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SHAPE OPTIMIZATION OF ELASTO-PLASTIC BODIES OBEYING HENCKY'S LAW

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Abstract. A minimization of a cost functional with respect to a part of the boundary, where the body is fixed, is considered. The criterion is defined by an integral of a yield function. The principle of Haar-Kármán and piecewise constant stress approximations are used to solve the state problem. A convergence result and the existence of an optimal boundary is proved.

Keywords: domain optimization, variational inequality, elasto-plastic bodies obeying Hencky's law.

AMS Subject class.: 65K10, 65N30, 73E99.

INTRODUCTION

It is the aim of the present paper to solve the following optimal design problem. Given body forces, surface loads and material characteristics of an elasto-plastic two-dimensional body, find the shape of a part of its boundary such that a cost functional is minimized. The cost functional is an integral of the yield function and zero displacements are prescribed on the unknown part of the boundary.

One of the simplest mathematical models describing the elasto-plastic behaviour of solid bodies is given by the constituent law of Hencky. The classical boundary value problems allow a variational formulation in terms of stresses, known by the name of Haar-Kármán principle. In the papers by Mercier [6] and Falk [4], [3], approximate solutions of two-dimensional problems have been studied, which consist of piecewise constant stress fields. Using the latter finite element model and piecewise linear approximations of the unknown boundary, we define some discrete optimization problem.

The main result of the paper is the convergence analysis of the solutions of discrete problems to a solution of the original continuous optimization problem.

1. FORMULATION OF THE OPTIMIZATION PROBLEM

First let us recall some basic relations of the elasto-plastic bodies obeying the law of Hencky.

Let $\Omega \subset \mathbb{R}^2$ be a given bounded domain with Lipschitz boundary $\partial\Omega$. Assume that

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_g\,, \quad \Gamma_u \cap \Gamma_g = \emptyset\,,$$

each of the parts Γ_u and Γ_g being open in $\partial\Omega$.

Let $\mathbb{R}_\sigma$ be the space of symmetric 2×2 matrices (stress or strain tensors). A repeated index implies summation over the range 1, 2. Assume that a yield function $f\colon \mathbb{R}_\sigma \to \mathbb{R}$ is given, which is convex, Lipschitz and

(1) $$f(\lambda\sigma) = |\lambda|\, f(\sigma) \quad \forall\lambda \in \mathbb{R}\,, \quad \forall\sigma \in \mathbb{R}_\sigma\,.$$

These conditions are fulfilled e.g. by the von Mises function

$$f(\sigma) = (\sigma^D_{ij}\sigma^D_{ij} + \tfrac{1}{9}(\sigma_{kk})^2)^{1/2}\,,$$

where

$$\sigma^D_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}\,.$$

We introduce the following spaces and notations:

$$S(\Omega) = \{\tau\colon \Omega \to \mathbb{R}_\sigma \mid \tau_{ij} \in L^2(\Omega)\ \forall i, j\}\,,$$

$$\langle\sigma, e\rangle_\Omega = \int_\Omega \sigma_{ij}e_{ij}\,\mathrm{d}x\,, \quad \|\sigma\|_{0,\Omega} = \langle\sigma, \sigma\rangle_\Omega^{1/2}\,.$$

In the space $S(\Omega)$ we introduce also the energy scalar product

$$(\sigma, \tau)_\Omega = \langle b\sigma, \tau\rangle_\Omega\,, \quad \|\sigma\|_\Omega = (\sigma, \sigma)_\Omega^{1/2}\,,$$

where

$$b\colon S(\Omega) \to S(\Omega)$$

is the isomorphism defined by the generalized Hooke's law

$$e = b\sigma \Leftrightarrow e_{ij} = b_{ijkl}\sigma_{kl}\,.$$

We assume that positive constants b_0, b_1 exist such that

(2) $$b_0\|\sigma\|^2_{0,\Omega} \leqq \langle b\sigma, \sigma\rangle_\Omega \leqq b_1\|\sigma\|^2_{0,\Omega} \quad \forall\sigma \in S(\Omega)$$

and

$$\langle b\sigma, \tau\rangle_\Omega = \langle\sigma, b\tau\rangle_\Omega \quad \forall\sigma, \tau \in S(\Omega)\,.$$

Assume that a body force $\mathbf{F} \in [L^2(\Omega)]^2$ and a surface traction $\mathbf{g} \in [L^2(\Gamma_g)]^2$ are given.

We define the set of plastically admissible stress fields

$$P(\Omega) = \{\tau \in S(\Omega) \mid f(\tau) \leqq 1 \text{ a.e. in } \Omega\}$$

and the set of statically admissible stress fields

$$E(\Omega) = \{\tau \in S(\Omega) \mid \langle\tau, e(\mathbf{v})\rangle_\Omega = L_\Omega(\mathbf{v})\ \forall\mathbf{v} \in V(\Omega)\}\,,$$

where

$$V(\Omega) = \{\mathbf{v} \in [H^1(\Omega)]^2 \mid \mathbf{v} = 0 \text{ on } \Gamma_u\},$$

$$e(\mathbf{v})_{ij} = \tfrac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i),$$

$$L_\Omega(\mathbf{v}) = \int_\Omega F_i v_i \,\mathrm{d}x + \int_{\Gamma_g} g_i v_i \,\mathrm{d}s.$$

The *Haar-Kármán principle* says that the actual stress field minimizes the complementary energy

$$\mathscr{S}(\tau) = \tfrac{1}{2}\|\tau\|_\Omega^2$$

over the set $E(\Omega) \cap P(\Omega)$.

