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ON THREE PROBLEMS OF NEUTRON TRANSPORT THEORY

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Summary. In this paper, the initial-value problem, the problem of asymptotic time behaviour of its solution and the problem of criticality are studied in the case of linear Boltzmann equation for both finite and infinite media. Space of functions where these problems are solved is chosen in such a way that the range of physical situations considered may be so wide as possible. As mathematical apparatus the theory of positive bounded operators and of semigroups are applied. Main results are summarized in three basic theorems.

Keywords: neutron transport, initial-value problem, criticality, asymptotic behaviour, neutron flux, analytical solution, cross sections.

INTRODUCTION

The behaviour of the neutron flux φ in a medium is well described by the equation

$$\frac{\partial\varphi}{\partial t} = \boldsymbol{T}\varphi + \boldsymbol{S}\varphi + \boldsymbol{F}\varphi \tag{1}$$

completed by some conditions (e.g. initial or boundary ones). Here, the following notation is used:

$$\boldsymbol{T}\varphi = -\{\sqrt{(2E)}\,\omega\nabla + \sqrt{(2E)}\sum_i N_i(\boldsymbol{x})\,\sigma_{ti}(E,\omega)\}\,\varphi(\boldsymbol{x}, E, \omega, t)\,;$$

$$\boldsymbol{S}\varphi = \sqrt{(2E)}\sum_i N_i(\boldsymbol{x})\int_0^\infty dE' \int_\Omega \mathrm{d}\omega'\,\sigma_{si}(E' \to E,\ \omega' \to \omega)\,\varphi(\boldsymbol{x}.\,E', \omega', t)\,;$$

$$\boldsymbol{F}\varphi = \sqrt{(2E)}\sum_i N_i(\boldsymbol{x})\int_0^\infty dE' \int_\Omega d\omega'\,\sigma_{fi}(E' \to E;\ \omega' \to \omega)\,\varphi(\boldsymbol{x}, E', \omega', t)\,;$$

$\boldsymbol{x}, E, \omega, t$ ... coordinates of location, energy, angle and time, respectively;

Ω ... surface area of the unit sphere;
$N_i(\boldsymbol{x})$ ... density of nuclei of the scatterer;
σ_{ti} ... microscopic total effective cross-section;
σ_{si}, σ_{fi} ... microscopic differential effective cross-section for scattering and fission, respectively.

In this paper, three problems will be studied:

a) the initial-value problem,

b) the problem of dominant time eigenvalues, and

c) the criticality problem.

The problem a) is formulated as follows: to find a solution φ of Eq. (1) in a class C_1 of complex functions if the initial distribution $\varphi(\boldsymbol{x}, E, \boldsymbol{\omega}, t = 0)$ belonging to a class C_2 is given.

Existence and uniqueness of the solution to the problem a) were shown e.g. in paper [1] for a convex homogeneous body $\mathbf{D}$ in the case $C_1 \equiv L_2\{\mathbf{D} \times (0, \infty) \times \times \Omega \times [0, \infty)\}$ and $C_2 \equiv L_2\{\mathbf{D} \times (0, \infty) \times \Omega\}$. As inelastic scattering for high energies and fission effective cross-sections have not appropriate properties, these processes were not considered there. Later, the result was generalized to the case of a convex and partwise homogeneous body including more general models of scattering [2]. The problem was studied in the space L_1, but it was assumed that the velocities of neutrons are bounded (the reason was essentially the same as in paper [1]). In a similar way, the problem was solved also in [3].

The problem b) has the following formulation: to find a real number λ and a non-trivial nonnegative function φ belonging to a class C_3 so that the equation

$$\lambda\varphi = \boldsymbol{T}\varphi + \boldsymbol{S}\varphi + \boldsymbol{F}\varphi \tag{2}$$

may be fulfilled, λ being the maximum of the possible values.

This problem is closely connected with the problem of asymptotic time behaviour of the solution to the problem a).

