial X_1}\bigg|_{T_k{}^0} \sum_{i,j=1}^{\varkappa} b_{jm}^{(1)} |\det \mathscr{F}| \int_{T_k{}^0} (K_{ij}^0(X) - K_{ij}^0(L_t^0))\, N_i^0(\mathbf{e}^0)\, \mathrm{d}X\,,$$

$$S_{m3} = \sum_{k=1}^{K_A} \frac{\partial v_m^0}{\partial X_1}\bigg|_{T_k{}^0} \sum_{i,j=1}^{\varkappa} b_{jm}^{(1)} |\det \mathscr{F}| \int_{T_k{}^0} K_{ij}^0(L_t^0) \left(N_i^0(\mathbf{e}^0) - N_i^0(\mathbf{e}^0)\bigg|_{G_k{}^0}\right) \mathrm{d}X\,.$$

We shall estimate the individual terms. Let us put

$$F_i^t(\mathbf{u}^0) = N_i^0(\mathbf{e}^0)|_{G_k{}^0} + N_i^0(\mathbf{e}^0)|_{G_k{}'^0}\,. \tag{2.10}$$

We show that

$$|F_i^t(\mathbf{u}^0)| \leqq Ch\|\mathbf{u}^0\|_{3,A_t{}^0} \leqq C_1 h\|\mathbf{u}\|_{3,A_t} \tag{2.11}$$

for any $t = 1, \ldots, K_A$. To this end we use the mapping

$$\hat{\mathbf{x}} = (\mathbf{X} - \mathbf{X}_0^t)/h \tag{2.12}$$

(see Fig. 2.2) and define

$$\hat{\varphi}(\hat{\mathbf{x}}) = \varphi^0(h\hat{\mathbf{x}} + \mathbf{X}_0^t)$$

for any function φ^0. By (2.12) any parallelogram A_0^t is mapped onto the reference "unit" parallelogram $\alpha = \tau \cup \tau'$, where τ and τ' denotes the upper and lower triangle respectively.

Let us drop the subscripts t for the time being. It follows easily that

$$N_i^0(\mathbf{e}^0) = \sum_{m=1}^{M} \left(\left(b_{im}^{(1)} \frac{\partial \hat{e}_m}{\partial \hat{x}_1} + b_{im}^{(2)} \frac{\partial \hat{e}_m}{\partial \hat{x}_2}\right)\Big/h + n_{im}\,\hat{e}_m\right), \tag{2.13}$$

where

$$\hat{e}_m = \hat{u}_m - (Pu)_m^{\wedge}\,.$$

Therefore, we may write

$$F_i(\mathbf{u}^0) = \frac{1}{h}\sum_{m=1}^{M} \hat{f}_{im}(\hat{u}_m) + \sum_{m=1}^{M} n_{im}\,\hat{g}_m(\hat{u}_m)\,, \tag{2.14}$$

$$\hat{f}_{im}(\hat{u}_m) = b_{im}^{(1)}\left(\frac{\partial \hat{e}_m}{\partial \hat{x}_1}(\gamma) + \frac{\partial \hat{e}_m}{\partial \hat{x}_1}(\gamma')\right) + b_{im}^{(2)}\left(\frac{\partial \hat{e}_m}{\partial \hat{x}_2}(\gamma) + \frac{\partial \hat{e}_m}{\partial \hat{x}_2}(\gamma')\right),$$

where γ and γ' are the centroids of τ and τ', respectively, and

$$\hat{g}_m(\hat{u}_m) = e_m(\gamma) - e_m(\gamma').$$

By the direct calculation we can verify that

$$\hat{f}_{im}(p) = 0 \quad \forall p \in P_2(\alpha), \quad \forall i, m. \tag{2.15}$$

In fact, let us choose e.g.

$$p(\hat{x}_1, \hat{x}_2) = \hat{x}_1^2.$$

Then

$$\hat{e}_m = \hat{x}_1^2 - \hat{x}_1,$$

$$\hat{f}_{im}(p) = b_{im}^{(1)}(2\gamma - 1 + 2\gamma' - 1) = 0, \quad \text{a.s.o.}$$

Since

$$|\hat{f}_{im}(\hat{u}_m)| \leqq C\|\hat{u}_m\|_{3,\alpha},$$

and (2.15) holds, we obtain

$$|\hat{f}_{im}(\hat{u}_m)| \leqq C|\hat{u}_m|_{3,\alpha} \tag{2.16}$$

using the Bramble-Hilbert lemma (see [6], Theorem 4.1.3).

Obviously, we have

$$\hat{g}_m(p) = 0 \quad \forall p \in P_1(\alpha),$$

$$|\hat{g}_m(\hat{u}_m)| \leqq C\|\hat{u}_m\|_{2,\alpha},$$

so that the Bramble-Hilbert lemma yields

$$|\hat{g}_m(\hat{u}_m)| \leqq C|\hat{u}_m|_{2,\alpha}. \tag{2.17}$$

For any $w^0 \in H^n(T_k^0)$, $n = 0, 1, 2, 3$, we have (see [6], pp. 118 and 122)

$$|\hat{w}|_{n,\tau} \leqq Ch^{n-1}|w^0|_{n,T_k^0} \quad \forall T_k^0 \in \mathcal{T}_h^0. \tag{2.18}$$

Substituting (2.16), (2.17) and (2.18) into (2.10), we arrive at

$$|F_i^t(\mathbf{u}^0)| \leqq h^{-1}\sum_{m=1}^{M} C_1 h^2 |u_m^0|_{3,A_t^0} + \sum_{m=1}^{M} C_2 h |u_m^0|_{2,A_t^0} \leqq C_3 h \|\mathbf{u}^0\|_{3,A_t^0} \tag{2.19}$$

and (2.11) follows immediately.

Since v_m is linear in T_k, we have

$$|v_m^0|_{1,T_k^0} = 2^{-1/2}h\|\nabla_X v_m^0|_{T_k^0}\|,$$

so that

$$\left|\frac{\partial v_m^0}{\partial X_1}\right| \leqq Ch^{-1}|v_m^0|_{1,T_k^0} \leqq C_1 h^{-1}|v_m|_{1,T_k} \tag{2.20}$$

holds in any triangle T_k^0.

