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Ivan Straškraba

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# SOLUTION OF A LINEAR MODEL OF A SINGLE-PISTON PUMP BY MEANS OF METHODS FOR DIFFERENTIAL EQUATIONS IN HILBERT SPACES

IVAN STRAŠKRABA

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*Summary.* A mathematical model of a fluid flow in a single-piston pump is formulated and solved. Variation of pressure and rate of flow in suction and delivery piping respectively is described by linearized Euler equations for barotropic fluid. A new phenomenon is introduced by a boundary condition with discontinuous coefficient describing function of a valve. The system of Euler equations is converted to a second order equation in the space $L^2(0, l)$, where $l$ is length of the pipe. The existence, unicity and stability of the solution of the Cauchy problem and the periodic solution is proved under explicit assumptions.

*Keywords:* Compressible fluid flow, telegraph equation, time-dependent boundary condition, stability, periodic solution.

*AMS Classification:* 35L20, 76N10, 35B35, 35B10.

## 1. FORMULATION OF THE PROBLEM

Consider the following system. We suppose that we have a pipeline placed in the $x$-axis at the beginning of which there is a piston pump while at its end there is a voluminous tank, where a constant pressure is maintained (see Fig. 1).

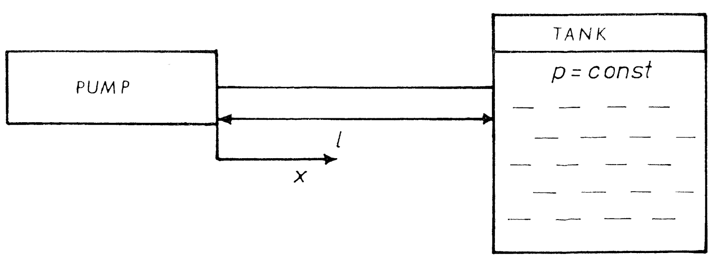


Fig. 1

We assume [1] that the flow in the pipeline is nonstationary, and that the flowing liquid is slightly viscous (e.g. water) and compressible. The flow is described by the Euler equations of one-dimensional hydrodynamics with a term caused by friction of the fluid near the wall of the pipe, due to its viscosity. Namely, the governing equations are

(1) $$u_t + uu_x + \frac{1}{\varrho} p_x = \frac{-k}{\varrho} |u| u ,$$

(2) $$\varrho_t + (\varrho u)_x = 0 .$$

Here $u = u(x, t)$ is the velocity, $p$ – pressure, $\varrho$ – density of the fluid at time $t$ and position $x$, $k$ is a constant. The equations (1), (2) are supplemented by a state equation of the form

$$p = p(\varrho) , \quad \text{where} \quad c^2 = p'(\varrho) > 0 , \quad (\varrho > 0) .$$

We assume that the variation of $u$ along the $x$-axis is not too large and, moreover, that the local Mach number $|u|/c$ is small. Then the term $u \,.\, u_x$ is small as compared with the other terms in (1) and so (1) can be written as

(3) $$u_t + \frac{1}{\varrho} p_x = - \frac{k}{\varrho} |u| u .$$

In the equation (2) we neglect the term $u\varrho_x$ supposing the variation of density along the $x$-axis is small, so that we get

(4) $$\varrho_t + \varrho u_x = 0 .$$

Set $Q = S \,.\, u$, where $S$ is the cross section of the pipeline. Inserting this into (3) we find

(5) $$Q_t + \frac{S}{\varrho} p_x = - \frac{k}{S\varrho} |Q| \,.\, Q .$$

Multiplying (4) by $p'(\varrho)$ yields

(6) $$p_t + \frac{\varrho}{S} p'(\varrho) \, Q_x = 0 .$$

Further approximation consists in taking $\varrho = \varrho_0 = \text{const.}$ – the density of the still medium – in (5), (6). Thus we have

(7) $$Q_t + \frac{S}{\varrho_0} p_x = - \frac{k}{S\varrho_0} |Q| \,.\, Q , \quad p_t + \frac{\varrho_0}{S} p'(\varrho_0) \, Q_x = 0 .$$

In order to establish the boundary conditions we follow the scheme of the pump as in Fig. 2.

The fluid flows into the reservoir with a rate of flow $Q_1$ and flows out of the working compartment of the pump with a rate of flow $Q_2$. The piston of the pump moves with a periodic speed causing variations of the rate of flow of the magnitude $f_0(t)$. A part of $Q_1$ might be either consumed or fortified by the variation of density due to the variation of pressure.

So we find

$$(8) \qquad Q = \Delta Q = Q_1 - Q_2 = \frac{d\Delta V}{dt} + f_0(t),$$

where $\Delta V$ is the variation of the volume of the fluid in the pump.

Define the capacity of the reservoir together with the working compartment as $C_r$, where $dV/dp = -C_r$, and assume that $C_r$ is constant.

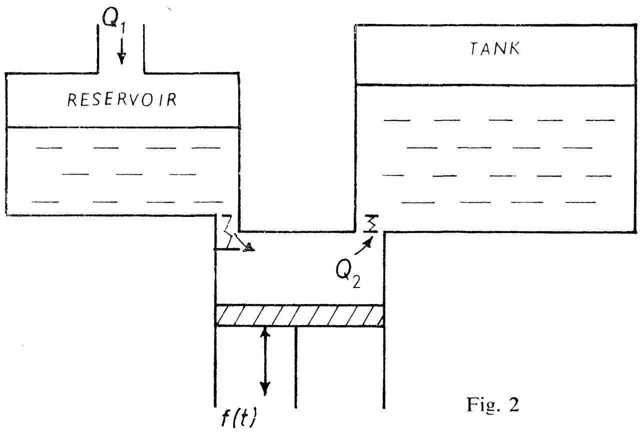


Fig. 2

Then (8) yields

$$(9) \qquad Q(0, t) + C_r\, p_t(0, t) = f_0(t).$$

This situation corresponds to an open valve. If the valve is closed we must consider just the capacity of the working compartment of the pump, which is, say, $C_0$. If the time of closing the valve is $t_1$, we have (9) in the form

$$(10) \qquad \varkappa(t)\, p_t(0, t) + Q(0, t) = f(t), \quad t \geqq 0,$$

where

$$(11) \qquad \varkappa(t) = \begin{cases} C_r & 0 \leqq t < t_1, \\ C_0 & t_1 \leqq t < \omega, \end{cases} \quad \varkappa(t + \omega) = \varkappa(t),$$

$$f(t) = \begin{cases} f_0(t) & 0 \leqq t < t_1, \\ 0 & t_1 \leqq t < \omega, \end{cases} \quad f(t + \omega) = f(t), \quad t \geqq 0$$

($\omega$ is the period of a cycle of the pump).

