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LOCATION OF POLYGON VERTICES ON CIRCLES AND ITS APPLICATION IN TRANSPORT STUDIES

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Summary. The paper deals with the problem how to locate a set of polygon vertices on given circles fulfilling some oriteria of "regularity" of individual and composed polygons. Specifying the conditions we can obtain a lot of particular versions of this general problem. Some of them are already solved, the others are not.

Applications of this theory can be found in scheduling of periodically repeating processes, e.g. in coordination of several urban lines on a common leg, in optimization of the rhythm of a marshalling yard etc.

Keywords: Polygons on circles, regularity measures, optimal location, coordination, transport, common leg, marshalling yard.

Class. AMS: 90 B 35, 05 B 99

Let us begin with two practical examples:

First, let us suppose that in a town there are, n urban transport lines with a common leg (e.g. in the main street of the town). Moreover, let us suppose that there are many passengers which are interested to take a vehicle of only one "proper" line, but there are also many others, which can use a vehicle of any line (traveling in the main street only).

Every passenger would prefer to have a regular flow of "his" vehicles, i.e. to have equal intervals between them. Unfortunately, it is usually difficult to satisfy these requirements in practice: if the flow of the vehicles of every line is regular, the flow in the common leg is generally not regular.

If e.g. we have three lines:

Line 1 with 6-minute intervals between vehicles,
line 2 with 10-minute intervals between vehicles,
line 3 with 15-minute intervals between vehicles,
then the departures of the vehicles from the first station of the common segment can be:

Line 1: 6;00, 6;06, 6;12, 6;18, 6;24, 6;30, 6;36, 6;42, ...
Line 2: 6;00, 6;10, 6;20, 6;30, 6;40, ...

Line 3: 6;00, 6;15, 6;30, 6;45, ...

The intervals between the vehicles on the common leg would be

$$0, 0, 6, 4, 2, 3, 3, 2, 4, 6/0, 0, 6, 4, 2, \ldots$$

We can see that after the period of 30 minutes the sequence of intervals repeats. Further, it is evident that the regularity on the common segment is bad – the intervals change from 0 to 6 minutes. One can obviously ask whether there exists a better solution. Let us try to find another one:

Line 1: 6;00, 6;06, 6;12, 6;18, 6;24, 6;30, ...

Line 2: 6;05, 6;15, 6;25, ...

Line 3: 6;04, 6;19, ...

The periodically repeated vector of ten intervals will now be

$$4, 1, 1, 6, 3, 3, 1, 5, 1, 5, \ldots$$

Then the question to be answered is whether this solution is better than the previous one? Probably it would depend on the criterion we have chosen. One of the possible criteria is the following: "The smaller the minimal interval the worse the solution". In this case the second solution is better than the first (1 against 0). Another criterion says that "The smaller the maximal interval – the better the solution". In this case both solutions are equivalent. A very reasonable criterion is the following "The smaller the sum of the interval length squares, the better the solution". It is based on the fact, explained in detail in [1], that the average total waiting time of passengers is proportional to the sum of the interval length squares. Then the results are the following:

$$0 + 0 + 36 + \quad 16 + 4 + 9 + 9 + \quad 4 + 16 + 36 = 130\,,$$
$$16 + 1 + \quad 1 + c\,36 + 9 + 9 + 1 + 25 + \quad 1 + 25 = 124\,,$$

and the second solution is again better than the first. However, it is is not the best one: e.g. choosing the first departures of line 1 at 6 : 00, of line 2 at 6 : 05 and of line 3 at 6 : 07 gives the sum of squares 116.

The second example concerns the rhythm of a marshalling yard.

Let us suppose that in a marshalling yard there are n tracks to which the wagons come from the hump. In many marshalling yards these tracks are used not only to carry out the humping process, but also to arrange wagons into trains which then depart out of the station. The train-forming process has approximately the same duration on every track and thus the departure times in time-tables have a big influence on the rhythm of the work and also a little on the humping process.

Let us suppose that from the i-th track we have m_i departures of trains daily and let

$$m = m_1 + \ldots + m_n\,.$$

Then m values of $t_{i,j}$ are to be determined ($t_{i,j}$ stands for the time of departure of the j-th train from the i-th track). These values should satisfy two rather antagonistic conditions:

1. The flow of trains from the i-th line should be regular, i.e.

$$t_{i,j+1} - t_{i,j} = 1440/m_i \quad \text{(minutes)}$$

This condition is desirable from the point of view of the wagon-gathering process on the i-th track

2. The two nearest departures from the different tracks should be as distant as possible, i.e. if $t_1, \ldots, t_m$ are the times t_{ij} written in their chronological order then

$$\min_{i=1,\ldots,n} (t_{i+1} - t_i)$$

must be as large as possible (where $t_{n+1} = t_1 + 1440$).

This condition follows from the fact that if some of the differences $t_{i+1} - t_i$ is too small, then either the station needs more employees for the marshalling of trains, or the i-th prepared train waits for its departure uselessly, or the $(i + 1)$-st train is overdue.

For example, let us have a station with 3 tracks for humping, with $m_1 = 5, m_2 = 3$ and $m_3 = 2$. Of course this example is a mere illustration, such a small station does not exist.