For the derivation of the principle – see [2] or [4], [3]. Note that the principle is equivalent to the following variational inequality: $\sigma \in E(\Omega) \cap P(\Omega)$,

(3) $$(\sigma, \tau - \sigma)_\Omega \geqq 0 \quad \forall \tau \in E(\Omega) \cap P(\Omega).$$

Passing to the shape optimization problem, we introduce the following set of admissible design variables

$$U_{ad} = \Big\{v \in C^{(0),1}([0, 1]) \quad \text{(i.e. Lipschitz functions)},$$

$$\alpha \leqq v \leqq \beta, \quad |\mathrm{d}v/\mathrm{d}x_2| \leqq C_1, \quad \int_0^1 v \,\mathrm{d}x_2 = C_2\Big\},$$

where α, β, C_1 and C_2 are given positive constants.

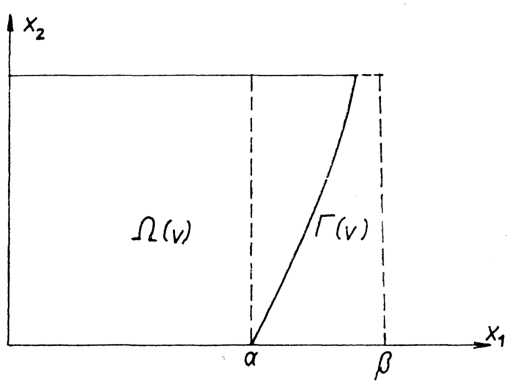

Fig. 1.

We shall consider a class of domains $\Omega = \Omega(v)$, (Fig. 1) where $v \in U_{ad}$ and

$$\Omega(v) = \{(x_1, x_2) \mid 0 < x_1 < v(x_2),\ 0 < x_2 < 1\}.$$

For any $v \in U_{ad}$, the graph $\Gamma(v)$ of v will coincide with the part Γ_u of $\partial\Omega(v)$.
The function v has to be determined from the *Optimization Problem*

(4) $$\mathscr{J}(\sigma(v)) = \min.$$

over the set of $v \in U_{ad}$, where

$$\mathscr{J}(\sigma(v)) = \int_{\Omega(v)} f^2(\sigma(v))\,\mathrm{d}x$$

and $\sigma(v)$ is the solution of the variational inequality (3) on the domain $\Omega \equiv \Omega(v)$.

In what follows we assume that $\boldsymbol{F} \in [L^2(\Omega_\delta)]^4$ and $\boldsymbol{F} \in [L^2(\partial\Omega_\delta - \Gamma_\delta)]^2$ are given, where $\Omega_\delta = (0, \delta) \times (0, 1)$, $\delta > \beta$ and

$$\Gamma_\delta = \{(x_1, x_2) |\, x_1 = \delta,\ 0 < x_2 < 1\}\,.$$

Moreover, *assume* that a tensor field $\sigma^0 \in E(\Omega_\delta)$ exists such that $\boldsymbol{x} \to \sigma^0(\boldsymbol{x})$ is a Lipschitz function in $\bar{\Omega}_\delta$ and

$$(1 + \varepsilon)\,\sigma^0 \in P(\Omega_\delta) \tag{5}$$

holds for some positive ε.

Note that (1) implies $f(0) = 0$ so that $0 \in P(\Omega_\delta)$ and $\sigma^0 \in P(\Omega_\delta)$ follows from (5), since $P(\Omega_\delta)$ is convex.

Remark 1.1. From the definition of $E(\Omega_\delta)$ we easily derive that

$$\operatorname{div} \sigma^0 + F = 0 \quad \text{in} \quad \Omega_\delta\,,$$
$$\sigma^0 \,.\, \nu = g \qquad \text{on} \quad \Omega_\delta - \Gamma_\delta\,.$$

Consequently, g is a Lipschitz function on any side of $\partial\Omega_\delta - \Gamma_\delta$.

Proposition 1.1. *The Haar-Kármán principle has a unique solution for any* $\Omega(v)$, $v \in U_{ad}$.

Proof. We can easily show that $P(\Omega(v)) \cap E(\Omega(v))$ is non-empty. In fact, the restriction of σ^0 onto $\Omega(v)$ belongs to this intersection, since the extension $\tilde{\boldsymbol{w}}$ of any $\boldsymbol{w} \in V(\Omega(v))$ by zero belongs to $V(\Omega_\delta)$ and we may write

$$\langle e(\boldsymbol{w}), \sigma^0 \rangle_{\Omega(v)} = \langle e(\tilde{\boldsymbol{w}}), \sigma^0 \rangle_{\Omega_\delta} = L_{\Omega_\delta}(\tilde{\boldsymbol{w}}) = L_{\Omega(v)}(\boldsymbol{w})\,.$$

The sets $E(\Omega(v))$ and $P(\Omega(v))$ are convex and closed in $S(\Omega(v))$, the functional $\mathscr{S}(\sigma)$ is quadratic, strictly convex. Hence the existence and uniqueness follow.