As pressure of the fluid in the tank at the end of the pipeline is constant, we are justified to set

$$(12) \qquad p(l, t) = \text{const.}, \quad t \geqq 0.$$

Our model does not include the function of the valve on the delivery side of the pump.

Finally, we are interested in the stabilized régime of the pump when $Q$ and $p$ vary periodically, i.e.

$$Q(x, t + \omega) = Q(x, t), \qquad p(x, t + \omega) = p(x, t), \quad x \in [0, l], \quad t \geqq 0. \tag{13}$$

Our final simplification is that we substitute $-(k/S\varrho_0) . |\bar{Q}| . Q$ for the right-hand side in the equation $(7_1)$, where $\bar{Q}$ is the average rate of flow during the time period $\omega$. It is clear that it can be approximated by

$$\bar{Q} = \frac{2r . S_0}{\omega},$$

where $r$ is the radius of the crank and $S_0$ is the cross section of the piston. Thus the system (7) is now in the form

$$Q_t + \frac{S}{\varrho_0} p_x = -\frac{2krS_0}{S\omega\varrho_0} Q, \tag{14}$$

$$p_t + \frac{\varrho_0}{S} c_0^2 Q_x = 0. \tag{15}$$

It is seen from (10), (11) and (14) that the functions $p_t$ and $Q_x$ cannot be continuous. In order to reformulate the problem in a suitable way we shall seek a solution of the equations (14), (15) subject to conditions (10), (12) and (13) in the class of continuous functions $p$, $Q$ such that $Q_t$, $p_x$ are continuous ($x \in [0, l]$, $t \geqq 0$) and $p_t$, $Q_x$ are continuous on $[0, l] \times [0, t_1]$ and on $[0, l] \times [t_1, \omega]$. Let us make some formal arrangements. The operation

$$-\frac{S}{\varrho_0}\frac{\partial}{\partial x}(15) + \frac{\partial}{\partial t}(14) \quad \text{yields}$$

$$Q_{tt} + \frac{2krS_0}{S\omega\varrho_0} Q_t - c_0^2 Q_{xx} = 0. \tag{16}$$

Substituting from (15) into (10) we get

$$-\varkappa(t)\frac{\varrho_0}{S} c_0^2\, Q_x(0, t) + Q(0, t) = f(t), \quad t \geqq 0. \tag{17}$$

Further, differentiating (12) with regard to $t$ and using again (15), we find

$$Q_x(l, t) = 0, \quad t \geqq 0. \tag{18}$$

Finally, (13) is expressed by

$$Q(x, t + \omega) = Q(x, t), \quad x \in [0, l], \quad t \geqq 0.$$

Setting now

$$
(19)\qquad \begin{cases} u(x,t) = Q(x,t) - f(t)\,, \\ \gamma = \dfrac{krS_0}{\varrho_0 S\omega}\,, \\ \alpha(t) = \begin{cases} \left(\varkappa_1 \dfrac{\varrho_0 c_0^2}{S}\right)^{-1} \equiv \alpha_1\,, & 0 \leqq t < t_1\,, \\ \left(\varkappa_2 \dfrac{\varrho_0 c_0^2}{S}\right)^{-1} \equiv \alpha_2\,, & t_1 \leqq t < \omega\,, \end{cases} \\ \qquad\qquad \alpha(t+\omega) = \alpha(t), \quad t \in R \\ g(t) = -f''(t) - 2\gamma f'(t) \end{cases}
$$

we get for $u$ the equations

$$
(20)\qquad \begin{aligned} & u_{tt} + 2\gamma u_t - c_0^2 u_{xx} = g(t)\,, \quad 0 < x < l\,, \quad t > 0\,, \\ & u_x(0,t) - \alpha(t)\,u(0,t) = 0\,, \quad t \geqq 0\,, \end{aligned}
$$

$$
(21)\qquad u_x(l,t) = 0\,, \quad t \geqq 0\,,
$$

$$
(22)\qquad u(x, t+\omega) = u(x,t)\,, \quad 0 \leqq x \leqq l\,, \quad t \geqq 0\,.
$$

Consider the abstract version of the problem (20)–(22). Let $H = L_2(0, l)$. Define operators $A_1$, $A_2$ in $H$ by the following rule:

$$
(23)\qquad D(A_i^2) = \left\{ v \in H\,;\ \frac{\mathrm{d}v}{\mathrm{d}x}\,, \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \in H\,,\ v'(0) - \alpha_i\, v(0) = 0\,,\quad v'(l) = 0 \right\},
$$

$$
(A_i^2 v)(x) = -c_0^2\, v''(x)\,, \quad v \in D(A_i^2)\,, \quad x \in (0, l)\,, \quad i = 1, 2\,.
$$

The problem (20)–(22) has the following equivalent:

$$
(24)\qquad \begin{aligned} & u''(t) + 2\gamma\, u'(t) + A(t)^2\, u(t) = g(t)\,, \quad t \in R\,, \\ & u(t+\omega) = u(t)\,, \quad t \in R\,, \end{aligned}
$$

where

$$
(25)\qquad A(t) = \begin{cases} A_1\,, & 0 \leqq t < t_1\,, \\ A_2\,, & t_1 \leqq t < \omega\,, \quad A(t+\omega) = A(t)\,, \quad t \in R\,. \end{cases}
$$

## 2. AUXILIARY LEMMAS

Solving the problem (24) we shall need some auxiliary results which now follow.

**Lemma 1.** *The operators $A_1^2$, $A_2^2$ defined in* (23) *are selfadjoint and positive definite.*

Proof. It is clear that e.g. the operator $A = A_1^2$ is symmetric, since for $v_1, v_2 \in D(A)$ we have

$$(Av_1, v_2) = -c_0^2 \int_0^l v_1''(x)\, v_2(x)\, dx =$$

$$= -c_0^2[v_1' v_2]_{x=0}^l + c_0^2[v_1 v_2']_{x=0}^l - c_0^2 \int_0^l v_1(x)\, v_2''(x)\, dx =$$

$$= c_0^2 \left[ -v_1'(l)\, v_2(l) + v_1'(0)\, v_2(0) + v_1(l)\, v_2'(l) - v_1(0)\, v_2'(0) - \int_0^l v_1(x)\, v_2''(x)\, dx \right] =$$

$$= c_0^2 \left[ \alpha_1\, v_1(0)\, v_2(0) - \alpha_1\, v_1(0)\, v_2(0) - \int_0^l v_1(x)\, v_2''(x)\, dx \right] = (v_1, Av_2)\,.$$

We shall show that $D(A^*) \subset D(A)$. Let $w \in D(A^*)$. Then

$$(Av, w) = (v, A^*w) \quad \text{for all} \quad v \in D(A)\,.$$

Denote $z = A^*w$. Then

$$- c_0^2 \int_0^l v''(x) \,.\, w(x)\, dx = \int_0^l v(x)\, z(x)\, dx \quad \text{for} \quad v \in D(A) \supset C_0^\infty(0, l)\,.$$

This means that the second distributional derivative of $-c_0^2 w$ is $z \in L_2(0, l)$. Now we show that $w'(0) - \alpha_1\, w(0) = 0$, $w'(l) = 0$.

