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# SUPERCONVERGENCE OF EXTERNAL APPROXIMATION FOR TWO-POINT BOUNDARY PROBLEMS

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*Summary.* The superconvergence property of a certain external method for solving two point boundary value problems is established. In the case when piecewise polynomial spaces are applied, it is proved that the convergence rate of the approximate solution at the knot points can exceed the global one.

*Keywords:* Superconvergence of external approximations, two-point boundary value problems

## INTRODUCTION

Finite element approximate solutions of differential equations can possess a superconvergence property, that is, there are some distinguished sets of points at which the convergence rate of the solution or its gradient exceeds the global optimum. It has been observed (cf. De Boor-Swartz [2], Douglas-Dupont [4]) that certain collocation methods give a superconvergence of solution at the knot points of the splines used. Moreover, superconvergence at the knots has been established by Douglas-Dupont [5] for the Galerkin solution of the two point boundary value problem when the approximation subspaces consist of $C^0$-piecewise polynomials. A superconvergence phenomenon for the gradient of the finite element approximate solution of the second order elliptic boundary value problem was analysed e.g. by Zlámal [9] and Křížek-Neittaanmäki [7] where extensive references concerning this problem can be found.

The object of this paper is to establish some superconvergence properties of a certain external method for solving two point boundary value problems using piecewise polynomial spaces. The method considered is a generalization of the Galerkin method.

Let us consider the problem

$$(1.1) \qquad -u'' + bu = f \quad \text{on} \quad I = [0, 1]\,, \qquad u(0) = u(1) = 0\,.$$

We will assume that $b \geqq \beta > 0$, $b, f \in \mathscr{L}^2(I)$.

Let us introduce two families of spaces $\{V_h\}_{h\in\mathscr{H}}$, $\{W_h\}_{h\in\mathscr{H}}$ such that $V_h \subset H_0^1(I)$ and $W_h \subset \mathscr{L}^2(I)$. Let $\phi_h$ be the orthogonal projection of $\mathscr{L}^2(I)$ onto $W_h$, i.e.

$$(1.2) \qquad \forall w_h \in W_h\,, \quad \forall v \in \mathscr{L}^2(I)\,, \quad (v - \phi_h v, w_h) = 0\,,$$

where ( , ) denotes the scalar product in $\mathscr{L}^2(I)$.

The following approximate problem will be considered:

$$(1.3) \qquad \text{find} \quad u_h \in V_h \quad \text{such that}$$
$$(u_h', v_h') + (b\phi_h u_h, \phi_h v_h) = (f, \phi_h v_h) \quad \forall v_h \in V_h\,.$$

It is a certain kind of the external approximation of (1.1) – the solution $u$ is approximated by the pair $\{\phi_h u_h, u_h\}$. The problem (1.3) is another formulation of the partial approximation of (1.1) considered by Aubin in [1] (cf. Definition 2.1, Chap. XI) for a special choice of prolongations. This external approximation was also applied to solving eigenvalue problems (cf. [8]).

In Section 2 a certain generalization of the Cea lemma giving a relation between the error bound and the approximation properties of $V_h$ and $W_h$ is proved. The main result concerning the superconvergence property is presented in Section 3 for the case when $V_h$ and $W_h$ are piecewise polynomial spaces. It is proved that if the solution is sufficiently smooth then the error at the knots is bounded by the square of the possible global estimation. In Section 4 the explicit form of $\phi_h v_h$ for $v_h \in V_h$ is found for a special choice of piecewise polynomial spaces. The general case of external approximation for higher order equations applying more than one, not sufficiently orthogonal, projections $\phi_h$, will be considered in the subsequent paper.

## 2. ERROR ESTIMATION

The following notation will be used:

$$\begin{aligned} F &= \mathscr{L}^2 \times H_0^1\,, & F_n &= W_h \times V_h \subset F\,;\\ \omega&\colon H_0^1 \to F\,; & \omega u &= \{u, u\} \in F\,;\\ \omega_h&\colon V_h \to F_h\,; & \omega_h v_h &= \{\phi_h v_h, v_h\}\,. \end{aligned}$$

Let us introduce a bilinear form $\bar{a}$ on $F \times F$:

$$\bar{a}(\bar{u}, \bar{v}) = (u_1', v_1') + (bu_0, v_0)$$
$$\forall \bar{u}, \bar{v} \in F\,, \quad \bar{u} = \{u_0, u_1\}\,, \quad \bar{v} = \{v_0, v_1\}\,.$$

By the assumption $b \geqq \beta > 0$, $\bar{a}$ is $F$-elliptic, i.e.

$$\bar{a}(\bar{u}, \bar{u}) \geqq \beta \|\bar{u}\|_F^2 \quad \forall \bar{u} \in F\,,$$

and moreover, $\exists \alpha > 0\ \forall \bar{u}, \bar{v} \in F$

$$\bar{a}(\bar{u}, \bar{v}) \leqq \alpha \|\bar{u}\|_F\, \|\bar{v}\|_F\,,$$

where $\|\bar{u}\|_F^2 = \|u_0\|_0^2 + \|u_1\|_1^2$.

