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ESTIMATION OF PARAMETERS OF MEAN AND VARIANCE IN TWO-STAGE LINEAR MODELS

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Summary. The paper deals with the estimation of unknown vector parameter of mean and scalar parameters of variance as well in two-stage linear model, which is a special type of mixed linear model. The necessary and sufficient condition for the existence of uniformly best unbiased estimator of parameter of mean is given. The explicite formulas for these estimators and for the estimators of the parameters of variance as well are derived.

Key words: Two-stage linear model, mixed linear model, estimation of parameters, best unbiased estimator.

AMS Classification: 62F10, 62J05.

INTRODUCTION

The two-stage linear model is often modelled by random vectors Y_1, Y_2 Y_k of dimensions $n \times 1$, $m \times 1$, respectively, with

$$(1) \qquad \begin{aligned} \mathbf{Y}_1 &= \mathbf{X}_1\boldsymbol{\beta}_1 + \varepsilon_1\,, \quad \mathsf{E}(\varepsilon_1) = \mathbf{0}\,, \quad \mathsf{E}(\varepsilon_1\varepsilon_1') = \sigma_1^2\mathbf{H}_1 \\ \mathbf{Y}_2 &= \mathbf{X}_2\boldsymbol{\beta}_2 + \mathbf{D}\boldsymbol{\beta}_1 + \varepsilon_2\,, \quad \mathsf{E}(\varepsilon_2) = \mathbf{0}\,, \quad \mathsf{E}(\varepsilon_2\varepsilon_2') = \sigma_2^2\mathbf{H}_2\,, \end{aligned}$$

$\varepsilon_1, \varepsilon_2$ uncorrelated. The vectors $\mathbf{Y}_1, \mathbf{Y}_2$ are normally distributed. The matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{D}$ of the dimensions $n \times k$, $m \times p$, $m \times k$, respectively, are known, and $\mathbf{X}_1, \mathbf{X}_2$ are of full rank in columns. The vector parameters $\boldsymbol{\beta}_1 \in \mathscr{R}^k$, $\boldsymbol{\beta}_2 \in \mathscr{R}^p$ and the scalar parameters σ_1^2, σ_2^2 are all unknown. We denote $\boldsymbol{\theta} = (\sigma_1^2, \sigma_2^2)'$. The parameter $\boldsymbol{\theta} \in \mathscr{R}^{2+}$. $\mathbf{H}_1, \mathbf{H}_2$ are nonsingular.

If the vector $\mathbf{Y}_1$ is consideted separately, there exists BLUE $\hat{\boldsymbol{\beta}}_1$ of the vector parameter $\boldsymbol{\beta}_1$, based on the vector of measurements $\mathbf{Y}_1$ in the form

$$\hat{\boldsymbol{\beta}}_1 = \mathbf{QY}_1\,, \quad \mathbf{QX}_1 = \mathbf{I}\,, \quad \mathbf{I} \text{ is the identity matrix}\,.$$

We transform the vector $\mathbf{Y}_2$ by $\hat{\boldsymbol{\beta}}_1$ in the following way:

$$\mathbf{Y}_2^* = \mathbf{Y}_2 - \mathbf{DQY}_1 = \mathbf{Y}_2 - \mathbf{D}\hat{\boldsymbol{\beta}}_1\,.$$

This transformation makes it possible to form a model

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2^* \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1\boldsymbol{\beta}_1 \\ \mathbf{X}_2\boldsymbol{\beta}_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ -\mathbf{DQ}\ \varepsilon_1 + \varepsilon_2 \end{pmatrix},$$

which can be written in the form

$$(2) \qquad \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2^* \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{DQ} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

The covariance matrix of the vector $(\mathbf{Y}_1', \mathbf{Y}_2^{*\prime})'$ is

$$(3) \quad \mathbf{V}_\theta = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{DQ} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \sigma_1^2\mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & \sigma_2^2\mathbf{H}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{Q}'\mathbf{D}' \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \sigma_1^2 \begin{pmatrix} \mathbf{H}_1 & -\mathbf{C}' \\ -\mathbf{C} & \mathbf{CH}_1^{-1}\mathbf{C}' + \varrho\mathbf{H}_2 \end{pmatrix}$$

where

$$\mathbf{C} = \mathbf{DQH}_1\,, \quad \varrho = \frac{\sigma_2^2}{\sigma_1^2}\,.$$