Using (2.11), (2.20) and the estimate

$$|K_{ij}^0(L_t^0)| \leqq \|K_{ij}^0\|_{C(\bar{\Omega})} \leqq C_3 \quad \forall i, j\,,$$

we obtain

$$(2.21) \qquad |S_{m1}| \leqq \sum_{t=1}^{K_A} Ch^2 \|\boldsymbol{u}\|_{3,A_t} |v_m|_{1,A_t} \leqq Ch^2 \|\boldsymbol{u}\|_{3,\Omega} |v_m|_{1,\Omega}\,.$$

Let us pass to the term S_{m2}. Considering an arbitrary triangle T_k^0 and dropping the subscript k, we may write

$$|K_{ij}^0(\boldsymbol{X}) - K_{ij}^0(L_t^0)| \leqq \|K_{ij}^0\|_{C^1(\bar{\Omega}^0)} \,.\, h \leqq Ch\,,$$

$$(2.22) \qquad \left|\int_{T^0} (K_{ij}^0(\boldsymbol{X}) - K_{ij}^0(L_t^0))\, N_i^0(\mathbf{e}^0)\, \mathrm{d}X\right| \leqq Ch(\text{mes } T^0)^{1/2} \|N_i^0(\mathbf{e}^0)\|_{0,T^0}\,.$$

The well-known error estimate implies that

$$(2.23) \qquad \|N_i^0(\mathbf{e}^0)\|_{0,T^0} \leqq C \sum_{m=1}^{M} \| u_m^0 - (Pu_m)^0\|_{1,T^0} \leqq C_1 \sum_{m=1}^{M} h|u_m^0|_{2,T^0} \leqq C_2 h|\boldsymbol{u}|_{2,T}\,.$$

Combining (2.22), (2.23) and (2.20), we obtain

$$(2.24) \qquad |S_{m2}| \leqq \sum_k Ch^2 |v_m|_{1,T_k} |\boldsymbol{u}|_{2,T_k} \leqq C_1 h^2 |\boldsymbol{u}|_{2,\Omega} |v_m|_{1,\Omega}\,.$$

To estimate S_{m3}, we set

$$w^0 = N_i^0(\mathbf{e}^0)$$

and

$$E_k(w^0) = \int_{T_k^0} (w^0 - w^0(G_k^0))\, \mathrm{d}X\,.$$

Since $e_m^0 = u_m^0 - Pu_m^0 \in H^3(T_k^0)$, we have $N_i^0(\mathbf{e}^0) = w^0 \in H^2(T_k^0)$. Using the mapping (2.12), we come to

$$|E_k(w^0)| = \tfrac{1}{2}h^2 |\hat{E}(\hat{w})|\,, \quad \hat{E}(\hat{w}) = \int_\tau (\hat{w} - \hat{w}(\gamma))\, \mathrm{d}x\,.$$

It holds, however, that

$$\hat{E}(\hat{p}) = 0 \quad \forall p \in P_1(\tau)\,,$$

$$|\hat{E}(\hat{w})| \leqq \hat{C}\|w\|_{2,\tau}\,.$$

Thus the Bramble-Hilbert lemma yields

$$|\hat{E}(\hat{w})| \leqq \hat{C}|w|_{2,\tau}\,.$$

By virtue of (2.18), we obtain

$$(2.25) \qquad |E_k(w^0)| \leqq C_1 h^2 |\hat{w}|_{2,\tau} \leqq C_1 h^3 |w^0|_{2,T_k^0}\,.$$

It is readily seen that

$$(2.26) \qquad |w^0|_{2,T_k{}^0} = |N_i^0(\mathbf{e}^0)|_{2,T_k{}^0} \leqq C \sum_{m=1}^{M} (|e_m^0|_{3,T_k{}^0} + |e_m^0|_{2,T_k{}^0}) \leqq$$
$$\leqq C \sum_{m=1}^{M} \|u_m^0\|_{3,T_k{}^0} \leqq C_1 \|\mathbf{u}\|_{3,T_k} .$$

Inserting (2.26) into (2.25), using (2.20) and the boundedness of $K_{ij}^0(L_t^0)$, we get

$$(2.27) \qquad |S_{m3}| \leqq Ch^2 \sum_{k=1}^{K^A} \|\mathbf{u}\|_{3,T_k} |v_m|_{1,T_k} \leqq Ch^2 \|\mathbf{u}\|_{3,\Omega} |v_m|_{1,\Omega} .$$

It remains to estimate the term S_m^* in (2.6). Note that the estimate (2.20) holds for the triangles $B_k^0 \in \mathscr{T}_h^{i0}$ by virtue of the condition (ii) of the strong regularity of $\{\mathscr{T}_h\}$. For triangles B_k adjacent to tangent segments we apply the following argument. If we denote by $\boldsymbol{\Pi u}$ the linear interpolate of $\mathbf{u}$ in B_k, then for any node $\mathbf{x} \in \partial\Omega_h \cap \Omega$ $|\Pi u_m(\mathbf{x}) - Pu_m(\mathbf{x})| = |u_m(\mathbf{x}) - u_m(\mathbf{y})| \leqq \|\nabla u_m\|_{C(\bar\Omega)} Ch^2 \leqq C_1 h^2 \|u_m\|_{3,\Omega}$, $m = 1, \ldots, M$, as follows from (H 1) and the Sobolev embedding theorem. Consequently,

$$\|(\Pi u_m)^0 - (Pu_m)^0\|_{1,B_k{}^0} \leqq Ch^2 \|u_m\|_{3,\Omega} ,$$
$$\|u_m^0 - (Pu_m)^0\|_{1,B_k{}^0} \leqq \|u_m^0 - (\Pi u_m)^0\|_{1,B_k{}^0} + \|(\Pi u_m)^0 - (Pu_m)^0\|_{1,B_k{}^0} \leqq$$
$$\leqq C(h|u_m|_{2,B_k} + h^2 \|u_m\|_{3,\Omega}) .$$

Then (2.23) holds for any B_k^0 with an additional term $C_3 h^2 \|\mathbf{u}\|_{3,\Omega}$ on the right-hand side, where C_3 is positive only if B_k is adjacent to a tangent segment. Thus we obtain

$$(2.28) \qquad \left| \iint_{B_k{}^0} K_{ij}^0 \, N_i^0(\mathbf{e}^0) \, \mathrm{d}X \right| \leqq C(\operatorname{mes} B_k^0)^{1/2} \, \|N_i^0(\mathbf{e}^0)\|_{0,B_k{}^0} \leqq$$
$$\leqq C_1(h^2 |\mathbf{u}|_{2,B_k} + C_3 h^3 \|\mathbf{u}\|_{3,\Omega}) .$$

Combining (2.20) and (2.28), we arrive at the estimate

$$(2.29) \quad |S_m^*| \leqq \sum_{k=1}^{K_B} Ch^{-1} |v_m|_{1,B_k} \, h^2 |\mathbf{u}|_{2,B_k} + C_1 \sum_{k=1}^{K_B'} C_3 h^{-1} |v_m|_{1,B_k} \, h^3 \|\mathbf{u}\|_{3,\Omega} \leqq$$
$$\leqq Ch |\mathbf{u}|_{2,\Omega_B} |v_m|_{1,\Omega_B} + C_4 \, h^2 \|\mathbf{u}\|_{3,\Omega} \, h^{-1/2} |v_m|_{1,\Omega_B} ,$$

where Ω_B is the union of all triangles (both from $\mathscr{T}_h^*$ and from $\mathscr{T}_h^i$), which do not belong to any parallelogram A_t.

On the basis of the hypotheses (H 1) and (H 2) we conclude that

$$(2.30) \qquad \max_{x \in \bar\Omega_B} \operatorname{dist}(\mathbf{x}, \partial\Omega) \leqq 4h$$

holds for sufficiently small h.