The variational formulation of the problem (1.1) and the problem (1.3) can be written as follows:

(2.1) $$\bar{a}(\omega u, \omega v) = (f, v) \quad \forall v \in H_0^1 ,$$

(2.2) $$\bar{a}(\omega_h u_h, \omega_h v_h) = (f, \phi_h v_h) \quad \forall v_h \in V_h .$$

We will assume that $\forall v \in H_0^1$ $\|(1 - \phi_h) v\|_0 \leqq ch\|v\|_1$.

Under this assumption there exists a unique solution of (2.2) for $h < h_0$ since, by the $F$-ellipticity of $\bar{a}$, $\bar{a}(\omega_h v_h, \omega_h v_h) \geqq c\|v_h\|_1^2$ for $h < h_0$.

Putting $v = v_h$ in (2.1) and subtracting (2.2) from (2.1) we obtain

$$\bar{a}(\omega u - \omega_h u_h, \omega_h v_h) + \bar{a}(\omega u, \omega v_h - \omega_h v_h) = (f, v_h - \phi_h v_h) .$$

Since $\omega v_h - \omega_h v_h = \{v_h - \phi_h v_h, 0\}$ and $f - bu = -u''$ (from (1.1)), we have

(2.3) $$\bar{a}(\omega u - \omega_h u_h, \omega_h v_h) = (-u'', v_h - \phi_h v_h) \quad \forall v_h \in V_h .$$

The following theorem is similar to the Cea lemma for the Galerkin approximation (cf [3], Th. 2.4.1).

**Theorem 1.** *Let*

$$\|v - P_h v\|_0 \leqq ch\|v\|_1 \quad \forall v \in H^1(I) .$$

*If $u$ and $u_h$ are solutions of (1.1) and (1.3), respectively, then there exists a constant $c$ independent of the subspaces $V_h$ and $W_h$ such that*

$$\|\omega u - \omega_h u_h\|_F \leqq C \inf_{v_h \in V_h} \|u - v_h\|_1 + \inf_{w_h \in W_h} \|u - w_h\|_0 + h \inf_{w_h \in W_h} \|u'' - w_h\|_0\} .$$

Proof. From the $F$-ellipticity of $\bar{a}$ it follows that for an arbitrary element $y_h \in V_h$

$$\beta\|\omega u - \omega_h u_h\|_F^2 \leqq \bar{a}(\omega u - \omega_h u_h, \omega u - \omega_h u_h) =$$
$$= \bar{a}(\omega u - \omega_h u_h, \omega u - \omega_h y_h) + \bar{a}(\omega u - \omega_h u_h, \omega_h(y_h - u_h)) .$$

Thus, the application of the equality (2.3) to the second term of the right-hand side gives

(2.4) $$\beta\|\omega u - \omega_h u_h\|_F^2 \leqq$$
$$\leqq \alpha\|\omega u - \omega_h u_h\|_F \|\omega u - \omega_h y_h\|_F + |(u'', (1 - \phi_h)(y_h - u_h))| .$$

Since $\|\phi_h\| = 1$, thus $\forall y_h \in V_h$

$$\|\omega u - \omega_h y_h\|_F = \{\|u - y_h\|_1^2 + \|u - \phi_h y_h\|_0^2\}^{1/2} \leqq$$
$$\leqq \{\|u - y_h\|_1^2 + | \|u - \phi_h u\|_0 + \|u - y_h\|_0|^2\}^{1/2} \leqq$$
$$\leqq \sqrt{5} \max [\|u - y_h\|_1, \|u - \phi_h u\|_0] .$$

Moreover, by (1.2) and by the fact that $\forall v \in \mathscr{L}^2(I)$

$$\|(1 - \phi_h) v\|_0 = \inf_{w_h \in W_h} \|v - w_h\|_0$$

it follows that for any $z_h \in W_h$

$$|(u'', (1 - \phi_h)(y_h - u_h))| = |(u'' - z_h, (1 - \phi_h)(y_h - u + u - u_h)| \leqq$$
$$\leqq \|u'' - z_h\|_0 \{ \inf_{w_h \in W_h} \|(y_h - u) - w_h\|_0 + \inf_{w_h \in W_h} \|(u - u_h) - w_h\|_0\} \leqq$$
$$\leqq h\|u'' - z_h\|_0 \{\|y_h - u\|_1 + \|u - u_h\|_1\},$$

due to the assumption of Theorem 1. Thus, since $\|u - u_h\|_1 \leqq \|\omega u - \omega_h u_h\|_F$, (2.4) implies

$$\|\omega u - \omega_h u_h\|_F^2 \leqq C\|\omega u - \omega_h u_h\|_F \{\max(\|u - y_h\|_1, \|u - \phi_h u\|_0) +$$
$$+ h\|u'' - z_h\|_0\} + h\|u'' - z_h\|_0 \|u - y_h\|_1,$$

for some constant $C$. Solving this inequality and replacing the maximum of the two norms by the sum, we obtain

$$\|\omega u - \omega_h u_h\|_F \leqq C\{\|u - y_h\|_1 + \|u - \phi_h u\|_0 + h\|u'' - z_h\|_0 +$$
$$+ [(\|u - y_h\|_1 + \|u - \phi_h u\|_0 + h\|u'' - z_h\|_0)^2 +$$
$$+ 4h\|u'' - z_h\|_0 \|u - y_h\|_1]^{1/2}\} \leqq$$
$$\leqq C(1 + \sqrt{3})[\|u - y_h\|_1 + \|u - \phi_h u\|_0 + h\|u'' - z_h\|_0].$$