In the case that the parameters σ_1^2, σ_2^2 are unknown but the ratio $\varrho = \sigma_2^2/\sigma_1^2$ is known, the model (2) represents a usual linear model and for BLUE for the vector parameter $(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ based on the vector of measurements $(\mathbf{Y}_1', \mathbf{Y}_2^{*\prime})'$ the results of the linear theory of estimation can be used.

The matrix $\mathbf{V}_\theta$ can be expressed also in the form

$$(4) \qquad \mathbf{V}_\theta = \sigma_1^2 \begin{pmatrix} \mathbf{H}_1 & -\mathbf{C}' \\ -\mathbf{C} & \mathbf{CH}_1^{-1}\mathbf{C}' \end{pmatrix} + \sigma_2^2 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix}.$$

In the case that none of $\sigma_1^2, \sigma_2^2\ \varrho$, is known, $\sigma_1^2 \neq \sigma_2^2$ the model (2) represents the mixed linear model (see [1]).

1. ESTIMATION OF $(\beta_1', \beta_2')'$

Following [1] we immediately get the locally best linear unbiased estimator (LBLUE) $(\tilde{\boldsymbol{\beta}}_1', \tilde{\boldsymbol{\beta}}_2')'$ for the vector $(\boldsymbol{\beta}_1'. \boldsymbol{\beta}_2')'$ in model (2). It is given by the formula

$$(5) \qquad \begin{pmatrix} \tilde{\boldsymbol{\beta}}_1 \\ \tilde{\boldsymbol{\beta}}_2 \end{pmatrix} = (\mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{X})^{-1}\, \mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{Y}$$

where

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2^* \end{pmatrix}.$$

Using the expression for the inverse of a partitioned matrix (see [4]) we calculate expression (5).

$$(6) \qquad \mathbf{V}_\theta^{-1} = \sigma_1^{-2} \begin{pmatrix} \mathbf{H}_1 & -\mathbf{C}' \\ -\mathbf{C} & \mathbf{CH}_1^{-1}\mathbf{C}' + \varrho\mathbf{H}_2 \end{pmatrix}^{-1} =$$

$$= \frac{\sigma_1^{-2}}{\varrho} \begin{pmatrix} \varrho \mathbf{H}_1^{-1} + \mathbf{H}_1^{-1}\mathbf{C}'\mathbf{H}_2^{-1}\mathbf{C}\mathbf{H}_1^{-1} & \mathbf{H}_1^{-1}\mathbf{C}'\mathbf{H}_2^{-1} \\ \mathbf{H}_2^{-1}\mathbf{C}\mathbf{H}_1^{-1} & \mathbf{H}_2^{-1} \end{pmatrix} =$$

$$= \frac{1}{\varrho\sigma_1^2} \begin{pmatrix} \varrho \mathbf{H}_1^{-1} + \mathbf{Q}'\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{Q} & \mathbf{Q}'\mathbf{D}'\mathbf{H}_2^{-1} \\ \mathbf{H}_2^{-1}\mathbf{D}\mathbf{Q} & \mathbf{H}_2^{-1} \end{pmatrix}.$$

Then

$$(7) \quad \mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{X} = \frac{1}{\varrho\sigma_1^2} \begin{pmatrix} \mathbf{X}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \varrho \mathbf{H}_1^{-1} + \mathbf{Q}'\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{Q} & \mathbf{Q}'\mathbf{D}'\mathbf{H}_2^{-1} \\ \mathbf{H}_2^{-1}\mathbf{D}\mathbf{Q} & \mathbf{H}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{pmatrix} =$$

$$= \frac{1}{\varrho\sigma_1^2} \begin{pmatrix} \varrho \mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1 + \mathbf{D}'\mathbf{H}_2^{-1}\mathbf{D} & \mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D} & \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{X}_2 \end{pmatrix}.$$