For $v \in D(A)$ we have

$$\int_0^l v \,.\, z\, dx = -c_0^2 \int_0^l v'' w\, dx =$$

$$= c_0^2 \left[ -v'(l)\, w(l) + v'(0)\, w(0) + v(l)\, w'(l) - v(0)\, w'(0) - \int_0^l v w''\, dx \right].$$

As $c_0^2 w'' = -z$ and $v'(0) - \alpha_1\, v(0) = 0$, $v'(l) = 0$, we have

$$v(l)\, w'(l) - v(0)\, [w'(0) - \alpha_1\, w(0)] = 0 \quad \text{for} \quad v \in D(A)\,.$$

Since it is possible to choose functions $v \in D(A)$ so that $v(0)$ and $v(l)$ take arbitrary prescribed values, we necessarily have

$$w'(l) = 0\,, \quad w'(0) - \alpha_1\, w(0) = 0\,.$$

Hence indeed $w \in D(A)$.

Let us prove that $A$ is positive. If $v \in D(A)$ then

$$(26) \qquad (Av, v) = -c_0^2 \int_0^l v'' v\, dx - c_0^2[v'(l)\, v(l) - v'(0)\, v(0)] +$$

$$+ c_0^2 \int_0^l v'(x)^2\, dx = c_0^2 \alpha_1\, v(0)^2 + c_0^2 \int_0^l v'(x)^2\, dx\,.$$

Further,

$$v(x) = v(0) + \int_0^x v'(\xi)\,\mathrm{d}\xi\,,$$

whence

$$v(x)^2 \leqq 2\left[v(0)^2 + \left(\int_0^x v'(\xi)\,\mathrm{d}\xi\right)^2\right] \leqq 2\,v(0)^2 + 2\int_0^x 1\,\mathrm{d}x \int_0^x v'(\xi)\,\mathrm{d}\xi =$$

$$= 2\,v(0)^2 + 2x\int_0^x v'(\xi)^2\,\mathrm{d}\xi \leqq 2\,v(0)^2 + 2x\int_0^l v'(\xi)^2\,\mathrm{d}\xi\,.$$

Hence

$$\int_0^l v(x)^2\,\mathrm{d}x \leqq 2l\,v(0)^2 + \int_0^l 2x\int_0^l v'(\xi)^2\,\mathrm{d}\xi\,\mathrm{d}x = \tag{27}$$

$$= 2l\,v(0)^2 + l^2\int_0^l v'(\xi)^2\,\mathrm{d}\xi \leqq \max\left(\frac{2l}{\alpha_1}, l^2\right)\left[\alpha_1\,v(0)^2 + \int_0^l v'(\xi)^2\,\mathrm{d}\xi\right].$$

This together with (26) yields

$$(Av, v) \geqq c_0^2\left[\max\left(\frac{2l}{\alpha_1}, l^2\right)\right]^{-1}\int_0^l v(x)^2\,\mathrm{d}x \equiv c\|v\|^2\,. \tag{28}$$

The proof is complete.

**Lemma 2.** *We have* ([2])

$$D(A_1) = D(A_2) = H^1(0, l) = \{v \in H;\ v' \in H\}\,. \tag{29}$$

Proof. It is clear that e.g. the closure of $D(A_1^2)$ in $D(A_1)$ is $D(A_1)$. According to (26) we have

$$\|v\|^2_{D(A_1)} = c_0^2\alpha_1\,v(0)^2 + c_0^2\int_0^l |v'(x)|^2\,\mathrm{d}x \quad \text{for} \quad v \in D(A_1^2)\,. \tag{30}$$

It follows from (27) and (30) that there is a constant $k > 0$ such that

$$\|v\|^2_{D(A_1)} \geqq k\int_0^l [v(x)^2 + v'(x)^2]\,\mathrm{d}x\,, \quad v \in D(A_1^2)\,.$$

On the other hand, using in (30) the Sobolev embedding theorem we get

$$\|v\|^2_{D(A_1)} \leqq K\int_0^l [v(x)^2 + v'(x)^2]\,\mathrm{d}x\,, \quad v \in D(A_1^2)$$

with a constant $K > 0$. Thus the graph norm of $A_1$ is equivalent to the norm of $H^1(0, l)$ on $D(A_1^2)$ and $D(A_1)$ is equal to the closure of $D(A_1^2)$ in the norm of $H^1(0, l)$. Now it suffices to prove that the orthogonal complement in $H^1(0, l)$ of this closure is the trivial subspace $\{0\}$. So, let $w \perp \mathrm{cl}_{H^1} D(A_1^2)$, i.e. let

$$(v, w)_{H^1(0,l)} = 0 \quad \text{for all} \quad v \in D(A_1^2)\,.$$

Then

$$\int_0^l (v'w' + vw)\,\mathrm{d}x = 0 \quad \text{for all} \quad v \in H^2(0, l) \quad \text{such that} \quad v'(0) - \alpha_1 v(0) = 0\,, \quad v'(l) = 0\,.$$

Integrating by parts and using the boundary conditions we get

$$\int_0^l (-v'' + v)\, w\,\mathrm{d}x = \alpha_1 v(0)\, w(0) \quad \text{for} \quad v \in D(A_1^2)\,. \tag{30a}$$

The operator $-(\mathrm{d}^2/\mathrm{d}x^2) + \mathrm{Id}$ is selfadjoint on $D(A_1^2)$, and

$$(-v'' + v, v) = \alpha_1 v(0)^2 + \int_0^l v'(\xi)^2\,\mathrm{d}\xi + \int_0^l v(\xi)^2\,\mathrm{d}\xi \geqq \int_0^l v(\xi)^2\,\mathrm{d}\xi \quad \text{for} \quad v \in D(A_1^2)\,.$$

It follows that the range of $-(\mathrm{d}^2/\mathrm{d}x^2) + \mathrm{Id}$ is the whole $L_2(0, l)$. Hence, for any $\varphi \in L_2(0, l)$ there is a unique $v \in H^2(0, l)$ satisfying the equations

$$-v'' + v = \varphi(x)\,,$$
$$v'(0) - \alpha_1 v(0) = 0\,,$$
$$v'(l) = 0\,.$$

Write the solution in an explicit form. The solution of the initial problem is

$$v(x) = \operatorname{ch} x \,.\, v(0) + \operatorname{sh} x \,.\, v'(0) - \int_0^x \operatorname{sh}(x - \xi)\, \varphi(\xi)\,\mathrm{d}\xi\,.$$