The left-hand side does not depend on $y_h$ and $z_h$, thus the infimum over $y_h \in V_h$ and $z_h \in W_h$ of the right-hand side can be taken. Thus the theorem is established.

Now, our consideration will be restricted to the piecewise polynomial spaces. Let $h = 1/(n + 1)$ and let $\Delta_h$ (briefly $\Delta$) be the uniform partition of $I$:

$$\Delta_h = \{x_i = ih,\ i = 0, \ldots, n + 1\}, \quad I_i = (x_i, x_{i+1}).$$

Let $P_r(I_i)$ denote the class of polynomials of degree not greater than $r$ on the set $I_i$. Let

$$S_h(C^0, r) = \{v \in C(I),\ v \in \mathscr{P}_r(I_i)\ i = 0, \ldots, n\},$$
$$S_h(\mathscr{L}^2, r) = \{v \in \mathscr{L}^2(I),\ v \in \mathscr{P}_r(I_i)\ i = 0, \ldots, n\}.$$

It will be assumed that

(2.5) $$V_h = S_h(C^0, m) \cap H_0^1(I), \quad W_h = S_h(\mathscr{L}^2, s)$$

and $m > s \geqq 0$.

Let $v \in H_\Delta^{r+1}$, where

$$H_\Delta^{r+1} = \{v \in \mathscr{L}^2(I): v \in H^{r+1}(I_i),\ i = 0, \ldots, n\}$$

with the norm

$$\|v\|_{\Delta r+1}^2 = \sum_{i=0}^{n} \|v|_{I_i}\|_{r+1}^2.$$

Let us introduce $I_\Delta^r v$ as the spline interpolant to $v$ from $S_h(C^0, r)$ generated by the knots

$$x_{ij} = x_i + j\frac{h}{r}, \quad j = 0, \ldots, r - 1, \quad x_{ir} = x_{i+1,0} = x_{i+1}.$$

Thus

$$I_\Delta^r v(x) = \sum_{i=0}^{n} p_{ri}(x)\,\chi\left(\frac{x}{h} - i\right) \tag{2.6}$$

where $\chi(x)$ is the characteristic function of the interval $(0, 1)$ and $p_{ri} \in \mathscr{P}_r(I_i)$ satisfies

$$p_{ri}(x_{ij}) = v(x_{ij}) \quad j = 0, \ldots, r. \tag{2.7}$$

From the Peano Kernel Theorem ([6], Th. 3.7.1) it follows that if $p_{ri}$ satisfies (2.7) then

$$\|v - p_{ri}\|_{H^1(I_i)}^2 \leqq ch^{2r}\int_{x_i}^{x_{i+1}} |v^{(r+1)}|^2.$$

Thus

$$\|v - I_\Delta^r v\|_1^2 = \sum_{i=0}^{n} \|v - p_{ri}\|_{H^1(I_i)}^2 \leqq ch^{2r}\|v\|_{\Delta r+1}^2. \tag{2.8}$$

Similarly the spline interpolant $J_\Delta^r v$ to $v$ from $S_h(\mathscr{L}^2, r)$ can be constructed. Namely, let $y_{ij} = x_i + j(h/(r + 1))$, $j = 0, \ldots, r$ be the knot points of interpolation and let

$$J_\Delta^r v(x) = \sum_{i=0}^{n} q_{ri}(x)\,\chi\left(\frac{x}{h} - i\right) \tag{2.9}$$

where $q_{ri} \in \mathscr{P}_r(I_i)$ satisfies

$$q_{ri}(y_{ij}) = v(y_{ij}), \quad j = 0, \ldots, r. \tag{2.10}$$

Since $L(v) := (v - J_\Delta^r v)|_{I_i} = 0$ for all $v \in \mathscr{P}_k(I_i)$ where $k$ is an arbitrary integer not greater than $r$, then for $v \in H^{k+1}(I_i)$ the Peano theorem ([6], th. 3.7.1) implies that

$$\|v - J_\Delta^r v\|_{\mathscr{L}^2(I_i)} \leqq ch^{k+1}\|v^{(k+1)}\|_{\mathscr{L}^2(I_i)}.$$

Thus,

$$\|v - J_\Delta^r v\|_0 \leqq ch^{k+1}\|v\|_{\Delta k+1} \tag{2.11}$$

for $v \in H_\Delta^{k+1}(I)$ provided $k \leqq r$.