2. APPROXIMATIONS BY PIECEWISE CONSTANT STRESS FIELDS

Let N be a positive integer and $h = 1/N$. We denote by e_j, $j = 1, 2, \ldots, N$, the subintervals $[(j - 1)\,h, jh]$ and introduce the set

$$U_{ad}^h = \{v_h \in U_{ad} | v_{h|e_j} \in P_1(e_j)\ \forall j\}$$

where P_k denotes the set of polynomials of k-th degree. Let Ω_h denote the domain $\Omega(v_h)$, bounded by the graph Γ_h of $v_h \in U_{ad}^h$. The domain Ω_h will be carved into triangles as follows (see fig. 2).

We choose $\alpha_0 \in (0, \alpha)$ and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, independent of v_h if h is fixed.

In the remaining part $\Omega_h - \mathscr{R}$ let the nodal points divide the intervals $[\alpha_0, v_h(jh)]$ into M equal segments, where

$$M = 1 + [(\beta - \alpha_0) N]$$

and the square brackets denote the integer part.

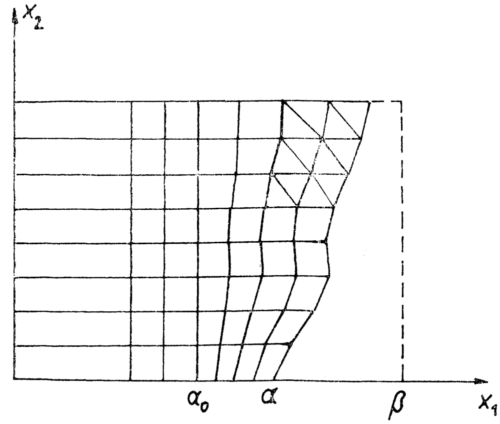

Fig. 2.

Thus we obtain a regular family $\{\mathscr{T}_h(v_h)\}$, $h \to 0$, $v_h \in U^h_{ad}$, of triangulations. Note that for any $v_h \in U^h_{ad}$ we construct a unique triangulation $\mathscr{T}_h(v_h)$. Denoting the triangles of $\mathscr{T}_h(v_h)$ by K, we define the finite element spaces

$$V_h(\Omega_h) = \{\mathbf{w}_h \in V(\Omega_h) |\ \mathbf{w}_{h|K} \in [P_1(K)]^2 \ \forall K \in \mathscr{T}_h(v_h)\}$$
$$H_h(\Omega_h) = \{\tau \in S(\Omega_h) \ |\tau|_K \in [P_0(K)]^4 \ \forall K \in \mathscr{T}_h(v_h)\}$$

and external approximation of the set $E(\Omega_h)$

$$E_h(\Omega_h) = \{\tau_h \in H_h(\Omega_h) |\ \langle \tau_h, e(\mathbf{w}_h)\rangle_{\Omega_h} = L_{\Omega_h}(\mathbf{w}_h) \ \forall \mathbf{w}_h \in V_h(\Omega_h)\} \, .$$

Instead of the problem (3) we introduce the following *approximate state problem*:

$$\text{find} \quad \sigma_h \in P(\Omega_h) \cap E_h(\Omega_h) \quad \text{such that}$$
$$(6) \qquad (\sigma_h, \tau_h - \sigma_h)_{\Omega_h} \geqq 0 \quad \forall \tau_h \in P(\Omega_h) \cap E_h(\Omega_h) \, .$$

Lemma 2.1. *The problem (6) has a unique solution for any $v_h \in U^h_{ad}$, $h > 0$.*

Proof. Like in the proof of Proposition 1.1 we can show that

$$\sigma^0|_{\Omega_h} \in P(\Omega_h) \cap E(\Omega_h) \, .$$

For any polygonal domain Ω and its triangulation $\mathscr{T}_h$ we define a projection mapping

$$r_h \colon S(\Omega) \to H_h(\Omega)$$

by means of the relation

(7) $$\langle \tau - r_h\tau, \sigma_h \rangle_\Omega = 0 \quad \forall \sigma_h \in H_h(\Omega) .$$

Henceforth we shall write

$$\sigma_0|_{\Omega_h} \equiv \sigma_0$$

and prove that $r_h\sigma^0 \in E_h(\Omega_h)$. In fact, given a $\mathbf{w}_h \in V_h(\Omega_h)$ we have $e(\mathbf{w}_h) \in H_h(\Omega_h)$ so that

$$\langle e(\mathbf{w}_h), r_h\sigma^0 \rangle_{\Omega_h} = \langle e(\mathbf{w}_h), \sigma^0 \rangle_{\Omega_h} = L_{\Omega_h}(\mathbf{w}_h) ,$$

since $w_h \in V(\Omega_h)$.

It is readily seen that

$$r_h\sigma^0 = (\text{mes } K)^{-1} \int_K \sigma^0 \, dx \quad \forall K \in \mathscr{T}_h(v_h) .$$

Consequently,

$$f(r_h\sigma^0) \leqq 1 \quad \text{in} \quad \Omega_h$$

and $r_h\sigma^0 \in P(\Omega_h)$ follows.