Inserting $v'(0) = \alpha_1 v(0)$ we get

$$v(x) = (\operatorname{ch} x + \alpha_1 \operatorname{sh} x)\, v(0) - \int_0^x \operatorname{sh}(x - \xi)\, \varphi(\xi)\,\mathrm{d}\xi\,,$$

while the derivative satisfies

$$v'(x) = (\operatorname{sh} x + \alpha_1 \operatorname{ch} x)\, v(0) - \int_0^x \operatorname{ch}(x - \xi)\, \varphi(\xi)\,\mathrm{d}\xi\,.$$

The boundary condition $v'(l) = 0$ is met if we set

$$v(0) = (\operatorname{sh} l + \alpha_1 \operatorname{ch} l)^{-1} \int_0^l \operatorname{ch}(l - \xi)\, \varphi(\xi)\,\mathrm{d}\xi\,.$$

Inserting this into (30a) yields

$$\int_0^l \varphi(x)\, w(x)\,\mathrm{d}x = \frac{\alpha_1}{\operatorname{sh} l + \alpha_1 \operatorname{ch} l} \int_0^l \operatorname{ch}(l - \xi)\, \varphi(\xi)\,\mathrm{d}\xi\; w(0)\,,$$

i.e.

$$\int_0^l \varphi(x) \left[ w(x) - \frac{\alpha_1 \operatorname{ch}(l - x)\, w(0)}{\operatorname{sh} l + \alpha_1 \operatorname{ch} l} \right] \mathrm{d}x = 0 \quad \text{for all} \quad \varphi \in L_2(0, l)\,.$$

We get

(30b)
$$w(x) = \frac{\alpha_1 \operatorname{ch}(l - x)\, w(0)}{\operatorname{sh} l + \alpha_1 \operatorname{ch} l}.$$

In particular, for $x = 0$

$$w(0) = \frac{\alpha_1 \operatorname{ch} l\, w(0)}{\operatorname{sh} l + \alpha_1 \operatorname{ch} l},$$

i.e.

$$w(0) \,.\, \operatorname{sh} l = 0$$

from where

(30c)
$$w(0) = 0\,.$$

Now (30b) and (30c) imply $w(x) \equiv 0$, q.e.d.

## 3. SOLUTION OF THE ABSTRACT PROBLEM

Continuing the investigation of the problem (24), let us consider at first the Cauchy problem

(31)
$$u''(t) + 2\gamma\, u'(t) + A(t)^2\, u(t) = g(t)\,, \quad t \geqq \tau\,,$$
$$u(\tau) = u_0\,,$$
$$u'(\tau) = u_1\,,$$

where

$$A(t) = \begin{cases} A_1\,, & 0 \leqq t < t_1\,, \\ A_2\,, & t_1 \leqq t < \omega\,, \end{cases} \quad A(t + \omega) = A(t)\,, \quad t \in R\,.$$

For $I \subset R$, $k \geqq 0$ integer denote by $C^k(I; H)$ the space of functions from $I$ into $H$ which have continuous derivatives up to order $k$ in the norm of $H$.

**Definition 3.** *A weak solution of the problem* (31) *is a function* $u \in C^1([0, \infty); H)$ *such that*

(32)
$$u(\tau) = u_0\,, \quad u'(\tau) = u_1\,,$$

$$u(t) = C_1(t - n\omega)\, u(n\omega) + S_1(t - n\omega)\,(u'(n\omega) + \gamma u(n\omega)) + \int_{n\omega}^{t} S_1(t - \tau)\, g(\tau)\, \mathrm{d}\tau$$

*for* $t \in [n\omega, n\omega + t_1]$ *and*

(33)
$$u(t) = C_2(t - n\omega - t_1)\, u(n\omega + t_1) + S_2(t - n\omega - t_1)$$
$$[u'(n\omega + t_1) + \gamma u(n\omega + t_1)] + \int_{n\omega + t_1}^{t} S_2(t - \tau)\, g(\tau)\, \mathrm{d}\tau$$

*for* $t \in [n\omega + t_1, (n + 1)\,\omega]\,, \quad n \in Z\,, \quad n\omega \geqq \tau\,,$

*where*

$$C_i(t) = e^{-\gamma t} \cos t \sqrt{(A_i^2 - \gamma^2)}\,,$$

$$S_i(t) = e^{-\gamma t} \frac{\sin t \sqrt{(A_i^2 - \gamma^2)}}{\sqrt{(A_i^2 - \gamma^2)}}\,, \quad i = 1, 2\,.$$

**Theorem 4.** *Let $u_0 \in D(A_1)$, $u_1 \in H$, $g \in C([0, \infty); H)$. Then there exists a unique weak solution $u(t)$ of the problem (31) and it is given by (32), (33). This solution belongs to*

$$C^1([0, \infty); H) \cap C([0, \infty); D(A_i))\,, \quad i = 1, 2\,.$$

Proof. For simplicity let $\tau = 0$. If $n = 0$ then (32) with $u(0) = u_0$, $u'(0) = u_1$ clearly represents a function from $C^1([0, t_1]; H) \cap C([0, t_1]; D(A_1))$ (this is standard). By Lemma 2 we have $D(A_1) = D(A_2)$; hence $u(t_1 -) \in D(A_2)$. Let $u(t_1 -) = u(t_1 +)$, $u'(t_1 -) = u'(t_1 +)$. Then (33) gives a function from

$$C^1([t_1, \omega]; H) \cap C([t_1, \omega]; D(A_2))\,.$$

Analogously, we can continue in $[\omega, \omega + t_1]$, $[\omega + t_1, 2\omega]$ and so on. It is easy to see that $u$ is continuous in both norms of $D(A_i)$, $i = 1, 2$ even at the contact points because it is continuous from the left and from the right with the same value of the limit.

Remark. It is clear that in Theorem 4 it suffices to assume $g \in L^\infty([0, \infty); H)$ instead of $g \in C([0, \infty); H)$. The assertion remains without any change.