The inverse of $\mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{X}$ is

$$(8) \quad (\mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{X})^{-1} = \varrho\sigma_1^2 \begin{pmatrix} \mathbf{V}_\varrho^- + \mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2\mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{V}_\varrho^- & -\mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2\mathbf{E}^- \\ -\mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{V}_\varrho^- & \mathbf{E}^- \end{pmatrix}$$

where

$$(9) \qquad \mathbf{E} = \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{X}_2 - \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2\,,$$

$$(10) \qquad \mathbf{V}_\varrho = \varrho\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1 + \mathbf{D}'\mathbf{H}_2^{-1}\mathbf{D}\,.$$

We express now the matrix expression $\mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{Y}$:

$$(11) \quad \mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{Y} = \frac{1}{\varrho\sigma_1^2} \begin{pmatrix} \mathbf{X}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \varrho \mathbf{H}_1^{-1} + \mathbf{Q}'\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{Q} & \mathbf{Q}'\mathbf{D}'\mathbf{H}_2^{-1} \\ \mathbf{H}_2^{-1}\mathbf{D}\mathbf{Q} & \mathbf{H}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2^* \end{pmatrix} =$$

$$= \frac{1}{\varrho\sigma_1^2} \begin{pmatrix} \mathbf{V}_\varrho\hat{\boldsymbol{\beta}}_1 + \mathbf{D}'\mathbf{H}_2^{-1}\mathbf{Y}_2^* \\ \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\hat{\boldsymbol{\beta}}_1 + \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{Y}_2^* \end{pmatrix}.$$

Finally, we have

$$(12) \qquad (\mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}_\theta^{-1}\mathbf{Y} =$$

$$= \begin{pmatrix} (\mathbf{V}_\varrho^- + \mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2\mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{V}_\varrho^-)(\mathbf{V}_\varrho\hat{\boldsymbol{\beta}}_1 + \mathbf{D}'\mathbf{H}_2^{-1}\mathbf{Y}_2^*) - \\ - \mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2\mathbf{E}^-(\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\hat{\boldsymbol{\beta}}_1 + \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{Y}_2^*) \\ -\mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{V}_\varrho^-(\mathbf{V}_\varrho\hat{\boldsymbol{\beta}}_1 + \mathbf{D}'\mathbf{H}_2^{-1}\mathbf{Y}_2^*) + \mathbf{E}^-(\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\hat{\boldsymbol{\beta}}_1 + \mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{Y}_2^*) \end{pmatrix}.$$

We can now state Theorem 1.1.

Theorem 1.1. LBLUE *of the vector* $(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ *in the model* (2) *is given by the formulas*

$$(13) \quad \tilde{\boldsymbol{\beta}}_1 = \mathbf{V}_\varrho^-\mathbf{V}_\varrho\hat{\boldsymbol{\beta}}_1 + \mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{Y}_2^* + \mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{X}_2\mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}(\mathbf{D}\mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1} - \mathbf{I})\,\mathbf{Y}_2^*\,,$$

$$\tilde{\boldsymbol{\beta}}_2 = \mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{Y}_2^* - \mathbf{E}^-\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{D}\mathbf{V}_\varrho^-\mathbf{D}'\mathbf{H}_2^{-1}\mathbf{Y}_2^*\,,$$

where $\mathbf{E}$ *and* $\mathbf{V}_\varrho$ *are given by* (9) *and* (10), *respectively.*

The proof of Theorem follows immediately from the expression (12).

Kleffe in [1] states the necessary and sufficient condition for the mixed linear model under which the uniformly best linear unbiased estimator (UBLUE) for the vector parameter of mean exists. We need the following notation.

$$\boldsymbol{V}_0 = \begin{pmatrix} \boldsymbol{H}_1 & -\boldsymbol{C}' \\ -\boldsymbol{C} & \boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}' + \boldsymbol{H}_2 \end{pmatrix}, \quad \text{i.e.} \quad \boldsymbol{V}_0 = \boldsymbol{V}_{\boldsymbol{\theta}} \quad \text{for} \quad \boldsymbol{\theta} = (1, 1)' .$$

Let the matrix $\boldsymbol{M}$ represent the projection operator onto the orthogonal complement of the space generated by the columns of the matrix $\boldsymbol{X}$, i.e.

$$\boldsymbol{M} = \boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^+ = \begin{pmatrix} \boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+ \end{pmatrix}.$$

The matrix $\boldsymbol{X}^+$ is the Moore-Penrose inverse of the matrix $\boldsymbol{X}$.