The set $P(\Omega_h) \cap E_h(\Omega_h)$ is therefore non-empty. Since it is also convex and closed in $S(\Omega_h)$, we obtain the unique solvability of (6).

Lemma 2.2. *Let $\{v_h\}$, $h \to 0$, be a sequence of $v^h \in U_{ad}^h$, $v_h \to v$ in $[0, 1]$ uniformly. Then*

(8) $$\tilde{\sigma}_h \to \sigma(v) \quad in \quad [L^2(\Omega_\delta)]^4 ,$$

where $\tilde{\sigma}_h$ is the solution of the approximate state problem (6) extended by zero to $\Omega_\delta - \Omega_h$ and $\sigma(v)$ is the solution of the problem (3), extended by zero to $\Omega_\delta - \Omega(v)$.

Proof. 1° The sequence $\{\tilde{\sigma}_h\}$ is bounded in $S(\Omega_\delta)$. In fact, we may insert

$$\tau_h = r_h\sigma^0 \in P(\Omega_h) \cap E_h(\Omega_h)$$

into the inequality (6) to obtain

$$\|\sigma_h\|_{\Omega_h}^2 \leqq (\sigma_h, r_h\sigma^0)_{\Omega_h} \leqq \|\sigma_h\|_{\Omega_h} \|r_h\sigma^0\|_{\Omega_h} .$$

Cancelling and using the inequalities (2) and (7), we may write

$$b_0^{1/2}\|\sigma_h\|_{0,\Omega_h} \leqq \|\sigma_h\|_{\Omega_h} \leqq \|r_h\sigma^0\|_{\Omega_h} \leqq$$
$$\leqq b_1^{1/2}\|r_h\sigma^0\|_{0,\Omega_h} \leqq b_1^{1/2}\|\sigma^0\|_{0,\Omega_h} \leqq b_1^{1/2}\|\sigma^0\|_{0,\Omega_\delta} .$$

Consequently,

(9) $$\|\tilde{\sigma}_h\|_{0,\Omega_\delta} \leqq C \quad \forall h$$

and there exists a subsequence (denote it by the same symbol) such that

(10) $$\tilde{\sigma}_h \to \sigma \text{ (weakly) in } S(\Omega_\delta) , \quad \sigma \in S(\Omega_\delta) .$$

2° We can show that

$$\sigma|_{\Omega(v)} \in E(\Omega(v)) \cap P(\Omega(v)) . \tag{11}$$

Let $\mathbf{w} \in V(\Omega(v))$ be given. Let us construct its extension $\tilde{\mathbf{w}}$ by zero into $\Omega_\delta - \Omega(v)$. There exists a sequence $\{\mathbf{w}_\varkappa\}$, $\varkappa \to 0$, such that

$$\mathbf{w}_\varkappa \in [C^\infty(\bar{\Omega}_\delta)]^2 , \quad \mathbf{w}_\varkappa = 0 \quad \text{in} \quad \bar{\Omega}_\delta - \Omega(v) ,$$

$$\operatorname{supp} \mathbf{w}_{\varkappa i} \cap \Gamma(v) = \emptyset , \quad i = 1, 2 ,$$

$$\mathbf{w}_\varkappa \to \tilde{\mathbf{w}} \quad \text{in} \quad H^1(\Omega_\delta) . \tag{12}$$

There exists $h_0(\varkappa)$ such that $\mathbf{w}_\varkappa = 0$ on Γ_h if $h < h_0(\varkappa)$, so that

$$\mathbf{w}_\varkappa|_{\Omega_h} \in V(\Omega_h) \quad \forall h < h_0(\varkappa) .$$

Let us construct the standard interpolation $\pi_h \mathbf{w}_\varkappa \in V_h(\Omega_h)$ and denote its extension by zero to $\Omega_\delta - \Omega_h$ by the same symbol.

For any σ_h we have

$$\langle \sigma_h, e(\pi_h \mathbf{w}_\varkappa) \rangle_{\Omega_h} = L_{\Omega_h}(\pi_h \mathbf{w}_\varkappa) ,$$

which can be rewritten as follows

$$\langle \tilde{\sigma}_h, e(\pi_h \mathbf{w}_\varkappa) \rangle_{\Omega_\delta} = L_{\Omega_\delta}(\pi_h \mathbf{w}_\varkappa) . \tag{13}$$

Note that

$$\pi_h \mathbf{w}_\varkappa \to \mathbf{w}_\varkappa \quad \text{for} \quad h \to 0 \quad \text{in} \quad H^1(\Omega_\delta)$$

and therefore

$$e(\pi_h \mathbf{w}_\varkappa) \to e(\mathbf{w}_\varkappa) \quad \text{in} \quad S(\Omega_\delta) .$$

Passing to the limit with $h \to 0$ in (13) and using (10), we thus obtain

$$\langle \sigma, e(\mathbf{w}_\varkappa) \rangle_{\Omega_\delta} = L_{\Omega_\delta}(\mathbf{w}_\varkappa) .$$

Passing to the limit with $\varkappa \to 0$ and using (12), we arrive at

$$\langle \sigma, e(\mathbf{w}) \rangle_{\Omega(v)} = \langle \sigma, e(\tilde{\mathbf{w}}) \rangle_{\Omega_\delta} = L_{\Omega_\delta}(\tilde{\mathbf{w}}) = L_{\Omega(v)}(\mathbf{w}) ,$$

so that $\sigma|_{\Omega(v)} \in E(\Omega(v))$.