For construction of a periodic solution to the equation (31) it is convenient to define the following operator $K(t, \tau)$: $H \to H$, $\tau \leqq t$. If $\tau \leqq t$, $v \in H$ and $u(t)$ is a weak solution of

$$\text{(34)} \qquad \begin{aligned} u''(t) + 2\gamma\, u'(t) + A(t)^2\, u(t) &= 0\,, \\ u(\tau) &= 0\,, \\ u'(\tau) &= v\,, \end{aligned}$$

then we define $K(t, \tau)\, v = u(t)$. According to Theorem 4 the operator $K(t, \tau)$ is defined and $K(t, \tau)\, v$, $(\mathrm{d}/\mathrm{d}t)\, K(t, \tau)\, v \equiv K_t(t, \tau)\, v$ and $A_i K(t, \tau)\, v$ are continuous functions of $\tau$ and $t$ in $H$ for any $v \in H$. From (32), (33) it is easy to see that for $0 \leqq \tau < n\omega \leqq t \leqq (n + 1)\, \omega$ we have

$$K(\tau, \tau) = 0\,, \quad K_t(\tau, \tau) = I\,,$$

$$\text{(35)} \qquad K(t, \tau) = C_1(t - n\omega)\, K(n\omega, \tau) + $$
$$+ S_1(t - n\omega)\, [K_t(n\omega, \tau) + \gamma K(n\omega, \tau)] \quad \text{if} \quad n\omega < t \leqq n\omega + t_1\,,$$

and

$$\text{(36)} \qquad K(t, \tau) = C_2(t - n\omega - t_1)\, K(n\omega + t_1, \tau) + $$
$$+ S_2(t - n\omega + t_1)\, [K_t(n\omega + t_1, \tau) + \gamma K(n\omega + t_1, \tau)]$$
$$\text{if} \quad n\omega + t_1 \leqq t \leqq (n + 1)\, \omega\,.$$

Let us consider the periodic problem

$$(37) \qquad u''(t) + 2\gamma\, u'(t) + A(t)^2\, u(t) = g(t)\,, \qquad u(t+\omega) = u(t)\,, \quad t \in R\,,$$

where $A(t)$ is as above and $g \in C(R; H)$ is a given $\omega$-periodic function. We will prove the following.

**Theorem 5.** *Let $g \in C(R; H)$ be an $\omega$-periodic function and let there exist constants $M > 0$ and $\delta > 0$ such that*

$$(38) \qquad \|A_1 K(t, \tau)\, v\| \leqq M e^{-\delta(t-\tau)} \|v\|\,,$$

$$(39) \qquad \|K_t(t, \tau)\, v\| \leqq M e^{-\delta(t-\tau)} \|v\|\,, \quad t \geqq \tau\,, \quad v \in H\,.$$

*Then there exists a unique weak solution of the problem* (37). *This solution is given by*

$$(40) \qquad u(t) = \int_{-\infty}^{t} K(t, \tau)\, g(\tau)\, d\tau\,, \quad t \in R\,,$$

*where the operator $K(t, \tau)$ is defined by* (35), (36).

Proof. The convergence of the integrals (40),

$$\int_{-\infty}^{t} A_t K(t, \tau)\, g(\tau)\, d\tau$$

as well as the validity of the formula

$$u'(t) = \int_{-\infty}^{t} K_t(t, \tau)\, g(\tau)\, d\tau$$

are guaranteed by (38), (39).

Let us show that $u(t)$ given by (40) is a weak solution of $(37_1)$ on $[0, \omega]$. This amounts to showing that

$$(41) \qquad u(t) = C_1(t)\, u(0) + S_1(t)\, [u'(0) + \gamma\, u(0)] + \int_0^t S_1(t-\tau)\, g(\tau)\, d\tau\,, \quad 0 \leqq t < t_1\,,$$

$$(42) \qquad u(t) = C_2(t - t_1)\, u(t_1-) + S_2(t - t_1)\, [u'(t_1-) + \gamma\, u(t_1-)] + \int_{t_1}^t S_2(t-\tau)\, g(\tau)\, d\tau\,, \quad t_1 \leqq t < \omega\,.$$

For $t \in [0, t_1]$ we have by (35)

$$u(t) = \int_{-\infty}^{0} K(t, \tau)\, g(\tau)\, d\tau + \int_0^t K(t, \tau)\, g(\tau)\, d\tau =$$

$$= \int_{-\infty}^{0} \{C_1(t)\, K(0, \tau) + S_1(t)\, [K_t(0, \tau) + \gamma K(0, \tau)]\}\, g(\tau)\, d\tau + \int_0^t S_1(t - \tau)\, g(\tau)\, d\tau .$$

Since clearly

$$u(0) = \int_{-\infty}^{0} K(0, \tau)\, g(\tau)\, d\tau , \quad u'(0) = \int_{-\infty}^{0} K_t(0, \tau)\, g(\tau)\, d\tau ,$$

(41) is thus proved. The formula (42) is verified quite analogously. Further, from the uniqueness part of Theorem 4 it follows that

$$K(t + \omega,\ \tau + \omega) = K(t, \tau) \quad \text{for all} \quad \tau \leqq t .$$

Besides, we have

$$u(t) = \int_{-\infty}^{t} K(t, \tau)\, g(\tau)\, d\tau = \int_0^{\infty} K(t, t - \sigma)\, g(t - \sigma)\, d\sigma .$$

Hence

$$u(t + \omega) = \int_0^{\infty} K(t + \omega,\ t + \omega - \sigma)\, g(t + \omega - \sigma)\, d\sigma =$$

$$= \int_0^{\infty} K(t, t - \sigma)\, g(t - \sigma)\, d\sigma = \int_{-\infty}^{t} K(t, \tau)\, g(\tau)\, d\tau = u(t)$$

and this completes the proof.

## 4. EXPONENTIAL STABILITY OF THE ABSTRACT PROBLEM

Now we shall derive some conditions guaranteeing (38), (39). We use the technique suggested in [2].

Let $u(t) = K(t, \tau)\, v$, where $v \in H$ and $K(t, \tau)$ is defined by (35), (36). Suppose first that $t \in [n\omega, n\omega + t_1]$ and $u(n\omega) = u_0 \in D(A_1^2)$, $u'(n\omega) = u_1 \in D(A_1)$. Then it is easy to show that $u \in C^2([n\omega, n\omega + t_1];\ H) \cap C([n\omega,\ n\omega + t_1];\ D(A_1^2))$.

Putting

(43) $$u(t) = e^{-\gamma(t-\tau)} w(t)$$

we get

(44) $$u'(t) = (-\gamma\, w(t) + w'(t))\, .\, e^{-\gamma(t-\tau)} ,$$

(45) $$u''(t) = (\gamma^2\, w(t) - 2\gamma\, w'(t) + w''(t))\, .\, e^{-\gamma(t-\tau)} .$$

As

(46) $$u''(t) + 2\gamma\, u'(t) + A_1^2\, u(t) = 0$$

we find for $\tau = n\omega$

(47) $$w''(t) + (A_1^2 - \gamma^2)\, w(t) = 0 ,\quad w(n\omega) = u_0 ,\quad w'(n\omega) = u_1 + \gamma u_0 .$$

Multiplying (47) by $w'(t)$ in $H$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w'(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(A(t)^2 - \gamma^2)^{1/2} w(t)\|^2 = 0 . \tag{48}$$

Suppose for simplicity

$$A_i^2 \geqq m > \gamma^2 , \quad i = 1, 2 . \tag{49}$$

(For the original problem this is satisfied.)