According to Lemma 2.2 in [1] the necessary and sufficient condition for the existence of UBLUE of $(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ is $\boldsymbol{M}\boldsymbol{V}_{\boldsymbol{\theta}}\boldsymbol{V}_0^{-1}\boldsymbol{X} = \boldsymbol{0}\ \forall \boldsymbol{V}_{\boldsymbol{\theta}}$. We check it in our case.

$$\boldsymbol{M}\boldsymbol{V}_{\boldsymbol{\theta}} = \sigma_1^2 \begin{pmatrix} (\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)\,\boldsymbol{H}_1 & -(\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)\,\boldsymbol{C}' \\ -(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{C} & (\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)(\boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}' + \varrho\boldsymbol{H}_2) \end{pmatrix}.$$

Further,

$$\boldsymbol{M}\boldsymbol{V}_{\boldsymbol{\theta}}\boldsymbol{V}_0^{-1} = \sigma_1^2 \begin{pmatrix} \boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+ & \boldsymbol{0} \\ -(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{D}\boldsymbol{Q} + \varrho(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{D}\boldsymbol{Q} & \varrho(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+) \end{pmatrix}$$

and finally,

$$\boldsymbol{M}\boldsymbol{V}_{\boldsymbol{\theta}}\boldsymbol{V}_0^{-1}\boldsymbol{X} = \sigma_1^2 \begin{pmatrix} (\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)\,\boldsymbol{X}_1 & \boldsymbol{0} \\ (\varrho - 1)(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{D} & \varrho(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{X}_2 \end{pmatrix} = \tag{14}$$

$$= \sigma_1^2 \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ (\varrho - 1)(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{D} & \boldsymbol{0} \end{pmatrix}.$$

Theorem 1.2. UBLUE *of the vector parameter* $(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ *under the model* (2) *exists if and only if* $\mathscr{R}(\boldsymbol{D}) \subset \mathscr{R}(\boldsymbol{X}_2)$, *i.e. the space generated by the columns of the matrix* $\boldsymbol{D}$ *is included in the space generated by the columns of the matrix* $\boldsymbol{X}_2$. *In case this condition is met,* LBLUE *is* UBLUE.

Proof. According to the properties of the g-inverse of a matrix (see [4]), and the necessary and sufficient condition given by [1], the matrix from the expression (14) is equal to zero-matrix if and only if $(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\,\boldsymbol{D} = \boldsymbol{0}$, and this immediately yields the statement of Theorem.

Remark. It is interesting to study the arrangement of the experiment and the conditions under which the assumptions stated in Theorem 1.2 are valid. Let us consider the etalon network (see [3]) arranged in the following way. Let the value of the basic etalon E be known. Let the values of the etalons $\beta_1^1, \beta_2^1, \ldots, \beta_k^1$ be unknown and let it be possible to measure the differences between the talons β_i^1 and E, and between the etalons β_i^1 and β_j^1. These are the etalons of the first stage. Let the etalons of the second stage be $\beta_1^2, \ldots, \beta_p^2$, which are to be derived from the etalons of the first stage, i.e., the differences between β_j^2 and β_i^1, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, p$ can be measured, as well as the differences between β_j^2 and β_l^2. It can be shown that the only way how to arrange this type of measurements so as to fulfil the necessary and sufficient conditions from Theorem 1.2 is the following. The only admissible measurements are: suppose the difference between β_j^2 and $\beta_{i_0}^1$ is measured. Further suppose there is a measurement of difference between β_l^2 $l \neq j$ and $\beta_{i_0}^1$. Then the measurement of difference between β_l^2 and β_j^2 is allowed, but if $i \neq i_0$ the measurement of difference between β_l^2 and β_i^1 is not admissible. Fig. 1 shows the situation described. These considerations follow from the theory of graphs (see [2]).