Since $P(\Omega_\delta)$ is closed and convex in $S(\Omega)$, it is weakly closed. Any $\tilde{\sigma}_h$ belongs to $P(\Omega_\delta)$ and hence the weak limit $\sigma \in P(\Omega_\delta)$, as well. Therefore $\sigma|_{\Omega(v)} \in P(\Omega(v))$.

3° We show that

$$\sigma = 0 \quad \text{a.e.} \quad \text{in} \quad \Omega_\delta - \Omega(v) . \tag{14}$$

Let $\sigma \neq 0$ on a set $M \subset \Omega_\delta - \Omega(v)$, mes $M > 0$. Introducing the characteristic function χ_M of M, we obtain

$$\langle \tilde{\sigma}_h, \chi_M \sigma \rangle_{\Omega_\delta} \to \langle \sigma, \chi_M \sigma \rangle_{\Omega_\delta} = \|\sigma\|_{0,M}^2 > 0 .$$

On the other hand,

$$\langle \tilde{\sigma}_h, \chi_M \sigma \rangle_{\Omega_\delta} = \langle \tilde{\sigma}_h, \sigma \rangle_{\Omega_h \cap M} \leqq \|\tilde{\sigma}_h\|_{0,\Omega_\delta} \|\sigma\|_{0,\Omega_h \cap M} \to 0$$

by virtue of (9) and mes $(\Omega_h \cap M) \to 0$.

Consequently, we arrive at a contradiction.

4° We show that the restriction $\sigma|_{\Omega(v)}$ solves the state problem (3).

Let a $\tau \in E(\Omega(v) \cap P(\Omega(v))$ be given. We consider a "shifted" domain $\Omega(v + \lambda) \equiv \equiv \Omega_\lambda$, where λ is a positive constant and construct a function

$$\tau^\lambda \in E(\Omega_\lambda) \cap P(\Omega_\lambda) \tag{15}$$

which tends to τ in $S(\Omega(v))$ for $\lambda \to 0$, as follows.

We define the function

$$\gamma(\lambda) = \frac{1 - \sqrt{(\lambda)/\varepsilon}}{1 + \sqrt{\lambda}} \tag{16}$$

(see (5), where ε has been introduced), and the extension $\tilde{\omega}$ of $\omega = \tau - \sigma^0$ by zero to the negative half-plane $x_1 < 0$.

Let us define

$$\begin{aligned} \omega^\lambda(x_1, x_2) &= \tilde{\omega}(x_1 - \lambda, x_2), \quad \mathbf{x} \in \Omega_\lambda, \\ \tau^\lambda &= \sigma_0 + \gamma(\lambda)\, \omega^\lambda. \end{aligned} \tag{17}$$

To prove that $\tau^\lambda \in E(\Omega_\lambda)$, it suffices to show that

$$\langle \omega^\lambda, e(\mathbf{w}) \rangle_{\Omega_\lambda} = 0 \quad \forall \mathbf{w} \in V(\Omega_\lambda),$$

since

$$\sigma^0 \in E(\Omega_\lambda)$$

follows by the argument used in proving Proposition 1.1.

We may use the transformation of coordinates

$$\begin{aligned} x_1 - \lambda = y_1, \quad x_2 = y_2, \\ \mathbf{w}(\mathbf{x}) = \mathbf{w}(y_1 + \lambda, y_2) = \hat{\mathbf{w}}(\mathbf{y}) \end{aligned} \tag{18}$$

and write

$$\langle \omega^\lambda, e(\mathbf{w}) \rangle_{\Omega_\lambda} = \int_{\Omega_\lambda} \tilde{\omega}(x_1 - \lambda, x_2)\, e(\mathbf{w}(\mathbf{x}))\, \mathrm{d}\mathbf{x} =$$

$$= \int_{\Omega_\lambda^*} \tilde{\omega}(y)\, e(\hat{\mathbf{w}}(\mathbf{y}))\, \mathrm{d}\mathbf{y} = \int_{\Omega(v)} \omega(y)\, e(\hat{\mathbf{w}}(\mathbf{y}))\, \mathrm{d}\mathbf{y} = 0\,.$$

Here we used the fact that $\hat{\mathbf{w}} \in V(\Omega(v))$ and

$$\omega = \tau - \sigma^0, \quad \tau \in E(\Omega(v)), \quad \sigma^0 \in E(\Omega(v)).$$

It remains to show that $\tau^\lambda \in P(\Omega_\lambda)$. Let us denote

$$\sigma_0(\mathbf{y}) + \gamma(\lambda)\,\omega(\mathbf{y}) \equiv \sigma^\lambda(\mathbf{y})\,,$$

Then

(19) $$\|\tau^\lambda(\mathbf{x}) - \sigma^\lambda(\mathbf{y})\| = \|\sigma_0(\mathbf{x}) - \sigma_0(\mathbf{y})\| \leqq C\lambda \quad \forall \mathbf{x} \in \Omega_\lambda - (0, \lambda) \times (0, 1)\,,$$

since σ_0 is Lipschitz and $\|\mathbf{x} - \mathbf{y}\| = \lambda$.