Integrating (48) over $[n\omega, t]$ we obtain

$$\begin{aligned} &\|w'(t)\|^2 + \|(A_1^2 - \gamma^2)^{1/2} w(t)\|^2 = \\ &= \|w'(n\omega)\|^2 + \|(A_1^2 - \gamma^2)^{1/2} w(n\omega)\|^2 . \end{aligned} \tag{50}$$

Now, if $u_0 \in D(A_1)$, $u_1 \in H$, then there exist $u_{0j} \in D(A_1^2)$, $u_{1j} \in D(A_1)$, $u_{0j} \xrightarrow{D(A_1)} u_0$, $u_{1j} \xrightarrow{H} u_1$ which yield the corresponding solutions $w_j(t)$ of (47) with $u_0 := u_{0j}$, $u_1 := u_{1j}$. Taking $w_{jk}(t) = w_j(t) - w_k(t)$ we get similarly as above

$$\begin{aligned} &\|w'_{jk}(t)\|^2 + \|(A_1^2 - \gamma^2)^{1/2} w_{jk}(t)\|^2 = \\ &= \|u_{1j} - u_{1k}\|^2 + \|(A_1 - \gamma^2)^{1/2} (u_{0j} - u_{0k})\|^2 , \end{aligned}$$

which implies that $w_j(t) \overset{D(A_1)}{\rightrightarrows} w(t)$, $w'_j(t) \overset{H}{\rightrightarrows} w'(t)$. Clearly, $w$ is the weak solution of (47). Suppose that e.g.

$$(n - 1)\,\omega + t_1 \leqq \tau \leqq (n + k)\,\omega + t_1 < t \leqq (n + k + 1)\,\omega$$

(other configurations of the numbers $\tau < t$ with respect to intervals of the type $[j\omega, j\omega + t_1]$, $[l\omega + t_1, (l + 1)\,\omega]$ can be treated quite analogously). The equality (48) holds in all intervals $[l\omega, l\omega + t_1]$, $[l\omega + t_1, (l + 1)\,\omega]$, $l = 0, 1, \ldots, k$ as well as in $[\tau, n\omega]$, $[(n + k)\,\omega + t_1, t]$. Integrating (48) over $[(n + k)\,\omega + t_1, t]$ we get

$$\begin{aligned} E(t) &\equiv \|w'(t)\|^2 + \|(A_2^2 - \gamma^2)^{1/2} w(t)\|^2 = \\ &= \|w'((n + k)\,\omega + t_1)\|^2 + \|(A_2^2 - \gamma^2)^{1/2} w((n + k)\,\omega + t_1)\|^2 . \end{aligned} \tag{51}$$

As the operator $(A_2^2 - \gamma^2)^{1/2} (A_1^2 - \gamma^2)^{-1/2}$ is everywhere defined and closed in $H$, it is bounded, so that

$$\begin{aligned} E(t) &\leqq \|w'((n + k)\,\omega + t_1)\|^2 + \\ &+ \|(A_2^2 - \gamma^2)^{1/2} (A_1^2 - \gamma^2)^{-1/2}\|^2 \,.\, \|(A_1^2 - \gamma^2)^{1/2} w((n + k)\,\omega + t_1)\|^2 \leqq \\ &\leqq \max\{1, \|(A_2^2 - \gamma^2)^{1/2} (A_1^2 - \gamma^2)^{-1/2}\|^2\} \,. \\ &\cdot [\|w'((n + k)\,\omega + t_1)\|^2 + \|(A_1^2 - \gamma^2)^{1/2} w((n + k)\,\omega + t_1)\|^2] . \end{aligned} \tag{52}$$

Integrating (48) over $[(n + k)\,\omega, (n + k)\,\omega + t_1]$ we find

$$(53)\qquad \|w'((n+k)\,\omega+t_1)\|^2+\|(A_1^2-\gamma^2)^{1/2}\,w((n+k)\,\omega+t_1)\|^2=$$
$$=\|w'((n+k)\,\omega)\|^2+\|(A_1^2-\gamma^2)^{1/2}\,w((n+k)\,\omega)\|^2\leqq$$
$$\leqq\max\{1,\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|^2\}\,.$$
$$.\,[\|w'((n+k)\,\omega)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w((n+k)\,\omega)\|^2]\,.$$

If we combine (52) and (53) we obtain

$$E(t)\leqq\max\{1,\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|^2\}\,.$$
$$.\,\max\{1,\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|^2\}\,.$$
$$.\,[\|w'((n+k)\,\omega)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w((n+k)\,\omega)\|^2]=$$
$$=q^2[\|w'((n+k)\,\omega)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w((n+k)\,\omega)\|^2]\,,$$

where

$$(54)\qquad q=\max\{\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|^2\,,$$
$$\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|^2\}\,.$$

Indeed, we have $[(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}]^{-1}=(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}$, from where $\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|\geqq\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|^{-1}$. If e.g. $\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|\leqq 1$ then $\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|\geqq 1$. Thus either

$$(55)\qquad \max\{1,\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|^2\}\,.$$
$$.\,\max\{1,\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|^2\}\leqq$$
$$\leqq\max\{\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|^2\,,$$
$$\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|^2\}=q\,,$$

if one of the numbers in parentheses is $\leqq 1$, or $\leqq q^2$ if both the numbers $\|(A_2^2-\gamma^2)^{1/2}\,(A_1^2-\gamma^2)^{-1/2}\|$ and $\|(A_1^2-\gamma^2)^{1/2}\,(A_2^2-\gamma^2)^{-1/2}\|$ are greater than 1. We proceed quite analogously in estimating the sum of the square powers of the derivative $w'(t)$ and the expressions of the type $(A_i^2-\gamma^2)^{1/2}\,w(t)$, where $t=j\omega$ or $t=j\omega+t_1$. We repeatedly use the trick with the operators $(A_i^2-\gamma^2)^{\pm 1/2}$. After $2k+2$ steps we get

$$\|w'(t)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w(t)\|^2\leqq$$
$$\leqq q^{2k+2}(\|w'(n\omega)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w(n\omega)\|^2)\,.$$

Integrating (48) over $[\tau, n\omega]$ we get

$$\|w'(n\omega)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w(n\omega)\|^2=$$
$$=\|w'(\tau)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w(\tau)\|^2=\|v\|^2\,.$$

So we eventually obtain

$$(56)\qquad \|w'(t)\|^2+\|(A_2^2-\gamma^2)^{1/2}\,w(t)\|^2\leqq q^{2k+2}\|v\|^2\,.$$

Clearly, $t - \tau = k\omega + \varepsilon$, where $0 \leqq \varepsilon \leqq 2\omega$. Hence $k = (t - \tau - \varepsilon)/\omega$ and (56) takes the form

$$(57) \qquad \|w'(t)\|^2 + \|(A_2 - \gamma^2)^{1/2} w(t)\|^2 \leqq e^{2\ln q} \,.\, e^{2(\ln q/\omega)(t-\tau)}\|v\|^2 \,.$$