Next we employ the following result of V. P. Iljin ([8], Theorem 4.4 and Remark 12.4). Let Ω^ε be a boundary strip of the width ε, lying in $\Omega \in \mathscr{C}^3(d)$, $0 < \varepsilon < d$. Then

$$(2.31) \qquad \|w\|_{0,\Omega^\varepsilon} \leqq C\varepsilon^{1/2} \|w\|_{1,\Omega}$$

holds for any $w \in H^1(\Omega)$, where C depends on d and the domain Ω only.

Using (2.30) and (2.31) with $\varepsilon = 4h$, we get

$$(2.32) \qquad |u_m|_{2,\Omega_B} \leqq Ch^{1/2} \|u_m\|_{3,\Omega} \, .$$

Inserting this into (2.29), we come to the estimate

$$(2.33) \qquad |S_m^*| \leqq Ch^{3/2} \|\mathbf{u}\|_{3,\Omega} \, |v_m|_{1,\Omega} \, .$$

Gathering the estimates (2.21), (2.24), (2.27) and (2.33), we obtain from (2.6) and (2.9)

$$(2.34) \qquad |a_1| \leqq Ch^{3/2} \|\mathbf{u}\|_{3,\Omega} \, |\mathbf{v}|_{1,\Omega} \, .$$

The term a_2 in the decomposition of $a(\mathbf{e}, \mathbf{v})$ can be estimated by an analogous way. The parallelograms A_t^0 are substituted by those with a diagonal parallel with the X_2-axis. Altogether, we get the same upper bound for $|a_2|$ as that in (2.34).

To estimate the last term a_3, we decompose it as follows

$$(2.35) \qquad a_3 = \sum_{m=1}^{M} (\bar{S}_m + \bar{S}_m^*) \, ,$$

where

$$\bar{S}_m = \sum_{t=1}^{K_A} \sum_{i,j=1}^{\varkappa} n_{jm} |\det \mathscr{F}| \int_{A_t{}^0} K_{ij}^0 v_m^0 \, N_i^0(\mathbf{e}^0) \, \mathrm{d}X \, ,$$

$$\bar{S}_m^* = \sum_{k=1}^{K_B} \sum_{i,j=1}^{\varkappa} n_{jm} |\det \mathscr{F}| \int_{B_k{}^0} K_{ij}^0 v_m^0 \, N_i^0(\mathbf{e}^0) \, \mathrm{d}X \, ,$$

with the same decomposition of $\bar{\Omega}_h$ as in (2.7) and (2.8). Next we write

$$(2.36) \qquad \bar{S}_m = \bar{S}_{m1} + \bar{S}_{m2} + \bar{S}_{m3} \, ,$$

$$\bar{S}_{m1} = \sum_{t=1}^{K_A} \sum_{i,j=1}^{\varkappa} n_{jm} |\det \mathscr{F}| \, K_{ij}^0(L_t) \, v_m^0 \, (L_t) \, \frac{h^2}{2} \, (N_i^0(e_m^0) \, |_{G_k{}^0} + N_i^0(e_m^0)|_{G_k{}'^0}) \, ,$$

$$\bar{S}_{m2} = \sum_{k=1}^{K^A} \sum_{i,j=1}^{\varkappa} n_{jm} |\det \mathscr{F}| \int_{T_k{}^0} (K_{ij}^0(\mathbf{X}) \, v_m^0(\mathbf{X}) - K_{ij}^0(L_t) \, v_m^0(L_t)) \, N_i^0(e_m^0) \, \mathrm{d}X \, ,$$

$$\bar{S}_{m3} = \sum_{k=1}^{K^A} \sum_{i,j=1}^{\varkappa} n_{jm} |\det \mathscr{F}| \int_{T_k{}^0} K_{ij}^0(L_t) \, v_m^0(L_t) \, (N_i^0(e_m^0) - N_i^0(e_m^0)|_{G_k}) \, \mathrm{d}X \, .$$

Since we have (see [6], p. 142)

$$(2.37) \qquad \|v_m^0\|_{C(T_k{}^0)} \leqq Ch^{-1} \|v_m^0\|_{0,T_k{}^0} \, ,$$

the estimate

$$(2.38) \qquad |\bar{S}_{m1}| \leqq Ch^2 \|\mathbf{u}\|_{3,\Omega} \, \|v_m\|_{0,\Omega}$$

follows by arguments similar to those of (2.11)–(2.21).

We can write

$$\int_{T_k^0} |K_{ij}^0(\mathbf{X})\, v_m^0(\mathbf{X}) - (K_{ij}^0 v_m^0)(L_t)|^2 \, \mathrm{d}X \leqq 2\int_{T_k^0} |K_{ij}^0(\mathbf{X}) - K_{ij}^0(L_t)|^2 \, (v_m^0)^2 \, \mathrm{d}X +$$
$$+ 2(K_{ij}^0(L_t))^2 \int_{T_k^0} |v_m^0(\mathbf{X}) - v_m^0(L_t)|^2 \, \mathrm{d}X \leqq Ch^2(\|v_m^0\|_{0,T_k^0}^2 + |v_m^0|_{1,T_k^0}^2) .$$

Combining this estimate with (2.23), we arrive at

$$(2.39) \qquad |\bar{S}_{m2}| \leqq C \sum_{k=1}^{K^A} h^2 \|v_m\|_{1,T_k} |\mathbf{u}|_{2,T_k} \leqq Ch^2 \|v_m\|_{1,\Omega} |\mathbf{u}|_{2,\Omega} .$$

Using the estimates (2.37), (2.25) and (2.26), we obtain

$$(2.40) \qquad |\bar{S}_{m3}| \leqq C \sum_{k=1}^{K^A} h^{-1} \|v_m\|_{0,T_k} h^3 \|\mathbf{u}\|_{3,T_k} \leqq C_1 h^2 \|\mathbf{u}\|_{3,\Omega} \|v_m\|_{0,\Omega} .$$

On the basis of (2.23) and the derivation of (2.28) we have

$$\left| \int_{B_k^0} K_{ij}^0 v_m^0 \, N_i^0(\mathbf{e}^0) \, \mathrm{d}X \right| \leqq C \|v_m^0\|_{0,B_k^0} \|N_i^0(\mathbf{e}^0)\|_{0,B_k^0} \leqq$$
$$\leqq \|v_m\|_{0,B_k}(C_1 h |\mathbf{u}|_{2,B_k} + C_3 h^2 \|\mathbf{u}\|_{3,\Omega}) .$$

Consequently,

$$|\bar{S}_m^*| \leqq Ch \|v_m\|_{0,\Omega_B} |\mathbf{u}|_{2,\Omega_B} + C_3 h^2 \|\mathbf{u}\|_{3,\Omega} \, Ch^{-1/2} \|\mathbf{v}_m\|_{0,\Omega_B}$$

follows easily. Using (2.30) and (2.31) twice, we are led to the estimate

$$(2.41) \qquad |\bar{S}_m^*| \leqq Ch^2 \|\mathbf{u}\|_{3,\Omega} \|v_m\|_{1,\Omega} .$$

From (2.35), (2.36), and the estimates (2.38)–(2.41), we get

$$|a_3| \leqq Ch^2 \|\mathbf{u}\|_{3,\Omega} \|\mathbf{v}\|_{1,\Omega} .$$

Combining the bounds for a_1, a_2, a_3, we arrive at the estimate (2.5). ■

Remark 2.2. Assume that the domain Ω has a polygonal boundary, which consists of line segments parallel with one of three different directions and the ratio of the lengths of any two parallel sides is rational. Then we can put

$$\mathscr{T}_h = \mathscr{T}_h^* , \quad \Omega = \Omega_h = \Omega_h^* ,$$

and a stronger estimate

$$(2.42) \qquad |a(\mathbf{u} - \mathbf{Pu}, \mathbf{v})| \leqq Ch^2 \|\mathbf{u}\|_{3,\Omega} \|\mathbf{v}\|_{1,\Omega}$$

holds for sufficiently small h.