Therefore, (2.8) and (2.11) imply that if $u \in H_\Delta^{m+1}$ then

$$\begin{cases} \inf\limits_{v_h \in V_h} \|u - v_h\|_1 \leqq ch^m\|u\|_{\Delta m+1}, \\ \inf\limits_{w_h \in W_h} \|u - w_h\|_0 \leqq ch^{s+1}\|u\|_{\Delta s+1}, \\ \inf\limits_{w_h \in W_h} \|u'' - w_h\|_0 \leqq ch^{\min(s+1, m-1)}\|u\|_{\Delta m+1}. \end{cases} \tag{2.12}$$

The following estimation is a simple consequence of Theorem 1 and (2.12)

**Corollary 1.** *If* $b, f \in H_{\Delta}^{m+1}(I)$ *and* $V_h = S_h(C^0, m) \cap H_0^1$ *and* $W_h = S_h(\mathscr{L}^2, s)$ for $m > s \geqq 0$, *then*

$$\|\omega u - \omega_h u_h\|_0 \leqq ch^{s+1} .$$

Convergence and error estimates of the external approximation of an elliptic operator were also considered by Aubin ([1], Th. 2.2. Chap. XI). The general results were obtained in another way.

## 3. SUPERCONVERGENCE AT THE KNOTS

Let $G(t, x)$ *be the Green function of the operator* $-u''$ *with the boundary conditions* $u(0) = u(1) = 0$.

Thus

$$G(t, x) = \begin{cases} (1-t)\,x & 0 \leqq x \leqq t \\ (1-x)\,t & t \leqq x \leqq 1 , \end{cases}$$

and

$$(u', G'(t, \cdot)) = -\int_0^1 u''(x)\, G(t, x)\, \mathrm{d}x = u(t) .$$

Let $\hat{v}(x) = G(x_i, x)$, where $x_i = ih$, $0 \leqq i \leqq n + 1$.

Then $\hat{v} \in V_h$ because, as was mentioned above, we consider now the case of the piecewise polynomial subspaces $V_h$ and $W_h$ given by (2.5) for $m \geqq 1$. The equation (2.3) for $v_h = \hat{v}$ takes the form

$$(u - u_h)(x_i) = (-u'', \hat{v} - \phi_h \hat{v}) + (b(u - \phi_h u_h), \phi_h \hat{v}) .$$

If $s \geqq 1$ then $\hat{v} \in S_h(\mathscr{L}^2, s)$ and $\hat{v} = \phi_h \hat{v}$. Thus

$$(3.1) \qquad |(u - u_h)(x_i)| \leqq \begin{cases} |(b(u - \phi_h u_h), \hat{v})| & \text{if} \quad s \geqq 1 , \\ |(b(u - \phi_h u_h), \phi_h \hat{v})| + |(u'', \hat{v} - \phi_h \hat{v})| & \text{if} \quad s = 0 . \end{cases}$$

The trick presented above was used by Douglas-Dupont [5] for establishing a superconvergence result for the Galerkin method.

**Lemma 1.** *Let* $u$ *and* $u_h$ *be the solutions of* (1.1) *and* (1.3), *respectively. If* $b, f \in H_{\Delta}^{m-1}(I)$ *and* $V_h$ *and* $W_h$ *are given by* (2.5) *then there exists a constant* $c$ *independent of* $h$ *such that*

$$|(u - u_h, b\hat{v})| \leqq \begin{cases} ch^{2(s+1)} & \text{if} \quad s < m - 1 , \\ ch^{2m-1} & \text{if} \quad s = m - 1 . \end{cases}$$

*If moreover* $b, f \in H_{\Delta}^{m}(I)$ *then for* $s = m - 1$,

$$|(u - u_h, b\hat{v})| \leqq ch^{2m} .$$

Proof. Let us introduce an auxiliary problem

$$(3.2) \qquad \text{find} \quad \psi \in H_0^1 , \quad \bar{a}(\omega v, \omega\psi) = (v, b\hat{v}) \quad \forall v \in H_0^1 .$$

By the $F$-ellipticity pf $\bar{a}$, there exists a unique solution of (3.2). If $b \in H_\Delta^\nu$ ($\nu = m - 1, m$), then $b\hat{v} \in H_\Delta^\nu$ and $\psi \in H_\Delta^{\nu+2}$ and by (2.7) and (2.9),

$$\|\psi - J_\Delta^m \psi\|_1 \leqq ch^m , \quad \|b\psi - J_\Delta^s b\psi\|_0 \leqq ch^{\min(s+1,\nu)} . \tag{3.3}$$