One can prove that

(20) $$f((1 + \sqrt{\lambda})\,\sigma^\lambda(\mathbf{y})) = (1 + \sqrt{\lambda})\,f(\sigma^\lambda(\mathbf{y})) \leqq 1$$

holds for sufficiently small λ and $\mathbf{y} \in \Omega(v)$.

In fact, we may write

$$(1 + \sqrt{\lambda})\,\sigma^\lambda = (1 + \sqrt{\lambda})\,[\sigma^0 + \gamma(\tau - \sigma^0)] =$$
$$= (1 + \sqrt{\lambda})\left[\sigma^0(1 + \varepsilon)\frac{1 - \gamma}{1 + \varepsilon} + \gamma\tau\right] = \sigma^0(1 + \varepsilon)\frac{\sqrt{\lambda}}{\varepsilon} + \tau\left(1 - \frac{\sqrt{\lambda}}{\varepsilon}\right)$$

and (20) holds for $\sqrt{\lambda} < \varepsilon$, since f is convex and both τ and $(1 + \varepsilon)\,\sigma^0$ belong to $P(\Omega(v))$.

Since f is Lipschitz, we have

$$f(\tau^\lambda(\mathbf{x})) \leqq f(\sigma^\lambda(\mathbf{y})) + \tilde{C}\|\tau^\lambda(\mathbf{x}) - \sigma^\lambda(\mathbf{y})\| \leqq (1 + \sqrt{\lambda})^{-1} + \tilde{C}C\lambda \leqq 1$$

for $\lambda \leqq \lambda_0(\tilde{C}C, \varepsilon)$ and any $\mathbf{x} \in \Omega_\lambda - (0, \lambda) \times (0, 1)$. In the strip $(0, \lambda) \times (0, 1)$ it holds

$$f(\tau^\lambda) = f(\sigma_0) \leqq 1\,.$$

Altogether, $\tau^\lambda \in P(\Omega_\lambda)$ and the proof of (15) is complete.

Furthermore, we have

(21) $$\|\tau^\lambda - \tau\|_{0,\Omega(v)} = \|\gamma(\lambda)\,\omega^\lambda - \omega\|_{0,\Omega(v)} \leqq$$
$$\leqq |\gamma(\lambda)|\,\|\omega^\lambda - \omega\|_{0,\Omega(v)} + |\gamma(\lambda) - 1|\,\|\omega\|_{0,\Omega(v)} \to 0$$

for $\lambda \to 0$. In fact,

$$|\gamma(\lambda)| \leqq 1\,, \quad |\gamma(\lambda) - 1| \to 0$$

and

$$\|\omega^\lambda - \omega\|^2_{0,\Omega(v)} \to 0 \quad \text{for} \quad \lambda \to 0\,.$$

(See [7] – Theorem 1.1.)

The function τ^λ will now be used to construct the test functions in the approximate problem (6).

It is obvious that

$$\Omega_h \subset \Omega_\lambda \quad \forall h < h_0(\lambda)\,.$$

Then

$$\tau^\lambda|_{\Omega_h} \in E(\Omega_h) \cap P(\Omega_h)$$

and we may construct the projection

$$r_h \tau^\lambda \in E_h(\Omega_h) \cap P(\Omega_h)$$

(cf. the proof of Lemma 2.1).

Let $\Omega_{\lambda H}$ be a polygonal domain inscribed into Ω_λ and such that

(i) $\Omega_h \subset \Omega_{\lambda H}$,

(ii) the partitions D_h of the interval $[0, 1]$ refine the partition D_H, (i.e. H is a multiple of h),

holds for the sequence of h under consideration. Let us consider extended triangulations

$$\mathscr{T}_{hH} \supset \mathscr{T}_h(v_h)$$

of the domain $\Omega_{\lambda H}$ and the projection mapping

$$r_h^{\lambda H}: S(\Omega_{\lambda H}) \to H_h(\Omega_{\lambda H})$$

defined on the triangulations $\mathscr{T}_{hH}$ by means of the relation (7). Obviously, $r_h^{\lambda H} \tau^\lambda$ is an extension of $r_h \tau^\lambda$ onto $\Omega_{\lambda H}$.

By definition

$$(\sigma_h, r_h \tau^\lambda)_{\Omega_h} \geqq \|\sigma_h\|^2_{\Omega_h} \,. \tag{22}$$

Passing to the limit with $h \to 0$, we may write

$$(\sigma_h, r_h \tau^\lambda)_{\Omega_h} = (\tilde{\sigma}_h, r_h^{\lambda H} \tau^\lambda)_{\Omega_{\lambda H}} \to (\sigma, \tau^\lambda)_{\Omega_{\lambda H}} = (\sigma, \tau^\lambda)_{\Omega(v)} \tag{23}$$

using (10), the relation

$$\lim_{h \to 0} \|r_h^{\lambda H} \tau^\lambda - \tau^\lambda\|_{0, \Omega_{\lambda H}} = 0$$

and (14).