In fact, the term S_m^* in (2.6) vanishes, since $v_m^0 = 0$ on $\partial\Omega$ implies that

$$\frac{\partial v_m^0}{\partial X_1} = 0 \quad \text{on any triangle} \quad B_k^0 .$$

Consequently, collecting the estimates (2.21), (2.24) and (2.27), (and their analogues[1]) for the terms a_2, a_3), we arrive at the estimate (2.42).

Lemma 2.2. *Let Ω belong to the class $\mathscr{C}^3(d)$, $f \in H^2(\Omega)$ and $v \in V_h$ for $M = 1$. Define the approximation*

$$(f, v)^*_{0,\Omega_h}$$

by the centroid rule on the triangulation $\mathscr{T}_h$. Then

$$|(f, v)_{0,\Omega} - (f, v)^*_{0,\Omega_h}| \leqq Ch^2 \|f\|_{2,\Omega} \|\mathbf{v}\|_{1,\Omega} \,.$$

Proof. Following the idea of [11], we define the local error on a single triangle

$$E_k(w) = \int_{T_k} w \, dx - w(G_k) \operatorname{mes} T_k \,, \quad T_k \in \mathscr{T}_h \,,$$

and write

$$|(f, v)_{0,\Omega} - (f, v)^*_{0,\Omega_h}| \leqq \sum_{k=1}^{K} \{|E_k(fv(G_k))| + |E_k(f(v - v(G_k)))|\} \,.$$

Applying the affine mapping

$$\mathbf{x} = \mathbf{B}_T \hat{\mathbf{x}} + \mathbf{b}_T \,,$$

which transforms the reference unit triangle τ onto T_k, we easily deduce

$$|E_k(w)| \leqq Ch^2 |\hat{E}(\hat{w})| \,, \tag{2.43}$$

where

$$\hat{E}(\hat{w}) = \int_\tau (\hat{w} - \hat{w}(\gamma)) \, d\hat{x} \,, \quad \hat{w}(\hat{\mathbf{x}}) = w(\mathbf{B}_T \hat{\mathbf{x}} + \mathbf{b}_T) \,.$$

Since

$$|\hat{E}(\hat{w})| \leqq C \|\hat{w}\|_{2,\tau} \,,$$
$$\hat{E}(p) = 0 \quad \forall p \in P_1(\tau) \,,$$

the Bramble-Hilbert lemma yields

$$|\hat{E}(\hat{w})| \leqq C |\hat{w}|_{2,\tau} \,.$$

Using the estimate (cf. (2.18))

$$|\hat{w}|_{2,\tau} \leqq Ch |w|_{2,T_k} \,,$$

we obtain

$$|E_k(w)| \leqq Ch^3 |w|_{2,T_k} \,. \tag{2.44}$$

Moreover, we have (by virtue of the strong regularity of $\{\mathscr{T}_h\}$)

$$|v(G_k)| \leqq \|v\|_{C(T_k)} \leqq Ch^{-1} \|v\|_{0,T_k} \quad \forall T_k \in \mathscr{T}_h \,, \tag{2.45}$$

---

[1]) Here suitable "regularized" domains $\tilde{\Omega} \neq \Omega$, $\tilde{\Omega} \in \mathscr{C}^3(d)$, have to be used when employing Iljin's inequality (2.31).

so that

$$(2.46) \qquad |E_k(f\,v(G))| = |E_k(f)|\,|v(G_k)| \leqq Ch^2\|f\|_{2,T_k}\,\|v\|_{1,T_k}$$

follows from (2.44) and (2.45).

Using again (2.43), we obtain

$$(2.47) \quad |E_k(f(v - v(G_k)))| \leqq Ch^2|\hat{E}(\hat{f}(\hat{v} - \hat{v}(\gamma)))| = Ch^2\left|\int_\tau \hat{f}(\hat{v} - \hat{v}(\gamma))\,\mathrm{d}\hat{x}\right|.$$

If we put

$$\mathscr{F}(\hat{f}) = \int_\tau \hat{f}(\hat{v} - \hat{v}(\gamma))\,\mathrm{d}\hat{x}\,.$$

then

$$\mathscr{F}(p) = 0 \quad \forall p \in P_0(\tau)\,,$$

since $\hat{v}$ is linear, and

$$|\mathscr{F}(\hat{f})| \leqq C\|\hat{v} - \hat{v}(\gamma)\|_{0,\tau}\,\|\hat{f}\|_{1,\tau}\,.$$

From the Bramble-Hilbert lemma

$$|\mathscr{F}(f)| \leqq C\|\hat{v} - \hat{v}(\gamma)\|_{0,\tau}\,|\hat{f}|_{1,\tau}$$

follows.

On the other hand, we have

$$|\hat{v}(\hat{x}) - \hat{v}(\gamma)| \leqq |\nabla v\,.\,(\hat{x} - \gamma)| \leqq \|\nabla\hat{v}\|\,, \quad \hat{x} \in \tau\,.$$

Consequently, we get

$$\|\hat{v} - \hat{v}(\gamma)\|_{0,\tau} \leqq |\hat{v}|_{1,\tau}\,,$$

$$|\mathscr{F}(\hat{f})| \leqq C|\hat{v}|_{1,\tau}\,|\hat{f}|_{1,\tau} \leqq C_1\|f\|_{1,T_k}\,\|v\|_{1,T_k}\,.$$

Substituting into (2.47), we arrive at

$$(2.48) \qquad |E_k(f(v - v(G_k)))| \leqq Ch^2\|f\|_{1,T_k}\,\|v\|_{1,T_k}\,.$$

Combining (2.48) with (2.46), we obtain

$$|(f, v)_{0,\Omega} - (f, v)^*_{0,\Omega_h}| \leqq C\sum_{k=1}^{K} h^2\|f\|_{2,T_k}\,\|v\|_{1,T_k} \leqq Ch^2\|f\|_{2,\Omega}\,\|v\|_{1\Omega}. \qquad \blacksquare$$

To introduce an approximation of the problem (2.1), let us assume that $\bar{\boldsymbol{u}} \in \boldsymbol{W} \cap \cap (C(\bar{\Omega}))^M$. Then the discrete problem can be defined as follows:

Find $\boldsymbol{u}_h \in \boldsymbol{P}\bar{\boldsymbol{u}} + \boldsymbol{V}_h$ such that

$$(2.49) \qquad a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h)^*_{0,\Omega_h} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,$$

where the right-hand side is defined as a sum over $m = 1, \ldots, M$, of the approximations defined in Lemma 2.2.