Formulation of the problem c) is as follows: to find a real number $\eta \geqq 0$ and a non-trivial nonnegative function φ belonging to a class C_4 so that the equation

$$\boldsymbol{T}\varphi + \boldsymbol{S}\varphi + \eta\boldsymbol{F}\varphi = 0 \tag{3}$$

may be fulfilled.

The problems b) and c) have been discussed in literature very often. Always a bounded and convex body is considered, $L_p(1 \leqq p < \infty)$ plays the role of classes $C_1 - C_4$ and the main task is to transform the problems in such a way that the theory of compact positive operators [4] may be applied directly. Clearly, for this purpose, the corresponding operators in the formulation must have the desired properties, which does not dispense with strong restrictions set on the medium characteristics. In this way, the range of physical applications gets narrower [1, 5–8].

In this paper we will deal with the problems a), b) and c) for both bounded and unbounded media. The plan is as follows: First of all, basic physical properties of the medium will be stated in a form of three generalizing suppositions (being satisfied in all known real cases, of course). Then a space of functions will be chosen appropriately with respect to these properties. Finally, the problems will be transformed in such a way that the theory of semigroups and of positive operators may be applied, and basic theorems will be proved.

Definition 1. *Let* $\mathbf{B}$ *be a metric space with a measure* μ *and* $\mathbf{M} \subset \mathbf{B}$ *a given set,* $\mu(\mathbf{M}) > 0$. *Let* $\{\mathbf{V}_j\}_{j \geq 0}$ *be a finite or countable decomposition of* $\mathbf{M}$ *such that*

i) $\mathbf{V}_i \cap \mathbf{V}_j = \emptyset$ *for* $i \neq j$ *and* $\mu(\mathbf{V}_0) = 0$;

ii) *for any* $j \geqq 1$ *the set* V_j *is open,* $\mu(\mathbf{V}_j) > 0$ *and there exists a constant* $r > 0$ *such that the inequality*

$$\mu(\mathbf{V}_j \cap \mathbf{K}(\boldsymbol{x}, \varepsilon)) > r\mu(\mathbf{K}(\boldsymbol{x}, \varepsilon))$$

holds for any $\varepsilon \in (0, 1)$ *and any* $\boldsymbol{x} \in \mathbf{V}_j$ ($\mathbf{K}(\boldsymbol{x}, \varepsilon)$ *is a sphere with radius* ε *and with centre* $\boldsymbol{x}$ *in the space* $\mathbf{B}$).

A complex function φ *defined in* $\mathbf{B}$ *is said to have* $PS(\mathbf{M})$ *property if it is finite and continuous on any set* $\mathbf{V}_j$, $j \geqq 1$. *Similarly, a function* φ *is said to have* $SPS(\mathbf{M})$ *property if it is finite and uniformly continuous on any set* $\mathbf{V}_j$, $j \geqq 1$.

Supposition 1. *For any medium* $i = 1, 2, \ldots$, *we have*

a) *The total microscopic effective cross-section* $\sigma_{ti}(E, \omega)$ *is a real function,* $\sigma_{ti}\colon (0, \infty) \times \Omega \to (0, \infty)$, *which is bounded on the set* $(c, d) \times \Omega$ *for any* $c, d \in \in (0, \infty)$, $c < d$.

b) *The function* $\sqrt{(2E)}\, \sigma_{ti}(E, \omega)$ *has* $SPS((0, \infty) \times \Omega)$ *property and there exist bounded functions* $a_i(\omega) \geqq 0$ *and* $b_i(\omega) > 0$ *such that*

$$\lim_{j \to 0} \frac{\sqrt{(2E)}\, \sigma_{ti}}{a_i + \sqrt{(2E)}\, b_i} = \lim_{E \to 0} \frac{\sqrt{(2E)}\, \sigma_{ti}}{a_i + \sqrt{(2E)}\, b_i} = 1 \, .$$

c) *The functions* $\sqrt{(2E)}\, (\partial \sigma_{ti} / \partial \omega_j)$, $j = 1, 2, 3$, *have* $SPS((0, \infty) \times \Omega)$ *property.*