Put $v = u - u_h$ Since $u - u_h \in H_0^1$ thus (3.2) implies that

$$\begin{aligned}(u - u_h, b\hat{v}) &= \bar{a}(\omega(u - u_h), \omega\psi) = \\ &= \bar{a}(\omega u - \omega_h u_h, \omega(\psi - y)) + \bar{a}(\omega_h u_h - \omega u_h, \omega\psi) + \\ &+ \bar{a}(\omega u - \omega_h u_h, \omega y - \omega_h y) + \bar{a}(\omega u - \omega_h u_h, \omega_h y)\end{aligned}$$

for arbitrary $y \in H_0^1$. Let $y = I_\Delta^m \psi$ (cf. (2.6)). Thus

$$\bar{a}(\omega u - \omega_h u_h, \omega(\psi - I_\Delta^m \psi)) \leqq \alpha \sqrt{(2)} \, \|\omega u - \omega_h u_h\|_F \, \|\psi - I_\Delta^m \psi\|_1 .$$

Next, since $u\omega_{hh} - \omega u_h = \{\phi_h u_h - u_h, 0\}$, by (1.2) for $w_h = J_\Delta^s b\psi$, we have

$$\begin{aligned}\bar{a}(\omega_h u_h - \omega u_h, \omega\psi) &= (\phi_h u_h - u_h, b\psi - J_\Delta^s b\psi) \leqq \\ &\leqq \|\phi_h u_h - u_h\|_0 \, . \, \|b\psi - J_\Delta^s b\psi\|_0 ,\end{aligned}$$

and moreover,

$$\begin{aligned}\bar{a}(\omega u - \omega_h u_h, \omega y - \omega_h y) &= (b(u - \phi_h u_h), y - \phi_h y) = \\ &\leqq c\|\omega u - \omega_h u_h\|_F \, . \, \|I_\Delta^m \psi - \phi_h I_\Delta^m \psi\|_0 .\end{aligned}$$

Finally, (2.3) and (1.2) imply

$$\begin{aligned}\bar{a}(\omega u - \omega_h u_h, \omega_h y) &= (u'', \phi_h y - y) = (u'' - J_\Delta^s u'', \phi_h y - y) \leqq \\ &\leqq \|u'' - J_\Delta^s u''\|_0 \, . \, \|I_\Delta^m \psi - \phi_h I_\Delta^m \psi\|_0 .\end{aligned}$$

Thus, (3.3), (2.9) and Corollary 1 yield

$$|(u - u_h, b\hat{v})| \leqq c\{h^{(s+1)m} + h^{\min(s+1,\nu)}[\|\phi_h u_h - u_h\|_0 + \|I_\Delta^m \psi - \phi_h I_\Delta^m \psi\|_0]\}$$

if $f \in H_\Delta^\nu$. Now, it remains to observe that

$$\|u_h - \phi_h u_h\|_0 \leqq \|u_h - u\|_0 + \|u - \phi_h u\|_0 + \|\phi_h(u - u_h)\|_0$$

and

$$\|I_\Delta^m \psi - \phi_h I_\Delta^m \psi\|_0 \leqq \|I_\Delta^m \psi - \psi\|_0 + \|\psi - \phi_h \psi\|_0 + \|\phi_h(\psi - I_\Delta^m \psi)\|_0$$

yield

$$\|u_h - \phi_h u_h\|_0 \leqq c[h^{s+1} + \|u - \phi_h u\|_0] \leqq c[h^{s+1} + \|u - I_\Delta^s u\|_0] \leqq c_1 h^{s+1}$$

and

$$\|I_\Delta^m \psi - \phi_h I_\Delta^m \psi\|_0 \leqq c[h^{m+1} + \|\psi - \phi_h \psi\|_0] \leqq c[h^{m+1} + \|\psi - I_\Delta^s \psi\|_0] \leqq c_1 h^{s+1} ,$$

and the lemma is proved.

**Theorem 2.** *Let $u$ and $u_h$ be the solutions of* (1.1) *and* (1.3), *respectively. If $b, f \in H_\Delta^{m-1}$ and $V_h$ and $W_h$ are given by* (2.5) *then for $x_i = ih$, $i = 1, \ldots, n$,*

$$|u(x_i) - u_h(x_i)| \leqq \begin{cases} ch^{2(s+1)} & \text{if} \quad s < m - 1\,, \\ ch^{2m-1} & \text{if} \quad s = m - 1\,. \end{cases}$$

*If moreover* $b, f \in H_\Delta^m$ *then for* $s = m - 1$,

$$|u(x_i) - u_h(x_i)| \leqq ch^{2m}\,.$$

Proof. Let us observe that

$$|(b(u - \phi_h u_h), \phi_h \hat{v})| \leqq c\|u - \phi_h u_h\|_0 \, \|\hat{v} - J_\Delta^s \hat{v}\|_0 + |(b(u - \phi_h u_h), \hat{v})|$$

and

$$|(b(u - \phi_h u_h), \hat{v})| = |(u - \phi_h u, b\hat{v}) + (u - u_h, \phi_h b\hat{v} - b\hat{v}) + (u - u_h, b\hat{v})| \leqq$$
$$\leqq \|b\hat{v} - J_\Delta^s b\hat{v}\|_0 \left[\|u - J_\Delta^s u\|_0 + \|u - u_h\|_0\right] + |(u - u_h, b\hat{v})|\,.$$