The departures from the three tracks can be chosen as follows:

1st track: 0(0 : 00), 288(4 : 48), 576(9 : 36), 864(14 : 24), 1152(19 : 12),

2nd track: 240(4 : 00), 720(12 : 00), 1200(20 : 00),

3rd track: 192(3 : 12), 912(15 : 12).

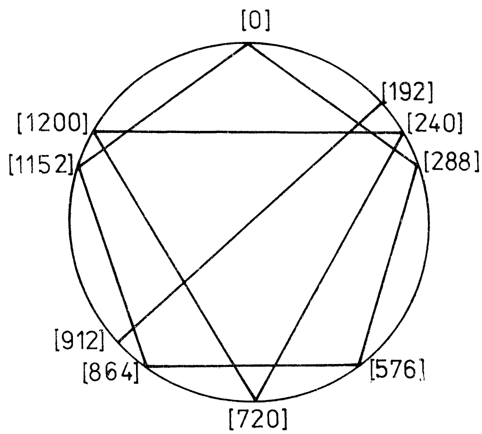

Fig. 1.

This solution is sketched in Fig. 1.

One can immediately observe that these numbers can be obtained from the second solution of the preceding example by multiplying the number of minutes after 6 : 00 by 48. Thus it is natural to look a common mathematical model for the both practical examples.

In both cases the rhythm was periodic with a certain period of T minutes ($T = 30$ or $T = 1440$, respectively). Hence it is quite natural to represent the departures t_{ij} (the j-th vehicle on the i-th line or the j-th train from the i-th track) on a circle c of the length T. The vector $(t_{i1}, \ldots, t_{im_i})$ is represented by an m_i-gon which is regular if

$$t_{i,j+1} - t_{i,j} = T/m_i \quad \text{for} \quad i = 1, \ldots, n - 1 .$$

The whole set

$$\{t_{ij} \mid j = 1, \ldots, n;\ j = 1, \ldots, m_i\}$$

is also represented by a total m-gon, $m = m_1 + \ldots + m_n$. The representation of a number t by a point $[t]$ on a circle is found as follows: an origin $[0]$ is chosen on c and t is represented by a point $[t]$ with the property that $|[0]\,[t]| = t$ (where $|[0]\,[t]|$ is the length of the arc $[0]\,[t]$ in the positive sense).

Thus the problem can be formulated as follows:

On the circle with the length T with a natural coordination, for every $i = 1, \ldots, n$ find an m_i-gon $A_i = \{[t_{i1}], \ldots, [t_{im_i}]\}$ such that the individual polygons $A_1, \ldots, A_n$ and the total m-gon $A = \{[t_{ij}]\ i = 1, \ldots, n; j = 1, \ldots, m_i\}$ fulfil some "regularity" conditions.

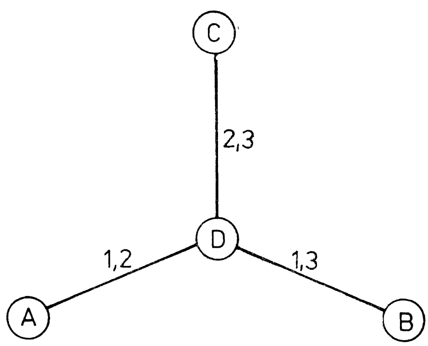

Fig. 2.

There are some practical situations, however, which need a more general model. Let us suppose that we have a net of three urban lines from Fig. 2.

Line 1 goes from A through D to B and back,
line 2 goes from A through D to C and back,
line 3 goes from B through D to C and back.

Let us also suppose that the intervals between the vehicles on these lines are equal to those from the first example. Then we again have the common period $T = 30$ minutes. The essential difference is in the fact that we have three different segments which are common for different pairs of lines. Thus we have to consider three circles

c_1, c_2, c_3 with the length T. On c_1 (for AD) we locate the vertices of a pentagon and a triangle,

on c_2 (for BD) we locate the vertices of a pentagon and a chord ("a 2-gon"),

on c_3 (for CD) we locate the vertices of a triangle and a chord.

Moreover, the pentagons on c_1 and c_2 must be congruent and turned by a given angle depending on the running time of line 1 vehicle from D through A to D. Analogous statements must hold for the triangle on c_1 and c_3 and the chord on c_2 and c_3.

The general formulation of the problem of polygon location on circles (a PLC-problem) can be now formulated as follows:

Let T be a positive real number and let $c_1, \ldots, c_p$ be circles of the length T. Let $n, m_1, \ldots, m_n$ be positive integers. Let $K_1, \ldots, K_n$ be subsets of $\{1, \ldots, p\}$. Let $r_i = \min K_i$, $i = 1, \ldots, n$ and let s_{ij} be a nonnegative real number for $i = 1, \ldots, n$ and $j \in K_i - \{r_i\}$. Let $s_{ij} \in \langle 0; T)$.