In the case of the differential effective cross-section, we use a common notation σ_{ri} for processes of scattering and of fission. Next, let $\sigma_{ri} = \sigma_{ri1} + \sigma_{ri2}$ be a decomposition of σ_{ri} and $\varphi\colon (0, \infty) \times \Omega \to \mathbf{E}_1$ a real function. For brevity we denote:

$$F_\varphi^{ik}(E, \omega, E', \omega') \equiv \sigma_{rik}(E' \to E, \ \omega' \to \omega)\, \varphi(E', \omega') \, , \quad k = 1, 2 \, ,$$

$$g_{\varphi,1}^{ik} \equiv \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega\, F_\varphi^{ik} \, , \quad g_{\varphi,2}^{ik} \equiv \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' |F_\varphi^{ik}| \, , \quad g_{\varphi,1}^{ikl} \equiv \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' \frac{\partial}{\partial \omega_l} F_\varphi^{ik} \, ,$$

$$g_{\varphi,2}^{ikl} \equiv \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' \left| \frac{\partial}{\partial \omega_l} F_\varphi^{ik} \right| \quad \text{and} \quad g_{\varphi,1}^{ikl'} \equiv \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' \frac{\partial}{\partial \omega_l'} F_\varphi^{ik} \, ,$$

$$l = 1, 2, 3 \, .$$

Supposition 2. *There exist a bounded function* $f\colon (0, \infty) \to (0, \infty)$ *having property* $SPS((0, \infty))$ *and a decomposition* $\sigma_{ri} = \sigma_{ri1} + \sigma_{ri2}$ ($r = s, f$; $i = 1, 2, \ldots$) *such that we have*

a) σ_{ri1} *is a real function,* $\sigma_{ri1}: (0, \infty) \times (0, \infty) \times \Omega \times \Omega \to [0, \infty)$. *There exist constants* ε, $R \in (0, \infty)$ *and a point* $\{E_0, \omega_0\} \in (0, \infty) \times \Omega$ *such that the inequality*

$$\int_{|E_0 - E'| < \varepsilon} \mathrm{d}E' \int_{|\omega_0 - \omega'| < \varepsilon} \mathrm{d}\omega' \, F_f^{i1} > R$$

is fulfilled for any point $\{E, \omega\} \in (0, \infty) \times \Omega$, $|E - E_0| < \varepsilon$, $|\omega - \omega_0| < \varepsilon$.

b) *The functions* $(\sigma_{ti}f)^{-1} g_{f,2}^{i1}$, $(\sigma_{ti}f)^{-1} g_{f,2}^{i1l}$ *and* $(\sigma_{ti}f)^{-1} g_{f,2}^{i1l'}$ *are bounded in* $(0, \infty) \times \Omega$ *a.e. while the functions* $(\sigma_{ti}f)^{-1} g_{f,1}^{i1l}$ *and* $(\sigma_{ti}f)^{-1} g_{f,1}^{i1l'}$ $(l, l' = 1, 2, 3)$ *have* $PS((0, \infty) \times \Omega)$ *property. Furthermore,*

$$\lim_{\delta \to 0^+} \sup_{\substack{(0,\infty)\times\Omega \\ |E_1 - E_2| < \delta}} \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' |(\sigma_{ti}f)^{-1} F_f^{i1}(E_1, \omega, E', \omega') - \\ - (\sigma_{ti}f)^{-1} F_f^{i1}(E_2, \omega, E', \omega')| = 0 \, .$$