The weak convergence (10) implies

$$\liminf_{h \to 0} \|\sigma_h\|^2_{\Omega_h} = \liminf_{h \to 0} \|\tilde{\sigma}_h\|^2_{\Omega_\delta} \geqq \|\sigma\|^2_{\Omega_\delta} = \|\sigma\|^2_{\Omega(v)} \,. \tag{24}$$

Using (23) and (24) in (22), we obtain

$$(\sigma, \tau^\lambda)_{\Omega(v)} \geqq \|\sigma\|^2_{\Omega(v)} \,.$$

Passing to the limit with $\lambda \to 0$, we arrive at

$$(\sigma, \tau - \sigma)_{\Omega(v)} \geqq 0 \,,$$

by virtue of (21).

Using also (11), we conclude that the restriction $\sigma|_{\Omega(v)}$ coincides with the solution $\sigma(v)$ of the state problem (3). Since the latter problem is uniquely solvable – cf. Proposition 1.1 – the whole sequence $\{\tilde{\sigma}_h\}$ tends to $\sigma(v)$ weakly in $S(\Omega_\delta)$.

It remains to prove the strong convergence. Inserting $\tau = \sigma(v)$ into the previous argument and writing $\sigma(v) \equiv \sigma$, we obtain

$$\|\tilde{\sigma}_h\|^2_{\Omega_\delta} \leqq (\sigma_h, r_h \sigma^\lambda)_{\Omega_h} .$$

Passing to the limit with $h \to 0$, we arrive at

$$\limsup \|\tilde{\sigma}_h\|_{\Omega_\delta} \leqq (\sigma, \sigma^\lambda)_{\Omega(v)} \quad \forall \lambda .$$

Passing to the limit with $\lambda \to 0$ and using (21), we obtain

$$\limsup_{h \to 0} \|\tilde{\sigma}_h\|^2_{\Omega_\delta} \leqq \|\sigma\|^2_{\Omega_\delta} .$$

Combining this estimate with (24), we are led to the relation

$$\lim_{h \to 0} \|\tilde{\sigma}_h\|^2_{\Omega_\delta} = \|\sigma(v)\|^2_{\Omega_\delta} ,$$

which together with the weak convergence implies

$$\lim_{h \to 0} \|\tilde{\sigma}_h - \sigma(v)\|_{\Omega_\delta} = 0 .$$

On the basis of the equivalence of the norms, the strong convergence in $S(\Omega_\delta)$ follows.

Lemma 2.3. *Let* $\{v_h\}$, $h \to 0$, *be a sequence of* $v_h \in U^h_{ad}$, $v_h \to v$ *in* $[0, 1]$ *uniformly. Then*

$$\mathscr{J}(\sigma_h(v_h)) \to \mathscr{J}(\sigma(v)) ,$$

where $\sigma_h(v_h)$ *and* $\sigma(v)$ *are the solutions of the problems* (6) *and* (3) *on the domain* Ω_h *and* $\Omega(v)$, *respectively.*

Proof. Since $f(0) = 0$, we may write

$$\mathscr{J}(\sigma_h(v_h)) = \int_{\Omega_\delta} f^2(\tilde{\sigma}_h) \, \mathrm{d}\mathbf{x} , \quad \mathscr{J}(\sigma(v)) = \int_{\Omega_\delta} f^2(\sigma(v)) \, \mathrm{d}\mathbf{x} .$$

By assumption, we have

$$|f^2(\tilde{\sigma}_h) - f^2(\sigma)| \leqq |f(\tilde{\sigma}_h) - f(\sigma)| \, |f(\tilde{\sigma}_h) + f(\sigma)| \leqq$$
$$\leqq C\|\tilde{\sigma}_h - \sigma\| \, (2f(\sigma) + C\|\tilde{\sigma}_h - \sigma\|) .$$

Therefore, we may write

$$\mathscr{J}(\sigma_h) - \mathscr{J}(\sigma) = \left| \int_{\Omega_\delta} (f^2(\tilde{\sigma}_h) - f^2(\sigma) \, \mathrm{d}\mathbf{x} \right| \leqq$$
$$\leqq C_3 \int_{\Omega_\delta} \|\tilde{\sigma}_h - \sigma\| \, f(\sigma) \, \mathrm{d}\mathbf{x} + C_4 \int_{\Omega_\delta} \|\sigma_h - \sigma\|^2 \, \mathrm{d}\mathbf{x} \leqq$$
$$= C_5 \|\tilde{\sigma}_h - \sigma\|_{0,\Omega_\delta} + C_6 \|\tilde{\sigma}_h - \sigma\|^2_{0,\Omega_\delta} \to 0 ,$$

using Lemma 2.2.

We define the Approximate Optimization Problem:

$$\text{find} \quad u_h \in U^h_{ad} \quad \text{such that}$$

$$\mathscr{J}(\sigma_h(u_h)) \leqq \mathscr{J}(\sigma_h(v_h)) \quad \forall v_h \in U^h_{ad} \,. \tag{25}$$

Lemma 2.4. *The Approximate Optimization Problem has a solution for any h.*

Proof. Denoting by $\boldsymbol{a} \in \mathbb{R}^{N+1}$ the vector of nodal values

$$v_h(ih) = a_i \,, \quad i = 0, 1, \ldots, N \,,$$

it is easy to see that

$$v_h \in U^h_{ad} \Leftrightarrow \boldsymbol{a} \in \mathscr{A} \,,$$

where $\mathscr{A}$ is a compact set.