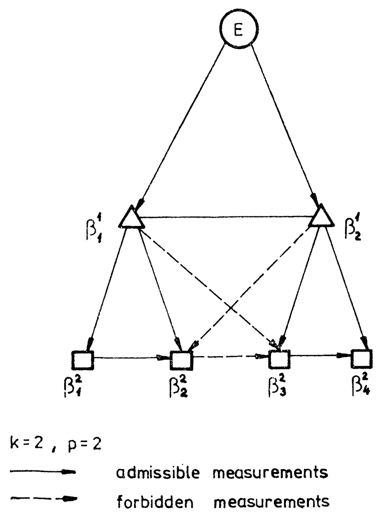

Fig. 1

2. ESTIMATION OF σ_1^2, σ_2^2

As we have mentioned, the model (2) forms a mixed linear model, with unknown parameters $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \sigma_1^2$ and σ_2^2, $\sigma_1^2 \neq \sigma_2^2$. We now find the conditions under which the parameters σ_1^2, σ_2^2 are unbiasedly estimable, and we find the "optimal" estimators for them. Due to the normality assumption it is enough to check whether the conditions for the existence of MINQE(U, I) for σ_1^2, σ_2^2, developed by Rao in [5], are fulfilled (see [6]). We turn our attention to the estimators which are unbiased. The statistic $\mathbf{Y'AY}$, where $\mathbf{A}$ is a symmetric $(n + m) \times (n + m)$ matrix, is unbiased for the parametric function $f_1\sigma_1^2 + f_2\sigma_2^2$, $(f_1, f_2)' \in \mathscr{R}^2$, if and only if

$$\mathsf{E}_{\beta,\theta}(\mathbf{Y'AY}) = f_1\sigma_1^2 + f_2\sigma_2^2 \quad \forall \boldsymbol{\beta} \in \mathscr{R}^{k+p}, \quad \forall \boldsymbol{\theta} \in \mathscr{R}^{2+} .$$

It is said to be invariant, if and only if

$$\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} = (\boldsymbol{Y} + \boldsymbol{X}\boldsymbol{\beta})' \boldsymbol{A}(\boldsymbol{Y} + \boldsymbol{X}\boldsymbol{\beta}) \quad \forall \boldsymbol{\beta} \in \mathscr{R}^{k+p} .$$

As shown in [6], MINQE (U, I) (minimum norm quadratic unbiased invariant estimator) is the locally best unbiased estimator for $f_1\sigma_1^2 + f_2\sigma_2^2$, $(f_1, f_2)' \in \mathscr{R}^2$.

Lemma 2.1. (See [6]). *A necessary and sufficient condition for* $f_1\sigma_1^2 + f_2\sigma_2^2$ *to be* MINQE (U, I)-*estimable is that the vector* $(f_1, f_2)'$ *belongs to the column space of the matrix* $\mathscr{A}$,

$$\mathscr{A} = (a_{ij}), \quad a_{ij} = \operatorname{tr}(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_i(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_j \quad i, j = 1, 2 ,$$
$$\boldsymbol{V}_{\theta_0} = \sigma_{1_0}^2 \boldsymbol{V}_1 + \sigma_{2_0}^2 \boldsymbol{V}_2 , \quad \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^+ .$$

We check this condition in our case. Our considerations will imply the condition $\mathscr{R}(\boldsymbol{D}) \subset \mathscr{R}(\boldsymbol{X}_2)$. First we check MINQE(U, I)-estimability of the parameter σ_1^2, i.e. $f_1 = 1$ $f_2 = 0$.