Theorem 2.1. *Let Ω belong to the class $\mathscr{C}^3(d)$. Let $\mathbf{u} \in (H^3(\Omega))^M$ and $\mathbf{u}_h$ be the solution of* (2.1) *and* (2.49), *respectively, where $\mathbf{f} \in (H^2(\Omega))^M$. Then*

$$\|\mathbf{u}_h - \mathbf{Pu}\|_{1,\Omega} \leqq Ch^{3/2}(\|\mathbf{u}\|_{3,\Omega} + \|\mathbf{f}\|_{2,\Omega})$$

holds for sufficiently small h.

Proof. As $\mathbf{u} \in \bar{\mathbf{u}} + \mathbf{V}$, $\mathbf{u}_h \in \mathbf{P}\bar{\mathbf{u}} + \mathbf{V}_h$, and the mapping $\mathbf{P}$ is additive, it is readily seen that

$$\mathbf{v}_h \equiv \mathbf{Pu} - \mathbf{u}_h \in \mathbf{V}_h\,.$$

Thus employing the inequality of Korn's type (2.4) and the definitions (2.1), (2.49), we come to

$$(2.50) \qquad \|\mathbf{Pu} - \mathbf{u}_h\|_{1,\Omega}^2 \leqq Ca(\mathbf{Pu} - \mathbf{u}_h, \mathbf{v}_h) \leqq C(|a(\mathbf{Pu} - \mathbf{u}, \mathbf{v}_h)| + $$
$$+ |a(\mathbf{u}, \mathbf{v}_h) - a(\mathbf{u}_h, \mathbf{v}_h)|) = C(|a(\mathbf{Pu} - \mathbf{u}, \mathbf{v}_h)| + |(\mathbf{f}, \mathbf{v}_h)_{0,\Omega} - (\mathbf{f}, \mathbf{v}_h)_{0,\Omega_h}^*|)\,.$$

Estimating the terms in the right-hand side with the help of Lemmas 2.1 and 2.2, we obtain

$$\|\mathbf{Pu} - \mathbf{u}_h\|_{1,\Omega} \leqq Ch^{3/2}(\|\mathbf{u}\|_{3,\Omega} + \|\mathbf{f}\|_{2,\Omega})\,. \qquad \blacksquare$$

Remark 2.3. Let the domain Ω have a polygonal boundary which consists of line segments parallel with one of three different directions and the ratio of the lengths of any two parallel sides is rational. Assume that the solution $\mathbf{u}$ of (2.1) belongs to $(H^3(\Omega))^M$ and the right-hand side $\mathbf{f} \in (H^2(\Omega))^M$.

Then we put $\mathscr{T}_h = \mathscr{T}_h^*$, $\Omega = \Omega_h^*$, and

$$(2.51) \qquad \|\mathbf{u}_h - \mathbf{Pu}\|_{1,\Omega} \leqq Ch^2(\|\mathbf{u}\|_{3,\Omega} + \|\mathbf{f}\|_{2,\Omega})$$

holds for sufficiently small h.

This result follows from Remark 2.2, Lemma 2.2 and (2.50).

3. AVERAGED GRADIENT AND ITS APPROXIMATION PROPERTIES

The results of this section will be used later to obtain better approximation of the first derivatives of the solution $\mathbf{u}$ than the derivatives of $\mathbf{u}_h$ (cf. (2.49)). For the time being, however, let $\mathbf{v}_h = (v_{1h}, \ldots, v_{Mh}) \in (W_h)^M$ be arbitrary. The gradient of its m-th component v_{mh} (the subscript m will be dropped in the whole section) is a piecewise constant vector function. This enables us to define an averaged gradient $G_h(v_h)$ on the domain Ω_h^* as the linear interpolation of $\mathscr{T}_h^*$ of the nodal values

$$(3.1) \qquad (G_h(v_h))(\mathbf{x}) = \sum_{j=1}^{m_n} w_j^x \operatorname{grad} v_h|_{T_j^x}\,.$$

Here $n = n(\mathbf{x}) \in \{1, \ldots, 6\}$ is the number of triangles of $\mathscr{T}_h^*$ which contain the node

$\boldsymbol{x}$, m_n is the number of the triangles T_j^x sketched in Fig. 3.n, the weights w_j^x, $j \in \{1, \dots, m_n\}$, are real numbers the values of which are marked also in Fig. 3.n, in the corresponding triangles T_j^x.

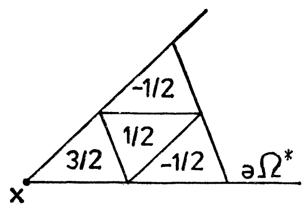

Fig. 3.1.

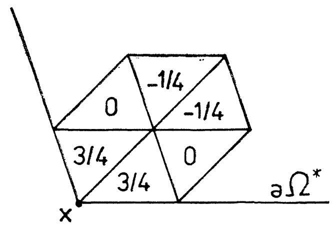

Fig. 3.2.

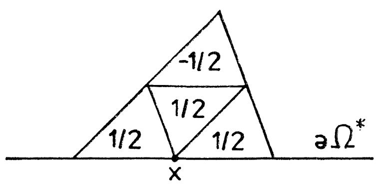

Fig. 3.3.

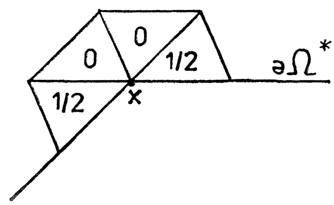

Fig. 3.4.

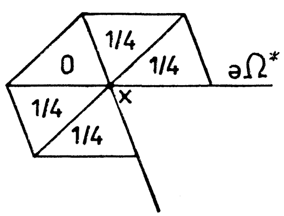

Fig. 3.5.

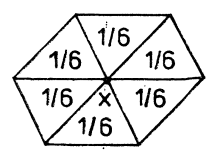

Fig. 3.6.

A crucial point of our analysis will be the proof of the estimate

$$\|\text{grad } v - G_h(Pv)\|_{0,\Omega_h^*} = O(h^2)$$

for any sufficiently smooth scalar function v (note that we have only $\|\text{grad } v - \text{grad } Pv\|_{0,\Omega_h^*} = O(h)$).