Using (57), (49) and (44) we get

$$(58) \qquad \|(A_2^2 - \gamma^2)^{1/2} K(t, \tau)\| \leqq q \,.\, \exp\left[\left(-\gamma + \frac{\ln q}{\omega}\right)(t - \tau)\right],$$

$$\|K(t, \tau)\| \leqq q \left(\frac{1}{m - \gamma^2}\right)^{1/2} \exp\left[\left(-\gamma + \frac{\ln q}{\omega}\right)(t - \tau)\right], \quad t \geqq \tau\,,$$

$$\|K_t(t, \tau)\| \leqq q\left(1 + \frac{\gamma}{\sqrt{(m - \gamma^2)}}\right) \exp\left[\left(-\gamma + \frac{\ln q}{\omega}\right)(t - \tau)\right], \quad t \geqq \tau\,.$$

It is important to notice that if one of the numbers

$$(59) \qquad q_1 = \|(A_1^2 - \gamma^2)^{1/2} (A_2^2 - \gamma^2)^{-1/2}\|^2\,, \quad q_2 = \|(A_2^2 - \gamma^2)^{1/2} (A_1^2 - \gamma^2)^{-1/2}\|^2$$

is smaller than 1, then in the estimates (58), $q$ can be replaced by $q^{1/2}$.

## 5. EXISTENCE OF SOLUTION UNDER EXPLICIT ASSUMPTIONS

As a direct consequence of Theorem 5 we have

**Theorem 6.** *Let $g \in C(R; H)$ be an $\omega$-periodic function and let either the numbers $q_1$, $q_2$ defined by (59) be both greater than 1 and*

$$(60) \qquad \gamma\omega > \ln q$$

*with*

$$q = \max\{q_1, q_2\}\,,$$

*or one of $q_1$, $q_2$ be less or equal to 1 and*

$$(61) \qquad \gamma\omega > \tfrac{1}{2} \ln q\,.$$

*Then there exists a unique weak solution of the problem (37). This solution is given by (40) with $K(t, \tau)$ given by (35), (36).*

To get a more explicit result we need

**Lemma 7.** *If $\alpha_1 > \alpha_2$ then $q_2 \leqq 1$, $q_1 \geqq 1$. If, moreover,*

$$c_0^2 > \frac{l^2\gamma^2}{\min\left(1, \dfrac{l\alpha_2}{2}\right)}$$

*then*

(62) $$q \leqq \frac{\alpha_1}{\alpha_2}\left[1 - \frac{l^2\gamma^2}{c_0^2 \,.\, \min\left(1, \frac{l\alpha_2}{2}\right)}\right]^{-1}.$$

Proof. Let $z \in H$ and $z_i = (A_i^2 - \gamma^2)^{-1/2} z$, $i = 1, 2$. Then we have

(63) $$\|(A_2^2 - \gamma^2)^{1/2}(A_1^2 - \gamma^2)^{-1/2} z\|^2 = \|(A_2^2 - \gamma^2)^{1/2} z_1\|^2 = ((A_2^2 - \gamma^2) z_1, z_1) =$$

$$= \alpha_2 c_0^2 z_1(0)^2 + c_0^2 \int_0^l z_1'(\xi)^2 \,d\xi - \gamma^2 \int_0^l z_1(\xi)^2 \,d\xi =$$

$$= (\alpha_2 - \alpha_1) c_0^2 z_1(0)^2 + \alpha_1 c_0^2 z_1(0)^2 + c_0^2 \int_0^l z_1'(\xi)^2 \,d\xi - \gamma^2 \int_0^l z_1(\xi)^2 \,d\xi =$$

$$= (\alpha_2 - \alpha_1) c_0^2 z_1(0)^2 + \|(A_1^2 - \gamma^2)^{1/2} z_1\|^2 =$$

$$= (\alpha_2 - \alpha_1) c_0^2 z_1(0)^2 + \|z\|^2 \leqq \|z\|^2 .$$

Thus we have $q_2 \leqq 1$. Since

$$(A_1^2 - \gamma^2)^{1/2}(A_2^2 - \gamma^2)^{-1/2} = [(A_2^2 - \gamma^2)^{1/2}(A_1^2 - \gamma^2)^{-1/2}]^{-1},$$

we have $q_1 \geqq (1/q_2) \geqq 1$ and consequently $q = q_1$. Let us estimate $q_1$:

(64) $$q_1 = \|(A_1^2 - \gamma^2)^{1/2}(A_2^2 - \gamma^2)^{-1/2} z\|^2 =$$

$$= \alpha_1 c_0^2 z_2(0)^2 + c_0^2 \int_0^l z_2'(\xi)^2 \,d\xi - \gamma^2 \int_0^l z_2(\xi)^2 \,d\xi \leqq \alpha_1 c_0^2 z_2(0)^2 + c_0^2 \int_0^l z_2'(\xi)^2 \,d\xi \leqq$$

$$\leqq \max\left(1, \frac{\alpha_1}{\alpha_2}\right)\left[\alpha_2 c_0^2 z_2(0)^2 + c_0^2 \int_0^l z_2'(\xi)^2 \,d\xi\right].$$

Now, if $v \in D(A_i)$ for some $i = 1, 2$ then

$$v(x)^2 = \left[v(0) + \int_0^x v'(\xi) \,d\xi\right]^2 \leqq 2\left[v(0)^2 + \left(\int_0^x 1 \,.\, v'(\xi) \,d\xi\right)^2\right] \leqq$$

$$\leqq 2 \,.\left[v(0)^2 + \left(\int_0^x 1 \,d\xi\right)\left(\int_0^x v'(\xi)^2 \,d\xi\right)\right] =$$

$$= 2\, v(0)^2 + 2x \int_0^x v'(\xi)^2 \,d\xi \leqq 2\, v(0)^2 + 2x \int_0^l v'(\xi)^2 \,d\xi .$$

Hence

(65) $$\int_0^l v(x)^2 \,dx \leqq 2l\, v(0)^2 + l^2 \int_0^l v'(\xi)^2 \,d\xi .$$

On the other hand,

$$(66)\qquad c_0^2 \int_0^l v'(\xi)^2\,d\xi + \alpha_i c_0^2\, v(0)^2 = \frac{c_0^2}{l^2}\left[l^2\alpha_i\, v(0)^2 + l^2 \int_0^l v'(\xi)^2\,d\xi\right] \geqq$$

$$\geqq \frac{c_0^2}{l^2}\min\left(1, \frac{l\alpha_i}{2}\right)\left[2l\, v(0)^2 + l^2 \int_0^l v'(\xi)^2\,d\xi\right] \geqq \frac{c_0^2}{l^2}\min\left(1, \frac{l\alpha_i}{2}\right)\int_0^l v(\xi)^2\,d\xi\,,$$

the last inequality being a consequence of (65). Here we use

**Proposition 8.** *Let* $a > c > 0$, $b > d > 0$, $x, y, z \geqq 0$ *and*

$$(67)\qquad z \leqq cx + dy\,.$$

*Then*

$$(68)\qquad ax + by \leqq \max\left\{\frac{a}{a-c}, \frac{b}{b-d}\right\}(ax + by - z)\,.$$