We have

$$\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M} =$$
$$= \sigma_{1_0}^2 \begin{pmatrix} \boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+ \end{pmatrix} \begin{pmatrix} \boldsymbol{H}_1 & -\boldsymbol{C}' \\ -\boldsymbol{C} & \boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}' + \varrho_0 \boldsymbol{H}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+ \end{pmatrix} =$$
$$= \sigma_{1_0}^2 \begin{pmatrix} (\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)\boldsymbol{H}_1(\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+) & -(\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)\boldsymbol{C}'(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+) \\ -(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+) & (\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)(\boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}' + \varrho_0\boldsymbol{H}_2)(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+) \end{pmatrix} .$$

For $\mathscr{R}(\boldsymbol{D}) \subset \mathscr{R}(\boldsymbol{X}_2)$ the matrix $\boldsymbol{C}'(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)$ as well as its transpose vanish, i.e. the Moore-Penrose inverse of $\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M}$ is

$$(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ = \frac{1}{\sigma_{1_0}^2} \begin{pmatrix} [(\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)\boldsymbol{H}_2(\boldsymbol{I} - \boldsymbol{X}_1\boldsymbol{X}_1^+)]^+ & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{\varrho_0}[(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)\boldsymbol{H}_2(\boldsymbol{I} - \boldsymbol{X}_2\boldsymbol{X}_2^+)]^+ \end{pmatrix} .$$

After some technical calculations we get the entries of the matrix $\mathscr{A}$ in the form

(15) $$a_{11} = \operatorname{tr}(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_1(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_1 =$$
$$= \frac{1}{\sigma_{1_0}^4} [\operatorname{tr}(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1 + 2\operatorname{tr}\frac{1}{\varrho_0}(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{C}'(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{C} +$$
$$+ \frac{1}{\varrho_0^2}\operatorname{tr}(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}'(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}'] .$$

Because of the identity $(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ = \boldsymbol{H}_2^{-1}(\boldsymbol{I} - \boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{H}_2^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{H}_2^{-1})$ we have

$$(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{C} = (\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{D}\boldsymbol{Q}\boldsymbol{H}_1 =$$
$$= \boldsymbol{H}_2^{-1}(\boldsymbol{I} - \boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{H}_2^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{H}_2^{-1})\boldsymbol{D}\boldsymbol{Q}\boldsymbol{H}_1 = \boldsymbol{0} .$$

Then we get

(16) $$a_{11} = \frac{1}{\sigma_{1_0}^4}\operatorname{tr}(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1 ,$$

(17)
$$a_{12} = a'_{21} = \mathrm{tr}(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_1(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_2 =$$
$$= \frac{1}{\sigma_{1_0}^4 \varrho_0^2} \mathrm{tr}(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}' (\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2 = \boldsymbol{0}\,,$$

and finally

(18)
$$a_{22} = \mathrm{tr}(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_2(\boldsymbol{M}\boldsymbol{V}_{\theta_0}\boldsymbol{M})^+ \boldsymbol{V}_2 =$$
$$= \frac{1}{\sigma_{1_0}^4 \varrho_0^2} \mathrm{tr}(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2\,.$$

The matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ are the projection matrices onto the orthogonal complements of the spaces generated by the columns of the matrices $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, respectively.

Then the criterion matrix for the existence of MINQE(U, I) is

(19)
$$\mathscr{A} = \frac{1}{\sigma_{1_0}^4}\begin{pmatrix} \mathrm{tr}(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{\varrho_0^2}\mathrm{tr}(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2 \end{pmatrix}.$$

$\mathscr{A}$ is a diagonal matrix and it is obvious that both the parameters σ_1^2 and σ_2^2 are MINQE(U, I)-estimable.

Now we consider the modified second stage, i.e. $\boldsymbol{Y}_2^* = \boldsymbol{X}_2\boldsymbol{\beta}_2 - \boldsymbol{D}\boldsymbol{Q}\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2$. The covariance matrix of $\boldsymbol{Y}_2^*$ is $\boldsymbol{W}_{\theta} = \sigma_1^2\boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}' + \sigma_2^2\boldsymbol{H}_2$. The criterion matrix for the MINQE(U, I)-estimability in the modified model is