Thus, due to Lemma 1, Corollary 1 and the estimate (2.11) it follows that

$$|(b(u - \phi_h u_h), \phi_h \hat{v})| \leqq ch^{(s+1)\min(s+1, m-1)}\,.$$

Now, the theorem follows from (3.1) and the estimate

$$|(u'', \hat{v} - \phi_h \hat{v})| \leqq \|u'' - J_\Delta^s u''\|_0 \, \|\hat{v} - J_\Delta^s \hat{v}\|_0 \leqq$$
$$\leqq \begin{cases} 0 & \text{if} \quad s > 0\,, \\ h & \text{if} \quad s = 0 \quad \text{and} \quad b, f \in \mathscr{L}^2(I)\,, \\ h^2 & \text{if} \quad s = 0 \quad \text{and} \quad b, f \in H_\Delta^1(I)\,. \end{cases}$$

A comparison of the convergence properties of the external method (1.3) and the corresponding Galerkin method is presented in the next remark.

Remark 1. Let $z_h$ be the Galerkin approximate solution to $u$ generated by $V_h = S_h(C^0, m) \cap H_0^1(I)$ and let $u_h$ be the solution of the external approximation equation (1.3) generated by the same $V_h$ and $W_h = S_h(\mathscr{L}^2, m - 1)$. Then $\|u - u_h\|_1$ is of the same order $(0(h^m))$ as $\|u - z_h\|_1$ and moreover, for $x_i = ih$, $i = 1, \ldots, n$

$$|u(x_i) - u_h(x_i)| \leqq \begin{cases} ch^{2m-1} & \text{if} \quad b, f \in H_\Delta^{m-1}\,, \\ ch^{2m} & \text{if} \quad b, f \in H_\Delta^m\,, \end{cases}$$

while

$$|u(x_i) - z_h(x_i)| \leqq ch^{2m} \quad \text{if} \quad b, f \in H_\Delta^{m-1}\,.$$

It is easy to see that the same result can be obtained if the nonselfadjoint equation

$$u \in H_0^1\colon (u', v') + (au, v') + (bu, v) = (f, v) \quad \forall v \in H_0^1$$

is approximated by

$$u_h \in V_h\colon (u_h', v_h') + (a\phi_h u_h, v_h') + (b\phi_h u_h, \phi_h v_h) = (f, \phi_h v_h) \quad \forall v_h \in V_h\,,$$

where $V_h$, $W_h$ and $\phi_h$ are defined as before.

Moreover, the method can be extended to the case of higher order differential equations. For example, let us consider the problem

$$u \in H_0^k\,, \quad (u^{(k)}, v^{(k)}) + (bu, v) = (f, v) \quad \forall v \in H_0^k\,,$$

and the approximate equation

$$u_h \in V_h\colon (u_h^{(k)}, v_h^{(k)}) + (b\phi_h u_h, \phi_h v_h) = (f, \phi_h v_h) \quad \forall v_h \in V_h$$

where $V_h = S_h(C^{k-1}, k + r) \cap H_0^k$ and $\phi_h$ is the orthogonal projection of $\mathscr{L}^2$ onto $W_h = S_h(\mathscr{L}^2, s)$. Then it can be proved that

$$\|u - u_h\|_k \leqq c\{ \inf_{v_h \in V_h} \|u - v_h\|_k + \inf_{w_h \in W_h} \|u - w_h\|_0 + h^s \inf_{w_h \in W_h} \|f - bu - w_h\|_0\,.$$

In the case $r \geqq k - 1$, the superconvergence at the knot points $x_i = ih$, $i = 1, \ldots, n$ can be established under the condition of sufficient regularity of the solution. For the proof of that property, the function $\hat{v}(x) = G(x_i, x)$ is applied, where $G(t, x)$ is the Green function of the problem$(-1)^k y^{(2k)} = f$, $y \in H_0^k$. If $r \geqq k - 1$ then $\hat{v} \in V_h$.

## 4. THE FINITE DIMENSIONAL PROBLEM

In this section only the spaces $V_h$ and $W_h$ given by (2.5) for $s = m - 1$ will be considered.

In order to obtain a matrix equation implied by (1.3) one needs an explicit form of $\phi_h v_h$ for $v_h \in S_h(C^0, m)$.

Let $x_i$ be the knot points considered above.

Let $\{x_{ij}\}_{j=0}^m$ be the uniform partition of the subinterval $[x_i, x_{i+1}]$: $x_{ij} = x_i + (j/m)\, h$, $j = 0, \ldots, m - 1$ and $x_{im} = x_{i+10}$, $x_{00} = 0$, $x_{nm} = 1$.