One can prove that the function

$$\boldsymbol{a} \mapsto \mathscr{J}(\sigma_h(\boldsymbol{a})) \tag{26}$$

is continuous on the set $\mathscr{A}$. In fact, the condition

$$\langle e(\mathbf{w}_h), \sigma_h \rangle_{\Omega_h(\boldsymbol{a})} = L_{\Omega_h(\boldsymbol{a})}(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in V_h(\Omega_h(\boldsymbol{a}))$$

is equivalent with a linear system

$$A(\boldsymbol{a}) \, \sigma = F(\boldsymbol{a})$$

with continuous functions $\boldsymbol{a} \mapsto A(\boldsymbol{a})$, $\boldsymbol{a} \mapsto F(\boldsymbol{a})$ (where σ denotes the vector of the values of σ_h in all triangles $K \in \mathscr{T}_h(\boldsymbol{a})$). The positive definite quadratic function

$$\mathscr{F}(\sigma) = \tfrac{1}{2} \|\sigma_h\|^2_{\Omega_h(\boldsymbol{a})}$$

has coefficients, which depend continuously on $\boldsymbol{a}$. Consequently, $\boldsymbol{a} \mapsto \sigma(\boldsymbol{a})$ is a continuous function, as well. The continuity of (26) follows from this fact and the properties of the yield function f.

Theorem 2.1. *Let $\{u_h\}$, $h \to 0$, be a sequence of solutions of the Approximate Optimization Problems (25).*

Then a subsequence $\{u_{\tilde{h}}\}$ exists such that

$$u_{\tilde{h}} \to u \quad in \quad C([0, 1]) \,,$$

$$\tilde{\sigma}_{\tilde{h}}(u_{\tilde{h}}) \to \sigma(u) \quad in \quad [L^2(\Omega_\delta)]^4 \tag{27}$$

holds for $\tilde{h} \to 0$, where $\tilde{\sigma}_{\tilde{h}}$ is the solution of the approximate problem (6), extended by zero to $\Omega_\delta - \Omega_{\tilde{h}}$, $\sigma(u)$ is the solution of the problem (3) on $\Omega(u)$, extended by zero to $\Omega_\delta - \Omega(u)$ and u is a solution of the Optimization Problem (4).

Any uniformly convergent subsequence of $\{u_h\}$ tends to a solution of (4) and (27) holds.

Proof. Let us consider a $v \in U_{ad}$. There exists a sequence $\{v_h\}$, $h \to 0$, such that $v_h \in U^h_{ad}$, $v_h \to v$ in $C([0, 1])$ (see [1] – Lemma 7.1).

Since U_{ad} is compact in $C([0, 1])$, a subsequence $\{u_{\tilde{h}}\}$ and $u \in U_{ad}$ exist such that

$$u_{\tilde{h}} \to u \quad \text{in} \quad C([0, 1]) .$$

By definition (25), we have

$$\mathcal{J}(\sigma_{\tilde{h}}(u_{\tilde{h}})) \leqq \mathcal{J}(\sigma_{\tilde{h}}(v_{\tilde{h}})) .$$

Applying Lemma 2.3 to both the sequences $\{u_{\tilde{h}}\}$ and $\{v_{\tilde{h}}\}$, we obtain

$$\mathcal{J}(\sigma(u)) \leqq \mathcal{J}(\sigma(v)) .$$

Consequently, u is a solution of the Optimization Problem (4). The convergence (27) follows from Lemma 2.2. The rest of the Theorem is obvious.

Corollary. *There exists at least one solution of the Optimization Problem* (4).

Proof is an immediate consequence of Lemma 2.4 and Theorem 2.1.

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Souhrn

OPTIMALIZACE TVARU PRUŽNĚ PLASTICKÝCH TĚLES PODLÉHAJÍCÍCH ZÁKONU HENCKYHO

Ivan Hlaváček

Minimalizuje se účelový funkcionál vzhledem k části hranice, na níž je těleso upevněno. Kritérium optimality je definováno integrálem funkce plasticity. K řešení stavové úlohy se užívá princip Haara-Kármána a po částech konstantní aproximace pole napětí na triangulacích. Dokazuje se existence řešení a konvergence přibližných řešení.

Резюме

ОПТИМИЗАЦИЯ ФОРМЫ УПРУГО-ПЛАСТИЧЕСКИХ ТЕЛ, ПОДЧИНЯЮЩИХСЯ ЗАКОНУ ХЕНКИ

IVAN HLAVÁČEK

Рассматривается минимизация целевой функции по отношению к той части границы, где тело фиксировано. Целевая функция определяется интегралом из функции пластичности. Для решения проблемы состояния применяется принцип Хара-Кармана и кусочно постоянные аппроксимации напряжений. Доказывается сходимость аппроксимаций в определённом смысле и существование оптимальной границы.

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