It is necessary to find an m_i-gon

$$A_{ij} = \{[t_{i,j,1}], \ldots, [t_{i,j,m_i}]\} \quad \text{on} \quad c_j \quad \text{for every} \quad i = 1, \ldots, n\,.$$

$j \in K_i$ such that

(1) A_{ij} are congruent for all $j \in K_i$,

(2) $t_{i,j,1} = t_{i,r_i,1} + s_{ij} \pmod{T}$,

(3) The individual polygons A_{ij} for every admissible i, j and the total polygons (for $j = 1, \ldots, p$) $A_j = \bigcup_{i;j \in K_i} A_{ij}$ fulfil some "regulatity" conditions.

We note that $j \in K_i$ means that the i-th polygon is located (also) on the j-th circle; s_{ij} means the angle (phase) shift of the i-th polygon on the j-th circle with respect to the i-th polygon on the r_i-th circle.

2. CLASSIFICATION OF PLC-PROBLEMS

It seems to be useful to classify the PLC-problems by a symbol $X|Z|\,U$ consisting of three characters:

X – a letter which characterizes the given data: the number of circles p, the number of polygons n and the sets K_i (i.e., X characterizes the type of the urban transport net if we deal with this application);

Z – a letter characterizing the type of the condition of the "regularity";

U – a letter characterizing the criterion of "regularity" used for polygons.

2.1 The letter at the first place is connected with the possible type of the transport net:

Y – expresses one circle with two polygons. It is derived from the Y-type urban net, which contains two lines with one common segment;

Ψ – expresses one circle with more than two polygons. It is derived from the Ψ-type urban net, which contains more than two lines with just one common segment;

Φ – expresses any general type of problem, not contained in the preceding ones.

2.2 The letter at the second place (the type of "regularity" condition) divides our problems into three groups:

I – in problems of this type we strictly demand every individual polygon to be regular and we optimize the "regularity"of the total polygons. It is derived from the case when every individual line has a regular flow of vehicles;

G – we demand every total polygon to be regular and we optimize the "regularity" of the individual polygons. It describes the case when (regardless of their numbers) the vehicles passing through every common segment form a regular flow;

N – no polygon needs to be obligatorily regular, we optimize "regularity" of the whole system.

2.3 The letter at the third place describes which kind of optimum we want to reach. The criterial functions give us, for every set of polygons $\{A_i = \{[t_{ij}], j = 1, \ldots, m_i\}$ $i = 1, \ldots, n\}$, the value characterizing the quality of the solution according to some criterion. We suppose that every polygon A_i may have a weight (importancy) p_i, the meaning of which we shall see later.

We consider the following criteria of optimality of a system of polygons:

Sq: $\frac{1}{2}\sum_{i=1}^{n} p_i \sum_{j=1}^{m_i} (t_{i,j+1} - t_{ij})^2$ to be minimum,

Mi: $\min_{i,j} p_i(t_{i,j+1} - t_{ij})$ to be maximum,

Mx: $\max_{i,j} p_i(t_{i,j+1} - t_{ij})$ to be minimum,

$$Sw: \sum_{i=1}^{n} p_i \sum_{j=k_i}^{k_i+m_i-1} \tfrac{1}{2}(t_{i,j+1} - t_{ij})^2 + (t_{i,j+1} - t_{ij}) \,.\, [(t_{,j} - t_{i,k_i}) - (j - k_i)(T/m_i)]\} =$$

$$= \sum_{i=1}^{n} p_i\{-T^2/2 + (T/m_i) \sum_{j=k_i+1}^{k_i+m_i} (t_{ij} - t_{ik_i})\} \quad \text{to be minimum},$$

where $[t_{i,k_i}]$ is such a vertex of A_i that for every $j = 1, \ldots, m_i$ we have $(t_{i,k_i+j} - t_{i,k_i}) \geqq j \,.\, T/m_i$. The existence of such k_i will be proved later.

Mw: $\max_{i,j} p_i([(t_{i,k_i+j} - t_{i,k_i}) - jT/m_i + T/m_i)$ to be minimum,

where k_i is defined as above.

In all these cases we consider $t_{i,m_i+j} = t_{i,j} + T$.

These criteria need some comments:

1. The polygons A_i need bot be the individual ones only. A_i can also be a total polygon on some circle. Even such a case is admissible that $A_1, \ldots, A_{n-1}$ are individual m_1-, $\ldots$, m_{n-1}-gons and A_n is the total m_n-gon where $m_n = m_1 + \ldots + m_{n-1}$.

2. In the case of urban transport lines (without oversaturation) the Sq-criterion expresses the total waiting time of passengers in the whole transport net during one period T, if p_i is the intensity of the passenger flow of the i-th line or i-th common segment; of the Mi-criterion (Mx-criterion) expresses the minimal (maximal, respectively) interval between two vehicles on the same line or segment. In the case of goods transport lines (with equally freighted vehicles and balanced capacities) the Sw-criterion expresses the total waiting time of goods in the whole transport net during one period T if p_i is the intensity of the goods flow of the i-th line or the common segment, the Mw-criterion expresses the maximal waiting time of some goods in the network.