c) *For any function* $\varphi: (0, \infty) \times \Omega \to E_1$ *for which the functions* φ/f *and* $(\partial/\partial\omega_l)(\varphi/f)$, $l = 1, 2, 3$ *are bounded a.e. and have* $PS((0, \infty) \times \Omega)$ *property, the functions* $(\sigma_{ti}f)^{-1} g_{\varphi,1}^{i2}$, $(\sigma_{ti}f)^{-1} g_{\varphi,1}^{i2l}$ *and* $(\sigma_{ti}f)^{-1} g_{\varphi,1}^{i2l'}$ $(l, l' = 1, 2, 3)$ *are bounded a.e. and have* $PS((0, \infty) \times \Omega)$ *property. Furthermore, if* $\varphi \geqq 0$ *then* $g_{\varphi,1}^{i2} \geqq 0$ *and if* φ *has* $SPS((0, \infty) \times \Omega)$ *property then* $g_{\varphi,1}^{i2}(\sigma_{ti}f)^{-1}$ *has this property, too.*

d) $$\lim_{E \to 0} g_{f,1}^{i1}(\sigma_{ti}f)^{-1} = \begin{cases} 0 & \text{for} \quad a_i(\omega) > 0 \\ d < \infty & \text{in the other case} \, , \end{cases} \quad \lim_{E \to \infty} g_{f,1}^{i1}(\sigma_{ti}f)^{-1} = 0 \, ,$$

$$\lim_{E \to 0} g_{f,1}^{i2}(f)^{-1} < \infty \quad \text{and} \quad \sup_{E,\omega} g_{f,1}^{i2}(\sigma_{ti}f)^{-1} < 1 \, .$$

We have $\sigma_{fi} = \sigma_{fi1}$ *and there exists a constant* $\gamma \in (0, \infty)$ *such that*

$$0 \leqq \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' \, \sigma_{fi}(E \to E', \, \omega \to \omega') \leqq \\ \leqq \gamma \left(\sigma_{ti}(E, \omega) - \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' \, \sigma_{si}(E \to E', \, \omega \to \omega') \right) .$$

e) *There exist a set* $\mathbf{M}_i \subset (0, \infty) \times \Omega$, $\mu(\mathbf{M}_i) > 0$ *and an integer* $n_i > 0$ *such that for any set* $\mathbf{M} \subset \mathbf{M}_i$ *and any function* $\psi: (0, \infty) \times \Omega \to [0, \infty)$, $|\psi| \leqq f$, $\psi > 0$ *on* $\mathbf{M}$, *the function*

$$G * \psi(E, \omega) = \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega' \, (\sigma_{ti}(E, \omega))^{-1} \, \sigma_{ri}(E' \to E, \, \omega' \to \omega) \, \psi(E', \omega')$$

is positive on $\mathbf{M}$, *and for any pair* $\{E, \omega\} \in \mathbf{M}_i$ *there exists an integer* $m > 0$ *such that* $(G^{(m)} * \psi)(E, \omega) > 0$. *If* $\psi = 0$ *on* $\mathbf{M}_i$ *a.e. then* $G^{(n_i)} * \psi = 0$ *on the set* $(0, \infty) \times \Omega$ *a.e.*

Supposition 3. *For any* $i = 1, 2, \ldots$ *the function* $N_i(\boldsymbol{x}): \mathbf{E}_3 \to [0, \infty)$ *is bounded and it has* $SPS(\mathbf{E}_3)$ *property while* $\partial N_i/\partial x_j$, $j = 1, 2, 3$, *have* $PS(\mathbf{E3})$ *property.*

Furthermore, for any $t \in [0, \infty)$, *the integrals*

$$\int_0^t \mathrm{d}t_1 \frac{\partial}{\partial x_j} \Big[N_i(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_1))\,\sqrt{(2E)}\,\sigma_{ti}(E, \boldsymbol{\omega})\,.$$

$$.\exp\Big(- \int_{t_1}^t \mathrm{d}t_2 \sqrt{(2E)}\,\sigma_{ti}(\varepsilon, \boldsymbol{\omega})\,N_i(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_2)) \Big) \Big],$$

$j = 1, 2, 3$, *are bounded functions of variables* $\boldsymbol{x}, E, \boldsymbol{\omega}$ *on the set* $\mathbf{E}_3 \times [0, \infty) \times \Omega$ *a.e.*

It can be easily shown that Suppositions 1 and 2 are true for all models of scattering and fission cross-sections usually employed (see e.g. [9]). As for Supposition 3, it is satisfied in all practical cases among which the case of partwise constant medium density is very important.