At first let us consider the difference grad $v - G_h(Pv)$ on a single triangle $T \in \mathcal{T}_h^*$. From (3.1) we see that the values of $G_h(v_h)$ on T depend only upon the values of grad v_h on $D(T)$, where

$$D(T) = \bigcup_{\substack{T' \cap T \neq \emptyset \\ T' \in \mathcal{T}_h^*}} T' . \tag{3.2}$$

Furthermore, we set $\Delta = D(\tau)$ for the reference triangle τ with the vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ – see Fig. 3.7 below. It is clear that for any $T \in \mathscr{T}_h^*$ there exists a regular 2×2 matrix $\mathbf{B}_T$ and a vector $b_T \in \mathbb{R}^2$ such that

$$F_T(\tau) = T,$$

where

$$F_T(\hat{\mathbf{x}}) = \mathbf{B}_T\hat{\mathbf{x}} + \mathbf{b}_T . \tag{3.3}$$

As the system $\mathfrak{M} = \{\mathscr{T}_h\}$ is regular (see [6] for Zlámal's condition), we get

$$\|\mathbf{B}_T\| \leqq Ch . \tag{3.4}$$

Moreover, the strong regularity of $\mathfrak{M}$ yields

$$\|\mathbf{B}_T^{-1}\| \leqq Ch^{-1} . \tag{3.5}$$

For $D(T)$ given by (3.2), we put

$$\delta_T = F_T^{-1}(D(T)) .$$

Obviously $\delta_T \subseteq \Delta$, and there is only a finite number of (reference) domains δ_T, when T passes through all triangulations $\mathscr{T}_h^*$.

Let $\hat{v}$ be a continuous and piecewise linear function on δ_T. Let the averaged gradient $\hat{G}_T(\hat{v})$ on the reference triangle τ be a linear function which is defined at the vertices $\hat{\mathbf{x}}$ of τ likewise (3.1), i.e.

$$(\hat{G}_T(\hat{v}))(\hat{\mathbf{x}}) = \sum_{j=1}^{m_n} w_j^x \operatorname{grad} \hat{v}|_{\hat{T}_j^x} , \tag{3.6}$$

where $\hat{T}_j^x = F_T^{-1}(T_j^x)$ and $\mathbf{x} = F_T(\hat{\mathbf{x}})$. There is again only a finite number of different formulae for $\hat{G}_T(\hat{v})$. For simplicity we drop the subscript $_T$ for the time being.

Lemma 3.1. *There exists a constant $\hat{C} > 0$ such that*

$$\|\operatorname{grad} \hat{v} - \hat{G}(\hat{P}\hat{v})\|_{0,\tau} \leqq \hat{C}|\hat{v}|_{3,\delta} \quad \forall \hat{v} \in H^3(\delta) , \tag{3.7}$$

where $\hat{P}\hat{v}$ is the linear interpolation of $\hat{v}$ on δ.

Proof. Let $\hat{v} \in H^3(\delta)$ be arbitrary and let $\mathbf{y}$ be that vertex of τ, where the convex function $\hat{\mathbf{x}} \to \|(\hat{G}(\hat{P}\hat{v}))(\hat{\mathbf{x}})\|^2$ attains its maximum over τ. Then for a suitable $k \in \in \{1, \ldots, 6\}$ we may write

$$\|\hat{G}(\hat{P}\hat{v})\|_{0,\tau} \leqq (\operatorname{mes} \tau)^{1/2} \|(\hat{G}(\hat{P}\hat{v}))(\mathbf{y})\| \leqq \tag{3.8}$$

$$\leqq (\operatorname{mes} \tau)^{1/2} \sum_{j=1}^{m_n} |w_j^y| \, \|\operatorname{grad} \hat{P}\hat{v}|_{\hat{T}_j^y}\| \leqq 3(\operatorname{mes} \tau)^{1/2} \|\operatorname{grad} \hat{P}\hat{v}|_{\hat{T}_k^y}\| \leqq$$

$$\leqq 3\|\operatorname{grad} \hat{P}\hat{v}\|_{0,\delta} ,$$

because $\sum_{j=1}^{m_n} |w_j^y| \leqq 3$ and $\hat{T}_k^y \subset \delta$.

For a fixed $\hat{g} \in L^2(\tau)$ we define a linear functional Φ by

$$\Phi(\hat{v}) = (\operatorname{grad} \hat{v} - \hat{G}(\hat{P}\hat{v}), \hat{g})_{0,\tau}, \quad \hat{v} \in H^3(\delta).$$

We apply (3.8) to see that Φ is bounded on $H^3(\delta)$, i.e.

$$\begin{aligned} |\Phi(\hat{v})| &\leqq \|\operatorname{grad} \hat{v} - \hat{G}(\hat{P}\hat{v})\|_{0,\tau} \|\hat{g}\|_{0,\tau} \leqq \\ &\leqq (\|\hat{v}\|_{1,\tau} + 3\|\operatorname{grad} \hat{P}\hat{v}\|_{0,\delta}) \|\hat{g}\|_{0,\tau} \leqq C\|\hat{v}\|_{3,\delta} \|\hat{g}\|_{0,\tau}. \end{aligned} \tag{3.9}$$

Next, we show that

$$\hat{G}(\hat{P}\hat{v}) = \operatorname{grad} \hat{v} \quad \text{on} \quad \tau \tag{3.10}$$

for all quadratics $\hat{v} \in P_2(\delta)$. Since both functions in the left- and right-hand side of (3.10) are then linear, it suffices to prove (3.10) for all vertices of τ. Let us confine ourselves to the vertex (0, 0) and to the first components of the gradients (the proof for the second components and other vertices is essentially the same). Let us write

$$\hat{v}(\hat{x}_1, \hat{x}_2) = c_0 + c_1\hat{x}_1 + c_2\hat{x}_2 + c_3\hat{x}_1^2 + c_4\hat{x}_1\hat{x}_2 + c_5\hat{x}_2^2,$$

and let us number the triangles from the union $\Delta = D(\tau)$ as marked in Fig. 3.7.

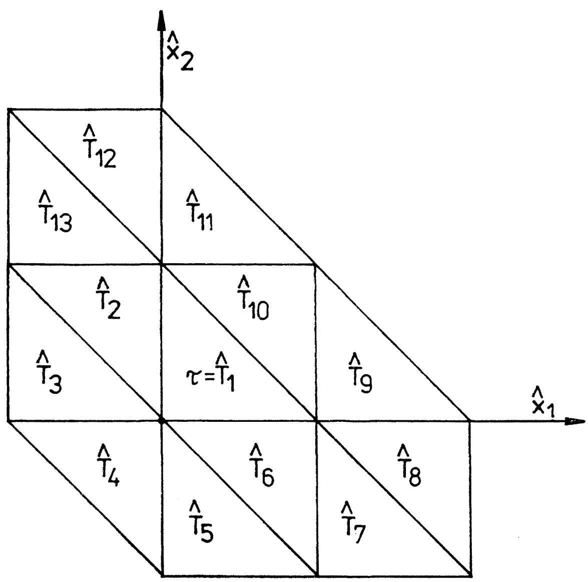

Fig. 3.7.