3. The existence of k_i ensues from thr following lemma:

Lemma 1. *Let c be a circle with a length T and let $B_1, \ldots, B_m$ be different points on c such that for every $i = 1, \ldots, m - 1$ the oriented arc $B_i B_{i+1}$ does not contain any other B_j, and for every $i = 1, \ldots, m$ let us have $B_{i+hm} \equiv B_i$ for every $h \geqq 1$. Then there exist $k \in \{1, \ldots, m\}$ such that for every $i = 1, \ldots, m$ we have*

$$|B_k B_{k+i}| \geqq iT/m\,.$$

Proof. Indirectly: Let us suppose that

$$(1)\qquad \begin{cases} \text{for } \ B_1 \ \text{ there exists } j_1 \text{ such that } \ |B_1 B_{1+j}| < j_1 T/m\,, \\ \text{for } \ B_{1+j_1} \ \text{ there exists } j_2 \text{ such that } \ |B_{1+j_1} B_{1+j_1+j_2}| < j_2 T/m\,, \\ \vdots \\ \text{for } \ B_{1+j_1+\ldots+j} \ \text{ there exists } j_{n+1} \text{ such that} \\ \qquad |B_{1+j_1+\ldots+j_n} B_{1+\ldots+jn+1}| < j_{n+1} T/m\,. \end{cases}$$

Because of the finiteness of the set $\{B_1, \ldots, B_m\}$ there exist such r, s that

$$B_{1+\ldots+j_r} \equiv B_{1+\ldots+j_r+\ldots+j_s}$$

and hence

$$(2)\qquad \sum_{i=r}^{s-1} |B_{1+\ldots+j_r+\ldots+j_i} B_{1+\ldots+j_r+\ldots+j_i+j_{i+1}}| = (j_{r+1} + \ldots + j_s)\, T/m\,.$$

On the other hand, (1) implies that the sum on the left hand side (2) must be smaller than the right hand side which contradicts (2).

The lemma is proved.

3. REVIEW OF THE PROBLEMS SOLVED

3.1 Problems of the $\cdot|G|\cdot$ type

The solutions of these problems are known only for the cases of the $Y|G|\cdot$ type. It is interesting that there exists a solution which is optimal for all five criterial functions mentioned above, but the proofs of optimality differ in the individual cases. The solution is independent of the optimizing coefficients p_i; the common solution is optimal for all couples (p_1, p_2). In these case of the $Y|G|\cdot$ type the general PLC-problem can be reduced in the following way:

We have a regular $(m_1 + m_2)$-gon A on the circle c and we have to determine which vertices will belong to the m_1-gon A_1 and which to the m_2-gon A_2 (or, in other words, we have to determine the sequence consisting only of m_1 numbers one and m_2 numbers two which defines the order of vertices of the polygons A_1 and A_2 in the common $(m_1 + m_2)$-gon). By the symbols $h[a]$ and $f[a]$ we shall denote the integer functions or a real number a which satisfy the inequalities

$$f[a] \leqq a < f[a] + 1\,,$$
$$h[a] - 1 < a \leqq h[a]\,.$$

The coordinates of the total $(m_1 + m_2)$-gon $A = \{[t_i],\ i = 1, \ldots, m = m_1 + m_2\}$ on the circle c satisfy the condition $t_i = i\,.\,T/(m_1 + m_2)$ for $i = 1, \ldots, m - 1$ and $t_m = T \equiv 0$. Now we determine the vertices of the m_1-gon $A_1 = \{[t_{1i}],\ i = 1, 2, \ldots \ldots, m_1\}$ in the following way. For all $i = 1, 2, \ldots, m_1 - 1$ let

$$t_{1i} = h[i(m_1 + m_2)/m_1]\,T/(m_1 + m_2) \tag{3}$$

and let $t_{1m_1} = T \equiv 0$.

Then it is not difficult to prove that for all $j = 1, 2, \ldots, m_2$

$$t_{2j+1} = f[j(m_1 + m_2)/m_2]\,T/(m_1 + m_2) + t_{21} \tag{4}$$

must hold in A_2 (see [5]), where $t_{21} = T/(m_1 + m_2)$.

In the next part we shall prove that this system A_1, A_2, A is the optimal solution of the problem $Y|G|\cdot$ for all five criterial functions.

We shall use the following lemma.

Lemma 2. *Let A_1, A_2, A be the polygons defined above, satisfying the conditions (3) and (4).*

Then there exist such integers i_0, j_0; $1 \leqq i_0 \leqq m_1$; $1 \leqq j_0 \leqq m_2$ that $t_{1i_0} - t_{2j_0} = T/(m_1 + m_2)$ and for all $i = 1, \ldots, m_1$,

$$t_{1\,i_0+1} = t_{1i_0} + f[i(m_1 + m_2)/m_1]\,T/(m_1 + m_2)$$

holds (where $t_{1m_1+1} = t_{1i} + T$), while for all $j = 1, \ldots, m_2$,

$$t_{2j_0+j} = t_{2j_0} + h[j(m_1 + m_2)/m_2]\,.\,T/(m_1 + m_2)$$

holds (where $t_{2m_2+j} = t_{2j} + T$).

The proof of this lemma is rather lengthy and can be found in [5].