Definition 2. *Let* $\mathbf{D} \subset \mathbf{E}_3$ *be a set of nonzero measure with respect to* $\mathbf{E}_3$. *We shall denote by* $C_1\{f; \mathbf{D}\}$ *the linear space of complex functions* Φ *with the domain* $\mathbf{M}_0 \equiv \mathbf{D} \times (0, \infty) \times \Omega$ *and such that the function* Φ/f *is bounded on* $\mathbf{M}_0$ *a.e. and continuous in the variable* $\boldsymbol{x}$ *on* $\mathbf{D}$ *for all pairs* $\{E, \boldsymbol{\omega}\} \in (0, \infty) \times \Omega$. *The norm is defined by*

$$\|\Phi\| \equiv \operatorname{vrai\,max}_{\mathbf{M}_0} \left|\frac{\Phi}{f}\right|.$$

Next, we shall denote by $C_2\{f; \mathbf{D}\}$ *the space of functions* $\psi \in C_1\{f; \mathbf{D}\}$ *for which the functions* $(1/f)\,(\partial\psi/\partial x_i)$ $(i = 1, 2, 3)$ *are finite on* $\mathbf{M}_0$ *a.e. and for which, under the condition that the interval* $(\boldsymbol{x} - \boldsymbol{\omega}a, \boldsymbol{x} - \boldsymbol{\omega}b)$ *belongs to* $\mathbf{D}$, *the integrals*

$$\int_a^b \mathrm{d}s \frac{1}{f} \frac{\partial\psi}{\partial x_i} (\boldsymbol{x} - \boldsymbol{\omega}s,\ E, \boldsymbol{\omega})\,, \quad i = 1, 2, 3\,,$$

are finite fuuctions of the variables $\boldsymbol{x}$, E *and* $\boldsymbol{\omega}$ *on* $\mathbf{M}_0$ *a.e.*

Definition 3. *Let* $\mathbf{M}$ *be a subset of* $\mathbf{M}_0$, $\mu(\mathbf{M}) > 0$. *A linear operator* A: $C_1\{f; \mathbf{D}\} \to$ $\to C_1\{f; \mathbf{D}\}$ *is said to have* $KL(\mathbf{M})$ *property if*:

i) *for any set* $\mathbf{M}_1 \subset \mathbf{M}$, $\mu(\mathbf{M}_1) > 0$ *and any nonnegative function* $\Phi \in C_1\{f; \mathbf{D}\}$ *which is positive on the set* $\mathbf{M}_1$, *the function* $A\Phi$ *is positive on* $\mathbf{M}_1$ *and, furthermore, for any triplet* $\{\boldsymbol{x}, E, \boldsymbol{\omega}\} \in \mathbf{M}$ *there exists an integer* $m > 0$ *such that* $A^m\Phi(\boldsymbol{x}, E, \boldsymbol{\omega}) > 0$;

ii) *for any* $\Phi \in C_1\{f; \mathbf{D}\}$, $\Phi = 0$ *on* $\mathbf{M}$ *a.e., there exists an integer* n *such that* $A^n\Phi = 0$ *on* $\mathbf{M}_0$ *a.e.*