Let $l_k^m$ denote the $k$-th Lagrange interpolate polynomial on the $m + 1$ knots $0 = x_{00}, x_{01}, \ldots, x_{0m}$, i.e.:

$$l_k^m(x) = \frac{(x - x_{00}) \ldots (x - x_{0k-1})(x - x_{0k+1}) \ldots (x - x_{0m})}{(x_{0k} - x_{00}) \ldots (x_{0k} - x_{0k-1})(x_{0k} - x_{0k+1}) \ldots (x_{0k} - x_{0m})}\,.$$

Similarly, let $l_k^{m-1}$ be the $k$-th Lagrange interpolate polynomial on the $m$ knots $x_{00}, \ldots, x_{0m-1}$. If $m = 0$ we put $l_0^0 = 1$. Then, for arbitrary $v_h \in V_h$ and $w_h \in W_h$,

$$v_h(x) = \sum_{i=0}^{n} \sum_{\nu=0}^{m} v_h(x_{i\nu})\, l_\nu^m(x - ih)\, \chi\left(\frac{x}{h} - i\right)$$

$$w_h(x) = \sum_{i=0}^{n} \sum_{\mu=0}^{m-1} w_h(x_{i\mu})\, l_\mu^{m-1}(x - ih)\, \chi\left(\frac{x}{h} - i\right).$$

Let $\mathfrak{a}_\nu = \{\alpha_{\nu j}\}_{j=0}^{m-1}$ be a solution of the matrix equation

$$A_h \mathfrak{a}_\nu = \mathfrak{b}_\nu$$

where

$$A_h = \{(l_\mu^{m-1}, l_j^{m-1})_{\mathscr{L}^2(0,h)}\}_{\mu,j=0}^{m-1}$$

and

$$\mathfrak{b}_\nu = \{(l_\nu^m, l_j^{m-1})_{\mathscr{L}^2(0,h)}\}_{j=0}^{m-1} .$$

Since $A_h$ is the Gramm matrix of a linearly independent set of functions on $[0, h]$, $A_h^{-1}$ exists and $\mathfrak{a}_\nu = A_h^{-1}\mathfrak{b}_\nu$. Let

$$q_\nu(x) = \sum_{j=0}^{m-1} \alpha_{\nu j}\, l_j^{m-1}(x) \quad \text{for} \quad x \in [0, h] .$$

It is easy to see that $q_\nu(x) = \phi_h l_\nu^m$ since

$$\int_0^h q_\nu(x)\, l_j^{m-1}(x) = \int_0^h l_\nu^m(x)\, l_j^{m-1}(x)\, \mathrm{d}x \quad \text{for} \quad j = 0, \ldots, m-1 .$$

Thus, due to the linearity of $\phi_h$, we have $\forall v_h \in V_h$

(4.1) $$\phi_h\, v_h(x) = \sum_{i=0}^{n} \sum_{\nu=0}^{m} v_h(x_{i\nu})\, \phi_h \left[ l_\nu^m(x - ih)\, \chi\left(\frac{x}{h} - i\right)\right] =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m-1} \{ \sum_{\nu=0}^{m} v_h(x_{i\nu})\, \alpha_{\nu j}\}\, l_j^{m-1}(x - ih)\, \chi\left(\frac{x}{h} - i\right).$$

As examples, the cases $m = 1$ and $m = 2$ will be considered. If $m = 1$ then $A_h = h$ $\mathfrak{b}_0 = \mathfrak{b}_1 = h/2$. Thus $\mathfrak{a}_0 = \mathfrak{a}_1 = \frac{1}{2}$ and

$$\phi_h \left[ \sum_{i=0}^{n} (v(x_i)\, l_0^1(x - ih) + v(x_{i+1})\, l_1^1(x - ih))\, \chi\left(\frac{x}{h} - i\right)\right] =$$

$$= \sum_{i=0}^{n} \tfrac{1}{2}(v(x_i) + v(x_{i+1}))\, \chi\left(\frac{x}{h} - i\right)$$

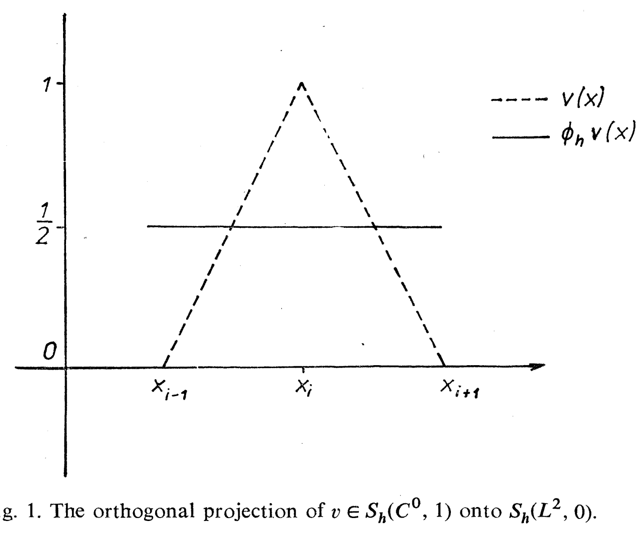


Fig. 1. The orthogonal projection of $v \in S_h(C^0, 1)$ onto $S_h(L^2, 0)$.