In the sequel we shall suppose that

$A'_1 = \{[t'_{11}], \ldots, [t'_{1m_1}]\}$ is an arbitrary m_1-gon on c and

$A'_2 = \{[t'_{21}], \ldots, [t'_{2m_2}]\}$ is an arbitrary m_2-gon on c such that A'_1 and A'_2 together form a regular $(m_1 + m_2)$-gon A'.

1. The proof of optimality in the problem $Y|G|\,Sq$ is based on Lemma 3 which is not difficult to prove by elementary means of mathematical analysis.

Lemma 3. *Let us have integers h, p, m $(0 < p \leqq m)$ and two integer sequences $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$ such that*

$$a_1 = a_2 \ldots = a_p = h\,; \quad a_{p+1} = a_{p+2} = \ldots = a_m = h + 1$$

and

$$\sum_{i=1}^m a_i = \sum_{i=1}^m b_i. \quad \textit{Then} \quad \sum_{i=1}^m a_i^2 \leqq \sum_{i=1}^m b_i^2\,.$$

As for all $i = 1, \ldots, m_s$ we have either

$$t_{1i+1} - t_{1i} = h[(m_1 + m_2)/m_1]\; T/(m_1 + m_2)$$

or

$$t_{1i+1} - t_{1i} = (h[(m_1 + m_2)/m_1] - 1)\; T/(m_1 + m_2)$$

(where $t_{1m_1+1} = t_{11} + T$), and for all $j = 1, \ldots, m_2$ we have either

$$t_{2j+1} - t_{2j} = f[(m_1 + m_2)/m_2]\; T/(m_1 + m_2)$$

or

$$t_{2j+1} - t_{2j} = (f[(m_1 + m_2)/m_2] + 1)\; T/(m_1 + m_2)$$

(where $t_{2m_2+1} = t_{21} + T$), we have by Lemma 3

$$\tfrac{1}{2}\sum_{i=1}^{2} p_i \sum_{j=1}^{m_i} (t_{ij+1} - t_{ij})^2 + \tfrac{1}{2}p \sum_{j=1}^{m_1+m_2} [T/(m_1 + m_2)]^2 \leqq$$

$$\tfrac{1}{2}\sum_{i=1}^{2} p_i \sum_{j=1}^{m_i} (t'_{ij+1} - t'_{ij})^2 + \tfrac{1}{2}p \sum_{j=1}^{m_1+m_2} [T/(m_1 + m_2)]^2$$

which was to be proved.

2. In order to solve the problem $Y|G|\,Mi$ we shall use an indirect proof. Let A'_1, A'_2, A' be a better solution of the problem $Y|G|\,Mi$ than A_1, A_2, A. Then either for all $i = 1, \ldots, m_1$ the inequality

$$p_1(t'_{1,i+1} - t'_{1i}) > \min_{j=1,\ldots,m_1} p_1(t_{1j+1} - t_{1j}) = p_1 f[(m_1 + m_2)/m_1]\,.\,T/(m_1 + m_2)$$

must hold or for all $i = 1, \ldots, m_2$ we have

$$p_2(t'_{2i+1} - t'_{2i}) > \min_{j=1,\ldots,m_2} p_2(t_{2j+1} - t_{2j}) = p_2 f[(m_1 + m_2)/m_2]\, T/(m_1 + m_2)\,.$$

As $(t'_{1i+1} - t'_{1i})$ and $(t'_{2i+1} - t'_{2i})$ are integer multiples of $T/(m_1 + m_2)$, we have either

$$t'_{1i+1} - t'_{1i} \geqq (f[(m_1 + m_2)/m_1] + 1)\; T/(m_1 + m_2)$$

or

$$t'_{2i+1} - t'_{2i} \geqq (f[(m_1 + m_2)/m_2] + 1)\; T/(m_1 + m_2)\,.$$

Hence either

$$T = \sum_{i=1}^{m_1} (t'_{1i+1} - t'_{1i}) \geqq m_1(f[(m_1 + m_2)/m_1] + 1)\, T/(m_1 + m_2) > T$$

or

$$T = \sum_{i=1}^{m_2} (t'_{2i+1} - t'_{2i}) \geqq m_2(f[(m_1 + m_2)/m_2] + 1)\, T/(m_1 + m_2) > T.$$

This is a contradiction which proves the assertion.

3. The proof of optimality in the case of the problem $Y|G|\, Mx$ is very similar to the previous one.

4. The optimality in the problem $Y|G|\, Sw$ and $Y|G|\, Mw$ will be proved together. Let us have A'_1, A'_2, A' as before. Then by Lemma 1 there exist t'_{1k_1} and t'_{2k_2} such that for all $i = 1, \ldots, m_1$ and for all $j = 1, \ldots, m_2$,

$$(t'_{1k_1+i} - t'_{1k_1}) \geqq i\, T/m_1 = i(m_1 + m_2)/m_1 \,.\, T/(m_1 + m_2)\,,$$

$$(t'_{2k_2+j} - t'_{2k_2}) \geqq j\, T/m_2 = j(m_1 + m_2)/m_2 \,.\, T/(m_1 + m_2)$$

holds. As we know that

$$t'_{1k_1+i} - t'_{1k_1} = k_{1i}\, T/(m_1 + m_2) \quad \text{and} \quad t'_{2k_2+j} - t'_{2k_2} = k_{2j}\, T/(m_1 + m_2)$$