$$\mathscr{B} = (b_{ij})\ b_{ij} = \mathrm{tr}(\boldsymbol{M}_2\boldsymbol{W}_{\theta_0}\boldsymbol{M}_2)^+ \boldsymbol{W}_i(\boldsymbol{M}_2\boldsymbol{W}_{\theta_0}\boldsymbol{M}_2)^+ \boldsymbol{W}_j \quad i = 1, 2\,, \quad j = 1, 2\,,$$

where $\boldsymbol{W}_1 = \boldsymbol{C}\boldsymbol{H}_1^{-1}\boldsymbol{C}'$ $\boldsymbol{W}_2 = \boldsymbol{H}_2$. Under the assumption $\mathscr{R}(\boldsymbol{D}) \subset \mathscr{R}(\boldsymbol{X}_2)$ the matrix $\mathscr{B}$ can be expressed in the form

(20)
$$\mathscr{B} = \frac{1}{\sigma_{1_0}^4\varrho_0^2}\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{tr}(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2 \end{pmatrix}.$$

In this case the parameter σ_1^2 is not MINQE(U, I)-estimable.

Theorem 2.1. *Under the model* (2) *and the condition* $\mathscr{R}(\boldsymbol{D}) \subset \mathscr{R}(\boldsymbol{X}_2)$ *the uniformly best unbiased invariant estimator for the parameter* σ_1^2 *is*

(21)
$$\hat{\sigma}_1^2 = \frac{1}{\mathrm{tr}(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1} \boldsymbol{Y}_1'(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{H}_1(\boldsymbol{M}_1\boldsymbol{H}_1\boldsymbol{M}_1)^+ \boldsymbol{Y}_1$$

and for the parameter σ_2^2,

(22)
$$\hat{\sigma}_2^2 = \frac{1}{\mathrm{tr}(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2} \boldsymbol{Y}_2'(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{H}_2(\boldsymbol{M}_2\boldsymbol{H}_2\boldsymbol{M}_2)^+ \boldsymbol{Y}_2\,,$$

which coincides with the uniformly best unbiased invariant estimator for σ_2^2 *under the modified second stage model.*

The proof follows from the expressions for MINQE(U, I) (see [6]).

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Súhrn

ODHAD PARAMETROV STREDNEJ HODNOTY A DISPERZIE V DVOJETAPOVÝCH LINEÁRNYCH MODELOCH

JÚLIA VOLAUFOVÁ

Dvojetapový lineárny model je charakterizovaný náhodnými vektormi $\mathbf{Y}_1$, $\mathbf{Y}_2$ nasledovne:

$$\mathbf{Y}_1 = \mathbf{X}_1 \boldsymbol{\beta}_1 + \varepsilon_1 \qquad \mathsf{E}(\varepsilon_1) = \mathbf{0} \quad \mathsf{E}(\varepsilon_1 \varepsilon_1') = \sigma_1^2 \mathbf{H}_1$$
$$\mathbf{Y}_2 = \mathbf{X}_2 \boldsymbol{\beta}_2 + \mathbf{D} \boldsymbol{\beta}_1 + \varepsilon_2 \quad \mathsf{E}(\varepsilon_2) = \mathbf{0} \quad \mathsf{E}(\varepsilon_2 \varepsilon_2') = \sigma_2^2 \mathbf{H}_2 \,,$$

$\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ nekorelované. Neznáme sú vektorové parametre $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ a skalárne parametre σ_1^2, σ_2^2. V práci je uvedená nutná a postačujúca podmienka pre existenciu rovnomerne najlepšieho nevychýleného odhadu pre parametre $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$. Uvedený je najlepší nevychýlený odhad pre parametre $\boldsymbol{\sigma}_1^2$, $\boldsymbol{\sigma}_2^2$.

Резюме

ОЦЕНИВАНИЕ ПАРАМЕТРОВ СРЕДНЕГО И ДИСПЕРСИИ В ДВУХЭТАПНОЙ ЛИНЕЙНОЙ МОДЕЛИ

JÚLIA VOLAUFOVÁ

В статье указано необходимое и достаточное условие для существования равномерно наилучших несмещенных оценок неизвестных параметров среднего. Выведены формулы для вычисления этих оценок. Указаны также наилучшие несмещенные оценки для параметров дисперсии.

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