Now let φ be a function belonging to the space $C_1\{f; \mathbf{E}_3\}$. We shall use the following notation:

$$S_0\varphi \equiv \sqrt{(2E)} \sum_i N_i(\boldsymbol{x}) \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\boldsymbol{\omega}'\,\sigma_{si2}(E' \to E,\ \boldsymbol{\omega}' \to \boldsymbol{\omega})\,\varphi(\boldsymbol{x}, E', \boldsymbol{\omega}')\,,$$

$$(P\varphi)(t) \equiv \varphi(x - \sqrt{(2E)}\,\omega t, E, \omega)\,.$$
$$.\exp\left(-\int_0^t \mathrm{d}t_1 \sqrt{(2E)} \sum_i N_i(x - \sqrt{(2E)}\,\omega(t - t_1))\,\sigma_{ti}(E, \omega)\right),$$
$$(Q\varphi)(t) \equiv \int_0^t \mathrm{d}t_1 (P\varphi)(t - t_1)\,,$$
$$(R_0\varphi)(t) \equiv (QS_0\varphi)(t)\,,$$
$$(R(\eta)\,\varphi)(t) \equiv (Q(S + \eta F)\,\varphi)(t)\,, \quad t \in [0, \infty)\,.$$

Clearly, taking into account Suppositions 1–3, we may consider P, Q, R_0 and R as linear bounded and positive operators which map the set $C_1\{f; \mathrm{E}_3\}$ into itself for any fixed $t \in [0, \infty)$.

Next, let us denote
$$R_1(\delta) \equiv \delta(R(1) - R_0) + R_0$$
for given $t, \delta \in [0, \infty)$.

We see that R_1 is a linear bounded operator and, due to Supposition 2d), there exist constants $\delta_1 > 0$ and $\alpha \in (0, 1)$ such that $R_1(\delta)$ is positive while

$$R_1(\delta) f \leqq \alpha(f - Pf) \tag{4}$$

for any $\delta \in [0, \delta_1]$. Using (4) we obtain
$$R_1^n(\delta)\, Pf \leqq R_1^n(\delta)\left(f - \frac{1}{\alpha} R_1(\delta) f\right)$$
for any integer $n > 0$ so that the operator $W_1(\delta, t)$,
$$W_1(\delta, t)\,\Phi \equiv \sum_{n=0}^{\infty} (R_1^n(\delta)\, P\Phi)(t)\,, \quad t \in [0, \infty)\,, \quad \Phi \in C_1\{f; \mathrm{E}_3\}\,,$$
is bounded and positive. Clearly,
$$W_1(\delta, t) f \leqq f - \frac{1 - \alpha}{\alpha} \sum_{n=1}^{\infty} R_1(\delta) f \leqq f$$
so that
$$f \geqq W_1(\delta, t) f \geqq \ldots \geqq W_1^n(\delta, t) f \geqq \ldots \geqq 0$$
and
$$W_1^n(\delta, t)\, R_1(\delta) f \leqq \frac{\alpha}{1 - \alpha} W_1^n(\delta, t)\,(1 - W_1(\delta, t)) f\,, \quad n = 1, 2, \ldots\,.$$

Therefore

$$\sum_{n=0}^{\infty} W_1^n(\delta, t)\, R_1(\delta) f \leqq \frac{\alpha}{1 - \alpha}\,(1 - \lim_{n\to\infty} W_1^n(\delta, t)) f \leqq \frac{\alpha}{1 - \alpha} f\,. \tag{5}$$

Now, from the definition of the operator $W_1(\delta, t)$ it is seen that $\varphi = W_1(\delta, t)\,\Phi$ is a solution of the problem

$$\varphi = R_1(\delta)\,\varphi + P\Phi\,, \quad t \in [0, \infty) \tag{6}$$

where $\Phi \in C_1\{f; \mathbf{E}_3\}$ and $\varphi \in C_1\{f; \mathbf{E}_3\}$ for any $t \in [0, \infty)$ fixed. This implies that $W_1(\delta, t)$ has the semigroup property with the parameter t:

$$\begin{aligned} &W_1(\delta, 0) = I \quad \text{(the unit operator)}\,, \\ &W_1(\delta, t)\,W_1(\delta, s) = W_1(\delta, t + s)\,, \quad t, s \in [0, \infty)\,. \end{aligned} \tag{7}$$