Then the averaging, which corresponds to the situation of Fig. 3.1, yields

$$\begin{aligned} ((\hat{G}(\hat{P}\hat{v}))(0,0))_1 &= \tfrac{3}{2}\partial_1\hat{P}\hat{v}|_{\hat{T}_1} + \tfrac{1}{2}(\partial_1\hat{P}\hat{v}|_{\hat{T}_{10}} - \partial_1\hat{P}\hat{v}|_{\hat{T}_9} - \partial_1\hat{P}\hat{v}|_{\hat{T}_{11}}) = \\ &= \tfrac{3}{2}(c_1 + c_3) + \tfrac{1}{2}(c_1 + c_3 + c_4 - (c_1 + 3c_3) - (c_1 + c_3 + c_4)) = \\ &= c_1 = (\partial_1\hat{v})(0,0). \end{aligned}$$

The point $\boldsymbol{x}$ at the Fig. 3.2 is common to two triangles. Mapping e.g. the left one to τ, we obtain

$$((\hat{G}(\hat{P}\hat{v}))(0,0))_1 = \tfrac{3}{4}(\partial_1 \hat{P}\hat{v}|_{T_1} + \partial_1 \hat{P}\hat{v}|_{T_6}) - \tfrac{1}{4}(\partial_1 \hat{P}\hat{v}|_{T_8} + \partial_1 \hat{P}\hat{v}|_{T_9}) =$$
$$= \tfrac{3}{4}(c_1 + c_3 + c_1 + c_3) - \tfrac{1}{4}(c_1 + 3c_3 + c_1 + 3c_3) = c_1 .$$

The other cases can be proved by an analogous way (see also [9], p. 108, for the situation of Fig. 3.6).

Thus (3.10) implies that the functional Φ vanishes for all $\hat{v}$ quadratic. From the Bramble-Hilbert lemma and (3.9), we come to

$$|\Phi(\hat{v})| \leqq \hat{C}|\hat{v}|_{3,\delta}\, \|\hat{g}\|_{0,\tau} ,$$

which proves the lemma. ■

Lemma 3.2. *There exists a constant $C > 0$ such that for any $T \in \mathscr{T}_h^*$ $(\mathscr{T}_h^* \subset \mathscr{T}_h \in \in \mathfrak{M})$ it holds that*

$$\|\operatorname{grad} v - G_h(Pv)\|_{0,T} \leqq Ch^2|v|_{3,D} \quad \forall v \in H^3(D) ,$$

where $D \equiv D(T)$ is defined by (3.2).

Proof. Let $T \in \mathscr{T}_h^*$ be given. In accordance with (3.3) we get

$$\text{(3.11)} \qquad \operatorname{grad} v(\boldsymbol{x}) = (\boldsymbol{B}_T^{-1})^{\mathrm{T}} \operatorname{grad} \hat{v}(\hat{\boldsymbol{x}}) ,$$

provided $v \in H^3(D)$ and $\hat{v} \in H^3(\delta)$ are coupled by the relation

$$\hat{v}(\hat{\boldsymbol{x}}) = v(\boldsymbol{x}) \quad (\boldsymbol{x} = F_T(\hat{\boldsymbol{x}}) \in D) .$$

Obviously, the formula (3.11) is valid for continuous and piecewise linear functions v and $\hat{v}$, too. Therefore, using (3.1), (3.6) and the fact that $F_T(\delta) = D$, we easily find that a formula similar to (3.11) holds also for the averaged gradients, i.e.

$$(G_h(v))(\boldsymbol{x}) = (B_T^{-1})^{\mathrm{T}} (\hat{G}(\hat{v}))(\hat{\boldsymbol{x}}) \quad (\boldsymbol{x} = F_T(\hat{\boldsymbol{x}}) \in T),$$

where v and $\hat{v}$ are continuous and piecewise linear on D and δ, respectively. Hence,

$$(G_h(Pv))(\boldsymbol{x}) = (\boldsymbol{B}_T^{-1})^{\mathrm{T}} (\hat{G}(\hat{P}v))(\hat{\boldsymbol{x}})$$

for all $\boldsymbol{x} \in T$, $v \in H^3(D)$ and the corresponding $\hat{\boldsymbol{x}} \in \tau$, $\hat{v} \in H^3(\delta)$. Combining this result with (3.11) and (3.7), and using the well-known relation $(Pv)^\wedge = \hat{P}\hat{v}$, we arrive at

$$\text{(3.12)} \qquad \|\operatorname{grad} v - G_h(Pv)\|_{0,T}^2 \leqq$$
$$\leqq \|\boldsymbol{B}_T^{-1}\|^2 \, \|\operatorname{grad} \hat{v} - \hat{G}(\hat{P}\hat{v})\|_{0,\tau}^2 \, |\det \boldsymbol{B}_T| \leqq \hat{C}\|\boldsymbol{B}_T^{-1}\|^2 \, |\det \boldsymbol{B}_T| \, |\hat{v}|_{3,\delta}^2 .$$

Since (see [6], p. 118)

$$|\hat{v}|_{3,\delta} \leqq C\|\boldsymbol{B}_T\|^3 \, |\det \boldsymbol{B}_T|^{-1/2} \, |v|_{3,D} ,$$

from the estimates (3.12), (3.4) and (3.5) the assertion of the lemma follows. ■■

The following estimate will be applied in the next section to each component of the solution $\boldsymbol{u} = (u_1, \ldots, u_M)$.

Theorem 3.1. *There exists a constant $C > 0$ such that*

$$\|\operatorname{grad} v - G_h(Pv)\|_{0,\Omega_h^*} \leqq Ch^2|v|_{3,\Omega_h^*} \quad \forall v \in H^3(\Omega)\,. \tag{3.13}$$

Proof. From Lemma 3.2 we have

$$\|\operatorname{grad} v - G_h(Pv)\|_{0,\Omega_h^*}^2 = \sum_{T\in\mathcal{T}_h^*} \|\operatorname{grad} v - G_h(Pv)\|_{0,T}^2 \leqq \tag{3.14}$$

$$\leqq C^2h^4 \sum_{T\in\mathcal{T}_h^*} |v|_{3,D}^2 \leqq 13C^2h^4 \sum_{T\in\mathcal{T}_h^*} |v|_{3,T}^2 = 13C^2h^4|v|_{3,\Omega_h^*}^2 \quad \forall v \in H^3(\Omega)\,,$$

since any $T \in \mathcal{T}_h^*$ is contained in at most 13 sets $D(T_i)$, $i = 1, \ldots, k$ $(k \leqq 13)$, where $T_2, T_3, \ldots, T_k \in \mathcal{T}_h^*$ are the "neighbouring" triangles to $T = T_1$ (cf. Fig. 3.7). ∎

4. THE MAIN RESULT ON THE SUPERCONVERGENCE

Now we are able to prove the main theorem of the paper. We collect the above results to obtain a superconvergence of the derivatives for our elliptic problem (2.1) with the solution $\boldsymbol{u} = (u_1, \ldots, u_M)$. Let us recall that $\boldsymbol{u}_h = (u_{1h}, \ldots, u_{Mh})$ denotes the solution of the discrete problem (2.49). We adopt the following notations. By $\mathcal{G}_h(\boldsymbol{u}_h)$ we denote $2 \times M$ matrix, the j-th column of which equals to $G_h(u_{jh})$; and let $\partial\boldsymbol{u}/\partial\boldsymbol{x}$ be $2 \times M$ matrix of the first partial derivarives of $\boldsymbol{u}$.