Let now $m = 2$. In this case

$$A_h = h \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{vmatrix}, \quad \mathfrak{b}_0 = h \begin{vmatrix} \frac{1}{6} \\ 0 \end{vmatrix}, \quad \mathfrak{b}_1 = h \begin{vmatrix} 0 \\ \frac{2}{3} \end{vmatrix}, \quad \mathfrak{b}_2 = h \begin{vmatrix} -\frac{1}{6} \\ \frac{1}{3} \end{vmatrix}$$

and thus

$$\mathfrak{a}_0 = \begin{vmatrix} \frac{2}{3} \\ \frac{1}{6} \end{vmatrix}, \quad \mathfrak{a}_1 = \begin{vmatrix} \frac{2}{3} \\ \frac{2}{3} \end{vmatrix}; \quad \mathfrak{a}_2 = \begin{vmatrix} -\frac{1}{3} \\ \frac{1}{6} \end{vmatrix}.$$

According to (4.1),

$$\phi_h \left[ \sum_{i=0}^{n} (v(x_{i0})\, l_0^2(x - ih) + v(x_{i1})\, l_i^2(x - ih) + v(x_{i2})\, l_2^2(x - ih))\, \chi\left(\frac{x}{h} - i\right) \right] =$$

$$= \sum_{i=0}^{n} \{ [\tfrac{2}{3}\, v(x_{i0}) + \tfrac{2}{3}\, v(x_{i1}) - \tfrac{1}{3}\, v(x_{i2})]\, l_0^1(x - ih) +$$

$$+ [\tfrac{1}{6}\, v(x_{i0}) + \tfrac{2}{3}\, v(x_{i1}) + \tfrac{1}{6}\, v(x_{i2})]\, l_0^1(x - ih) \}\, \chi\left(\frac{x}{h} - i\right)$$

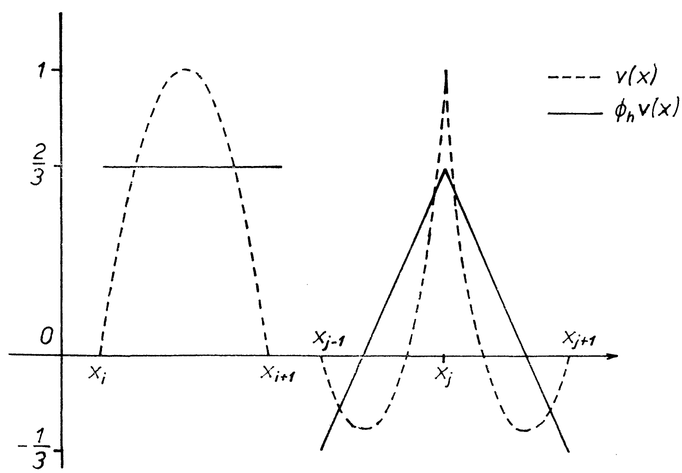


Fig. 2. The orthogonal projection of $v \in S_h(C^0, 2)$ onto $S_h(L^2, 1)$.

*References*


[1] *J. P. Aubin:* Approximation of elliptic boundary-value problems. Wiley-Interscience, New York, 1972.

[2] *C. De Boor, B. Swartz:* Collocation at Gausian points. SIAM J. Numer. Anal. 10 (1973), 582—606.

[3] *P. Ciarlet:* The finite element method for elliptic problems. North-Holland, Publishing Company (1978).
[4] *J. Douglas Jr., T. Dupont:* Collocation method for parabolic equations in a single space variable. Lecture Notes in Math., 385 (1974).
[5] *J. Douglas Jr., T. Dupont:* Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary problems — Topics in numerical analysis, ed. J.J.M.F. Miller, pp. 89—92 (1973).
[6] *P. J. Davis:* Interpolation and approximation. Blaisdell Publishing Company (1963).
[7] *M. Křížek, P. Neittaanmäki:* Superconvergence phenomenon in the finite element method arising from averaging gradients. Numer. Math. 45 (1984), pp. 105—116.
[8] *T. Regińska:* Superconvergence of eigenvalue external approximation for ordinary differential operators. IMA Jour. Numer. Anal. 6 (1986), pp. 309—323.
[9] *M. Zlámal:* Some superconvergence results in the finite element method — Mathematical Aspects of f.e.m.. Lecture Notes 606 (1977), pp. 353—362.

Souhrn

## SUPERKONVERGENCE VNĚJŠÍCH APROXIMACÍ PRO DVOJBODOVOU OKRAJOVOU ÚLOHU

TERESA REGIŃSKA

Dokazuje se vlastnost superkonvergence pro jistou vnější metodu řešení dvoubodové okrajové úlohy. Pro případ po částech polynomiálních prostorů je dokázáno, že rychlost konvergence přibližných řešení v uzlových bodech může přesáhnout globální rychlost konvergence.

Резюме

## СВЕРХСХОДИМОСТЬ ВНЕШНИХ АППРОКСИМАЦИЙ ДЛЯ ДВУХТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧИ

TERESA REGIŃSKA

Доказывается сверхсходимость одного внешнего метода решения двухточечной краевой задачи. Для случая кусочно полиномиальных пространств доказано, что скорость сходимости приближённых решений в узловых точках может превысить скорость сходимости в целом.

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