(where k_{1i} and k_{2j} are suitable positive integers), the inequalities

$$k_{1i} \geqq h[i(m_1 + m_2)/m_1] \quad \text{and} \quad k_2 j \geqq h[j(m_1 + m_2)/m_2]$$

necessarily hold. In addition, we know that for all $i = 1, \ldots, m_1$,

$$t_{1m_1+i} - t_{1m_1} = t_{1i} = h[i(m_1 + m_2)/m_1] \,.\, T/(m_1 + m_2) \geqq i\, T/m_1$$

holds, and for all $j = 1, \ldots, m_2$,

$$t_{2jo+j} - t_{2jo} = h[j(m_1 + m_2)/m_2]\, T/(m_1 + m_2) \geqq j\, T/m_2$$

holds.

This implies that

$$p_1\{-T^2/2 + (T/m_1) \sum_{i=1}^{m_1} (t_{1m_1+i} - t_{1m_1})\} \leqq$$

$$\leqq p_1\{-T^2/2 + (T(m_1) \sum_{i=1}^{m_1} (t'_{1k_1+i} - t'_{1k_1})\}$$

and

$$p_2\{-T^2/2 + (T/m_2) \sum_{j=1}^{m_2} (t_{2jo+j} - t_{2jo})\} \leqq$$

$$\leqq p_2\{-T^2/2 + (T/m_2) \sum_{j=1}^{m_2} (t'_{2k_2+j} - t'_{2k_2})\}\,,$$

which proves the optimality of the system A_1, A_2, A in the problem $Y|G|\, Sw$. The above argument also yields that

$$\max_{i=1,\dots,m_1} p_1\{(t_{1m_1+i} - t_{1m_1}) - i\,T/m_1 + T/m_1\} \leqq$$

$$\leqq \max_{i=1,m_1} p_1\{(t'_{1k_1+i} - t'_{1k_1}) - i\,T/m_1 + T/m_1\}$$

and

$$\max_{j=1,\dots,m_2} p_2\{(t_{2j_0+j} - t_{2j_0}) - j\,T/m_2 + T/m_2\} \leqq$$

$$\leqq \max_{j=1,m_2} p_2\{(t'_{2k_2+j} - t'_{2k_2}) - j\,T/m_2 + T/m_2\}$$

hold, which proves the optimality of the system A_1, A_2 A in the problem $Y|G|\,Mw$.

3.2 Problems of the $\cdot|I|\cdot$ type

The solutions of these problems are known only for the cases of the $Y|I|\cdot$ and $\cdot|I|\,Mi$ types. In the case of the $Y|I|\cdot$ type the general PLC-problem can be reduced in the following way:

We have a regular m_1-gon $A_1 = \{[t_{11}], \dots, [t_{1m_1}]\}$ and a regular m_2-gon $A_2 = \{[t_{2k}], \dots, [t_{2m_2}]\}$ on the circle c, which together form the $(m_1 + m_2)$-gon $A = [t_1, \dots, [t_{m_1+m_2}]$, and we have to determine only the angle (phase) shift between A_1 and A_2. The solution is independent of the optimizing coefficients p_i, because we optimize only the "regularity" of the total $(m_1 + m_2)$-gon A. For the sake of simplicity we shall look only for such solutions which satisfy $t_{11} = 0$,

$$t_{21} = \min_{\substack{i,j \\ t_{2i} \geqq t_{1j}}} (t_{2i} - t_{1j}) = \alpha$$

which can be proved to be equivalent to $0 \leqq t_{21} < T/n[m_1, m_2]$ where $n[m_1, m_2]$ is the least common multiple of m_1 and m_2.

Then $t_{1i} = (i - 1)\,T/m_1$ for all $i = 1, \dots, m_1$, and $t_{2j} = \alpha + (j - 1)\,T/m_2$ for all $j = 1, \dots, m_2$.

Let
$$m_1 = m\,.\,m'_1\,,$$
$$m_2 = m\,.\,m'_2\,,$$

where m'_1 and m'_2 satisfy

$$n[m'_1, m'_2] = m'_1\,.\,m'_2\,.$$

Then by Euclid there exist integers b_1, b_2 such that

$$0 < b_1 < m'_1\,,$$
$$0 < b_2 < m'_2$$

and in addition,

$$1 = b_1 m'_2 - b_2 m'_1$$

holds. Hence we have

$$T/n[m_1, m_2] - \alpha = T/(m'_1 m'_2 m) - \alpha = b_1\,T/(m m'_1) -$$

$$- b_2\, T/(mm_2') - \alpha = b_1\, T/m_1 - (b_2\, T/m_2 + \alpha) = t_{1b_1+1} - t_{2b_2+1}$$

and by [5, Theorem 5] we conclude

$$t_{1b_1+1} - t_{2b_2+1} = \min_{\substack{i,j \\ t_{1i}-t_{2j}>0}} t_{1i} - t_{2j} \quad (\text{for } j = 1, \dots, m_2 \quad i = 1, \dots, m_1 + 1$$