Theorem 4.1. *Let the assumptions of Theorem 2.1 be satisfied. Then*

$$\left\|\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}} - \mathcal{G}_h(\boldsymbol{u}_h)\right\|_{0,\Omega_h^*} \leqq Ch^{3/2}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{f}\|_{2,\Omega})$$

holds for sufficiently small h.

Proof. Obviously,

$$\left\|\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}} - \mathcal{G}_h(\boldsymbol{u}_h)\right\|_{0,\Omega_h^*} \leqq \sum_{j=1}^{M} \|\operatorname{grad} u_j - G_h(u_{jh})\|_{0,\Omega_h^*} \leqq \tag{4.1}$$

$$\leqq \sum_{j=1}^{M} \left(\|\operatorname{grad} u_j - G_h(Pu_j)\|_{0,\Omega_h^*} + \|G_h(Pu_j - u_{jh})\|_{0,\Omega_h^*}\right),$$

and from Theorem 3.1 we may easily bound the first terms

$$\|\operatorname{grad} u_j - G_h(Pu_j)\|_{0,\Omega_h^*} \leqq Ch^2|u_j|_{3,\Omega_h^*}\,, \quad j = 1, \ldots, M\,. \tag{4.2}$$

To bound the second terms we use some ideas of Sections 2 and 3. As $Pu_j - u_{jh}$ is a continuous piecewise linear function, we find likewise (3.8) that

$$\|G_h(Pu_j - u_{jh})\|_{0,T} \leqq 3\|\operatorname{grad}(Pu_j - u_{jh})\|_{0,D(T)}\,,$$

whenever $T \in \mathcal{T}_h^*$ and $j \in \{1, \ldots, M\}$. Any $T \in \mathcal{T}_h^*$ is contained in at most 13 sets

$D(T_i)$, $i = 1, \ldots, k \leqq 13$ (cf. (3.14)), and therefore

$$(4.3) \qquad \|G_h(Pu_j - u_{jh})\|_{0,\Omega_h^*} = 3\sqrt{(13)}\, \|\text{grad}\,(Pu_j - u_{jh})\|_{0,\Omega_h^*} \leqq \\ \leqq 3\sqrt{(13)}\, \|\boldsymbol{Pu} - \boldsymbol{u}_h\|_{1,\Omega},$$

which can be further estimated by Theorem 2.1. The theorem now follows from (4.1), (4.2), and (4.3). ■

Remark 4.1. In case of a polygonal domain discussed in Remark 2.3, the Theorem 4.1 holds even with the power h^2 instead of $h^{3/2}$ and we obtain the global superconvergence in the $\|\cdot\|_{0,\Omega}$-norm. The regularity assumption $\boldsymbol{u} \in (H^3(\Omega))^M$, however is unreasonable for general non-smooth boundaries and $\boldsymbol{f} \in (H^2(\Omega))^M$. For boundaries of the class $\mathscr{C}^{(\infty)}$, we have $\boldsymbol{u} \in (H^3(\Omega))^M$ if $\bar{\boldsymbol{u}} \in (H^3(\Omega))^M$ and $\boldsymbol{f} \in (H^1(\Omega))^M$, as follows from [1] and [17], Lemma 3.2, Chapt. 5.

Corollary 4.1. *Extending the definition of $\mathscr{G}_h(u_h)$ in the simple manner*

$$\mathscr{G}_h(\boldsymbol{u}_h) = \frac{\partial \boldsymbol{u}_n}{\partial \boldsymbol{x}} \quad in \quad \Omega_h^i = \Omega_h - \Omega_h^*,$$

one can deduce even the global superconvergence estimate

$$\left\|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}_h)\right\|_{0,\Omega_h} \leqq Ch^{3/2}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{f}\|_{2,\Omega}),$$

under the same assumptions as in Theorem 4.1.

Proof. By the above Theorem 4.1, it suffices to estimate the term

$$\left\|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \mathscr{G}_h(\boldsymbol{u}_h)\right\|_{0,\Omega_h^i} \leqq \sum_{j=1}^{M} \|\text{grad}\, u_j - \text{grad}\, u_{jh}\|_{0,\Omega_h^i} \leqq \\ \leqq \sum_{j=1}^{M} (\|u_j - Pu_j\|_{1,\Omega_h^i} + \|u_{jh} - Pu_j\|_{1,\Omega}).$$

The bound for the last term follows immediately from Theorem 2.1. Thus it remains to deal with the term $\|u_j - Pu_j\|_{1,\Omega_h^i}$. Denoting by π the standard interpolation operator, we easily find like in the proof of Lemma 2.1 (cf. the derivation of (2.28)) that

$$(4.4) \qquad \|u_j - Pu_j\|_{1,\Omega_h^i} \leqq \|u_j - \pi u_j\|_{1,\Omega_h^i} + \|\pi u_j - Pu_j\|_{1,\Omega_h^i} \leqq \\ \leqq C(h|u_j|_{2,\Omega_h^i} + h^{3/2}\|u_j\|_{3,\Omega}).$$

On the basis of the hypotheses (H 1) and (H 2) we conclude that

$$\max_{x \in \partial\Omega} \text{dist}\,(\boldsymbol{x}, \partial\Omega_h^*) \leqq Ch.$$

Consequently the use of the Iljin inequality (2.31) and (4.4) yields

$$\|u_j - Pu_j\|_{1,\Omega_h^i} \leqq Ch^{3/2}\|u_j\|_{3,\Omega}.$$ ■

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Souhrn

O JEDNOM SUPERKONVERGENTNÍM SCHÉMATU V METODĚ KONEČNÝCH PRVKŮ PRO ELIPTICKÉ SYSTÉMY

I. DIRICHLETOVY OKRAJOVÉ PODMÍNKY

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

V článku se uvažují systémy eliptických rovnic druhého řádu (zahrnující Lamého rovnice pružnosti) s nehomogenními Dirichletovými okrajovými podmínkami na omezené rovinné oblasti. Předkládá se jednoduché průměrující schéma, které zaručuje superkonvergenci derivací řešení při použití standardních lineárních trojúhelníkových prvků. Je dokázán globální odhad chyby v L^2-normě řádu $O(h^{3/2})$ pro oblasti s hladkou hranicí. Pro jistou třídu polygonálních oblastí se odvozuje globální odhad chyby řádu $O(h^2)$.

Резюме

ОБ ОДНОЙ СУПЕРСХОДЯЩЕЙСЯ СХЕМЕ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ

I. ГРАНИЧНЫЕ УСЛОВИЯ ДИРИХЛЕ

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

В работе рассматриваются системы эллиптических уравнений второго порядка (включающие уравнения упругости Ламе) с неоднородными граничными условиями Дирихле на ограниченной двумерной области. Предлагается простая осредняющая схема, которая гарантирует суперсходимость производных решения при использовании стандартных линейных треугольных элементов. Доказана глобальная оценка в L^2-норме порядка $O(h^{3/2})$ для областей с гладкой границей. Для специального класса многоугольных областей получена глобальная оценка ошибки порядка $O(h^2)$.

Authors' address: Ing. *Ivan Hlaváček*, DrSc., Dr. *Michal Křížek*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.