Proof of Proposition is quite elementary:

$$ax + by - z \geqq (a-c)\,x + (b-d)\,y =$$

$$= \frac{a-c}{a}\,.\,ax + \frac{b-d}{b}\,.\,by \geqq \min\left\{\frac{a-c}{a}, \frac{b-d}{b}\right\}(ax + by) =$$

$$= \left[\max\left\{\frac{a}{a-c}, \frac{b}{b-d}\right\}\right]^{-1}.\,(ax + by)\,.$$

From (66) we have

$$(69)\qquad \gamma^2 \int_0^l z_1(\xi)^2\,d\xi \leqq \frac{\alpha_2 l^2\gamma^2}{\min\left(1, \frac{l\alpha_2}{2}\right)}\, z_2(0)^2 + \frac{l^2\gamma^2}{\min\left(1, \frac{l\alpha_2}{2}\right)} \int_0^l z_2'(\xi)^2\,d\xi\,.$$

In Proposition 8 set

$$a = \alpha_2 c_0^2\,,\quad b = c_0^2\,,\quad c = \frac{\alpha_2 l^2\gamma^2}{\min\left(1, \frac{l\alpha_2}{2}\right)}\,,\quad d = \frac{l^2\gamma^2}{\min\left(1, \frac{l\alpha_2}{2}\right)}\,,$$

$$x = z_2(0)^2\,,\quad y = \int_0^l z_2'(\xi)^2\,d\xi\,,\quad z = \gamma^2 \int_0^l z_1(\xi)^2\,d\xi\,.$$

Take (69) instead of (67). Then (68) yields

$$(70)\qquad \alpha_2 c_0^2\, z_2(0)^2 + c_0^2 \int_0^l z_2'(\xi)^2\,d\xi \leqq$$

$$\leqq \max \left\{ \frac{\alpha_2 c_0^2}{\alpha_2 c_0^2 - \alpha_2 \dfrac{l^2 \gamma^2}{\min\left(1, \dfrac{l\alpha_2}{2}\right)}}, \frac{c_0^2}{c_0^2 - \dfrac{l^2 \gamma^2}{\min\left(1, \dfrac{l\alpha_2}{2}\right)}} \right\} .$$

$$\cdot \left[ \alpha_2 c_0^2 \, z_2(0)^2 + c_0^2 \int_0^l z_2'(\xi)^2 \, d\xi - \gamma^2 \int_0^l z_2'(\xi)^2 \, d\xi \right] =$$

$$= \frac{c_0^2}{c_0^2 - \dfrac{l^2 \gamma^2}{\min\left(1, \dfrac{l\alpha_2}{2}\right)}} \|z\|^2 .$$

From (64) and (70) we see that

$$q_1 \leqq \frac{\alpha_1}{\alpha_2} \cdot \left( 1 - \frac{l^2 \gamma^2}{c_0^2 \min\left(1, \dfrac{l\alpha_2}{2}\right)} \right)^{-1},$$

q.e.d.

It can be shown that the particular values of constants $\alpha_1, \alpha_2, l, \gamma, c_0, \omega$, we are insterested in satisfy the condition

$$\frac{1}{2} \ln \left[ \frac{\alpha_1}{\alpha_2} \left( 1 - \frac{l^2 \gamma^2}{c_0^2 \min\left(1, \dfrac{l\alpha_2}{2}\right)} \right)^{-1} \right] < \gamma\omega ,$$

which is sufficient for (61).

Remark. It should be mentioned that, in fact, the function $f(t)$ in (9) is of the form $f(t) = v(t) \, . \, f_0(t)$, where $v(t) = 0$ for $0 \leqq t < t_1$ and $v(t) = 1$ for $t_1 \leqq t < \omega$, $v(t + \omega) = v(t)$, and $f_0(t)$ is a smooth function describing the motion of the piston. This function has jumps at the points $t_1 + n\omega$, $n$ integer. So far we have supposed that $f(t)$ is approximated by a smooth function. This can be avoided by treating the problem (31) with $g(t) = f''(t) + 2\gamma f'(t)$, considering here the distributional derivatives of $f$. The "worse" function on the right hand side of (31) is then of the type $\delta'(t - t_1 - n\omega) \, . \, 1$, which still yields a weak solution of the abstract problem. Another possibility is to set, instead of $(19_1)$,

$$u(x, t) = Q(x, t) - \frac{\varkappa_1}{\varkappa_2 - \varkappa_1} f_0(t) - x \, . \, \frac{f_0(t) \, (l - x)^2}{l^2 (\varkappa_2 - \varkappa_1)} .$$

Then the problem (16), (17), (18), (13) is reduced to (31) with $g(t)$ sufficiently smooth and in the explicit formulas no distributions need be treated.

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Souhrn

## ŘEŠENÍ LINEÁRNÍHO MODELU JEDNOPÍSTOVÉHO ČERPADLA METODAMI PRO DIFERENCIÁLNÍ ROVNICE V HILBERTOVÝCH PROSTORECH

IVAN STRAŠKRABA

V práci je formulován a řešen matematický model proudění tekutin v jednopístovém čerpadle. Časový průběh tlaku a průtoku v sacím resp. výtlačném potrubí je popsán linearizovanými Eulerovými rovnicemi pro barotropní tekutinu. Nový jev přináší okrajová podmínka s nespojitým koeficientem popisující funkci ventilu. Soustava Eulerových rovnic je převedena na jednu rovnici druhého řádu v prostoru $L^2(0, l)$, kde $l$ je délka potrubí. Je dokázána existence, jednoznačnost a stabilita řešení Cauchyho úlohy a periodického řešení za explicitních předpokladů.

Резюме

## РЕШЕНИЕ ЛИНЕЙНОЙ МОДЕЛИ ОДНОПОРШНЕВОГО НАСОСА МЕТОДАМИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ГИЛЬБЕРТОВЫХ ПРОСТРАНСТВАХ

IVAN STRAŠKRABA

В работе сформулирована и решена математическая модель течения жидкости в однопоршневом насосе. Временной ход давления и протекания в спускном коллекторе описан линеаризованными уравнениями Эйлера для баротропной жидкости. Новое явление представляет собой краевое условие с разрывным коэффициентом, описывающее работу вентиля. Система уравнений Эйлера переведена на одно уравнение второго порядка в пространстве $L^2(0, l)$, где $l$ — длина коллектора. Доказаны существование, однозначность и устойчивость решения задачи Коши и периодического решения при выполнении явных предположений.

*Author's address:* RNDr. *Ivan Straškraba,* CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.