$$\text{where } t_{m_1+1} = T)\,.$$

1. In order to solve the problem $Y|I\; Sq$ we have to find a real number α, $0 \leqq \alpha < < T/n[m_1, m_2]$ minimizing the expression

$$(p/2) \sum_{j=1}^{m_1+m_2} (t_{j+1} - t_j)^2\,.$$

As the coordinates t_i depend only on α, we have

$$(p/2) \sum_{j=1}^{m_1+m_2} (t_{j+1} - t_j)^2 = r_0\alpha^2 + r_1\alpha + r_2$$

where the coefficients r_0, r_1, r_2 can be easily found for every particular m_1 and m_2. So we obtain the solution of the problem $Y|I|\; Sq$ by finding the minimum of the real function $r_0\alpha^2 + r_1\alpha + r_2$ on the interval $\langle 0, T/n[m_1, m_2])$.

2. To solve the problem $Y|I|\; Mi$ we have to determine such $a(0 < \alpha < < T/n[m_1, m_2])$ that $p \min_{j=1,m_1+m_2} (t_{j+1} - t_j)$ is maximal. The above considerations yield

$$\min_{j=1,m_1+m_2} (t_{j+1} - t_j) = \min_{\substack{i=1,\dots,m_1+1 \\ j=1,\dots,m_2}} |t_{2j} - t_{1i}| =$$

$$= \min \{ \min_{\substack{i,j \\ t_{2j} \geqq t_{1i}}} t_{2j} - t_{1i};\; \min_{\substack{i,j \\ t_{1i} > t_{2j}}} t_{1i} - t_{2j} \} =$$

$$= \min \{\alpha;\; T/n[m_1, m_2] - \alpha\}\,.$$

This minimum will be maximal if and only if $\alpha = T/(2\,.\,n[m_1, m_2])$, which determines the optimal solution.

3. When solving the problem $Y|I|\; Mx$ we have two possibilities. If $m_1 \neq m_2$, then it is trivial that the solution is every system consisting of a regular m_1-gon A_1 and of a regular m_2-gon A_2, which together from the $(m_1 + m_2)$-gon A.

If $m_1 = m_2$ then the optimal solution is such a system of polygons A_1, A_2, A that also A is regular, i.e.

$$t_{21} - t_{11} = T/(m_1 + m_2)\,.$$

4. Now we shall study the problems $Y|I|\; Sw$ and $Y|I|\; Mw$. In this part we shall use an integer function $c(i, j)$ such that $t_{c(i,j)} = t_{i,j}$, where $t_{c(i,j)}$ is the coordinate of the $c(i, j)$-th vertex in A and $t_{i,j}$ is the coordinate of the j-th vertex in A_i. Looking for the optimal solution of these problems we shall see that if $0 < \alpha < T/[m_1'(m_1 +$

$+ m_2)]$ then following [5], for all $j = 1, \dots, m_1 + m_2$ we have $t_j - t_2 \geqq (j - 2) \cdot T/(m_1 + m_2)\,(t_2 = t_{21} = \alpha)$ and moreover, for all $i = 1, \dots, m_1$ we have $t_{c(1,i)} - t_2 \geqq (c(1, i) - 2)\, T/(m_1 + m_2) + T/[m_1'(m_1 + m_2)] - \alpha$. If $T/(m_1'(m_1 + m_2)) < \alpha < T/n[m_1, m_2]$ then $t_j - t_{c(1,b_1+1)} \geqq [j - c(1, b_1 + 1)]\, T/(m_1 + m_2)$ holds for all $j = 1, \dots, m_1 + m_2$, while

$$t_{c(2,j)} - t_{c(1,b_1+1)} \geqq [c(2, j) - c(1, b_1 + 1)]\, T/(m_1 + m_2) + \alpha - T/(m_1'(m_1 + m_2))$$

holds for all $j = 1, \dots, m_2$. It follows that the value of the criterial function Sw is

$$p\{-T^2/2 + [T/(m_1 + m_2)] \sum_{j=1}^{m_1+m_2} (t_{j+k} - t_k)\} =$$

$$= Sw_0 + p|\alpha - T/[m_1'(m_1 + m_2)]|\, d\,,$$

where d is either m_1 or m_2 and Sw_0 is the value of the criterial function with $\alpha = T/(m_1'(m_1 + m_2))$. This shows that the optimal solution of the problem $Y|I|\,Sw$ will be obtained if $t_{21} = \alpha = T/(m_1(m_1 + m_2))$ holds.

From the preceding consideration it also follows that the value of the criterial function Mw is

$$p \max [(t_{k+j} - t_k) - j\, T/(m_1 + m_2) + T/(m_1 + m_2)] =$$

$$Mw_0 + p|\alpha - T/(m_1'(m_1 + m_2))|\,,$$

and this shows, that the optimal solution of the problem $Y|I|\,Mw$ is obtained if

$$t_{21} = \alpha = T/(m_1'(m_1 + m_2))\,.$$

5. The last problem which we will mention in this paper is $\psi|I|\,Di$ and its generalization $\phi|I|\,Mi$. The whole algorithm of the solution is rather complicated, so we will show only its substantial steps. The principal idea is to divide all possible systems of polygons into groups so that one group contains all such systems that the individual (regular) polygons form the same sequence of vertices in the total polygon. Then the algorithm consists of two parts:

a) the first is to find all "groups",

b) the second is to find the optimal solution

from the systems which belong to the same "group". The best solution of the problem is then chosen from all the solutions which were the best ones from the individual "groups". This algorithm was presented and proved in [6] and [7].

In [3] and [4] we can find a "pseudooptimal" algorithm for the $\psi|I|\,Mi$-problem. This algorithm is of great practical importance but it gives the optimal solution only in one class of problems (for small values n).

Let us have to locate an m_1-gon, …, m_s-gon on the circle c. Let $n = n[m_1, \dots, m_s]$ be the least common multiple of $m_1, \dots, m_s$. The set of regular n-gons $\{B_j, j = 1, \dots, p\}$ on c is said to be covered (partially covered) by the regular m_1-gon $A_1, \dots, m_s$-gon A_s if

(1) every vertex of every A_i is a vertex of some B_j;

(2) every vertex of every B_j is a vertex of exactly (not more than) one polygon A_i.

The pseudo-optimal algorithm in [3], [4] consists in finding the minimum p for the given $m_1, \ldots, m_s$ such that the regular $n[m_1, \ldots, m_s]$-gons $B_1, \ldots, B_p$ on c are partially covered by a regular m_1-gon $A_1, \ldots$, a regular m_s-gon A_s. Then $B_1, \ldots, B_p$ form a regular $(p \,.\, n)$-gon B.

Example. Let us have a marshalling yard with $s = 9$ tracks, for which $m_1 = 6$, $m_2 = m_3 = m_4 = m_5 = 4$, $m_6 = 3$, $m_7 = m_8 = m_9 = 2$.

Obviously, $p = 3$:
The regular 12-gon B_1 is partially covered by A_1, A_6,
the regular 12-gon B_2 is partially covered by A_5, A_7, A_8, A_9,
the regular 12-gon B_3 is covered by $A_2 = A_3 = A_4$.

Then $B_1 \cup B_2 \cup B_3$ form a regular 36-gon B with the distance of vertices $1440/36 = 40$ minutes. Thus the departures are as follows:

Track No 1: 0·00, 4·00, 8·00, 12·00, 16·00, 20·00;
track No 2: 1·20, 7·20, 13·20, 19·20;
track No 3: 3·20, 9·20, 15·20, 21·20;
track No 4: 5·20, 11·20, 17·20, 23·20;
track No 5: 0·40, 6·40, 12·40, 18·40;
track No 6: 2·00, 10·00, 18·00;
track No 7: 2·40, 14·40;
track No 8: 4·40, 16·40;
track No 9: 8·40, 20·40.

The covering problem of n-gons is related to that of arithmetical sequences in number theory – let us imagine the circle c with B_1 rolling on a straight line. Moreover, these two problems are connected with cuts of trees in the graph theory and with the theory of prefix codes in information theory. What an admirable example of relations between four absolutely different branches of mathematics.

3.3 Problems of the $\cdot N \cdot$ type

As concerns this part of the general problem only one result has been proved. In [1] and [2] an algorithm was shown which yields an optimal solution of the $YN\,Sq$ problem. All other problems, which are more difficult, together with the not mentioned cases of $\cdot|I|\cdot$ and $\cdot|G|\cdot$ types are under investigation and they are still waiting for the solution.

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Súhrn

ROZMIESTŇOVANIE VRCHOLOV MNOHOUHOLNÍKOV NA KRUŽNICIACH A JEHO APLIKÁCIA V DOPRAVE

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Článok nadväzuje na stať [7], ktorá opisovala problém rozmiestnenia pravidelných mnohouholníkov na kružnici. Táto úloha sa teraz zovšeobecňuje na prípad „skoro pravidelných" mnohouholníkov na jednej alebo viacerých kružniciach. Opisuje sa klasifikácia špeciálnych prípadov takýchto úloh podľa rôznych kritérií pravidelnosti individuálnych resp. zložených mnohouholníkov, podľa ich počtu a podľa toho, či ide o jednu alebo viac kružníc. Pre niekoľko špeciálnych úloh sa určujú optimálne riešenia.

Резюме

РАЗМЕЩЕНИЕ ВЕРШИН МНОГОУГОЛЬНИКОВ НА ОКРУЖНОСТЬЯХ И ЕГО ПРИЛОЖЕНИЕ К ТРАНСПОРТУ

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В статье рассматривается проблема распределения вершин многоугольников на данных окружностях, исполняя некоторые критерия „регулярности" индивидуальных и комплексных многоугольников. Уточнением условий получается ряд частых случаев общей проблемы. Некоторые из них уже решены, другие еще нет.

Приложения этой теории в подготовке расписаний периодически повторяющихся процессов, напр. в координации нескольких линий городского транспорта на общем пути или в оптимизации рифма сортировочной железногорожной станции.

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