Next, let $t_0 > 0$ be sufficiently small. Then the series

$$\left(\sum_{n=0}^{\infty} R^n(\eta)\,P\Phi\right)(t)$$

is convergent in norm for any $t \in [0, t_0]$, $\Phi \in C_1\{f; \mathbf{E}_3\}$ and it defines a linear operator $W(\eta, t)$: $C_1\{f; \mathbf{E}_3\} \to C_1\{f; \mathbf{E}_3\}$ (see e.g. [9]). Obviously, $\varphi = W(\eta, t)\,\Phi$ is a solution to the problem

$$\varphi = R(\eta)\,\varphi + P\Phi \tag{8}$$

in the interval $[0, t_0]$ where $\Phi \in C_1\{f; \mathbf{E}_3\}$ and $\varphi \in C_1\{f; \mathbf{E}_3\}$ for any $t \in [0, t_0]$ fixed. From (8) it follows that the operator $W(\eta, t)$ has the semigroup property (7) from which we infer that $W(\eta, t)$ exists as a bounded operator which maps the space $C_1\{f; \mathbf{E}_3\}$ into itself for any $t \in [0, \infty)$ and any finite complex η.

Finally, the validity of the following relations can be easily verified in the case of sufficiently small $t > 0$:

$$W(\eta, t)\,\Phi = \left(\sum_{k=0}^{\infty} \eta^k \left(\sum_{l=0}^{\infty} R^l(0)\,QF\right)^k \sum_{m=0}^{\infty} R^m(0)\,P\Phi\right)(t)\,, \tag{9}$$

$$W(\eta, t)\,\Phi = \left(\sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} R_1^l(\delta)\,(R(\eta) - R_1(\delta))\right)^k \sum_{m=0}^{\infty} R_1^m(\delta)\,P\Phi\right)(t)\,, \tag{10}$$

$$\Phi \in C_1\{f; \mathbf{E}_3\}\,, \quad \delta \in [0, \delta_0]\,, \quad |\eta| < \infty\,.$$

THEORY

In what follows we will assume that the basic suppositions 1–3 are satisfied.

Theorem 1. *Let $\psi \in C_2\{f; \mathbf{E}_3\}$ be a function for which $(1/f)(\partial\psi/\partial x_i)$, $i = 1, 2, 3$, are bounded on the set $\mathbf{E}_3 \times (0, \infty) \times \Omega$ a.e. Then Eq. (1) has a solution φ which belongs to the class $C_2\{f; \mathbf{E}_3\}$ for any fixed $t \in (0, \infty)$ and for which $\varphi(\mathbf{x}, E, \boldsymbol{\omega}, t \to 0) = \psi(\mathbf{x}, E, \boldsymbol{\omega})$. There is just one solution of the initial-value problem with such properties.*

Proof. Let φ be a solution to the initial-value problem with the properties stated in Theorem 1. Then, obviously, the integral equation

$$\varphi = \boldsymbol{R}(1)\,\varphi + \boldsymbol{P}\psi \tag{11}$$

is fulfilled for any $t \in [0, \infty)$. On the other hand, it is known that there is just one solution φ of Eq. (11) in the class $C_1\{f; \mathbf{E}_3\}$ and $\varphi = \boldsymbol{W}(1, t)\,\psi$ (see e.g. [9]). So we are to show that the solution of Eq. (11) belongs to the class $C_2\{f; \mathbf{E}_3\}$.

Supposing all necessary conditions are satisfied, we obtain

$$\frac{\partial \varphi}{\partial x_i} = \boldsymbol{R}(1)\,\frac{\partial \varphi}{\partial x_i} + \chi_i\,, \quad i = 1, 2, 3 \tag{12}$$

by differentiating (11). Here χ_i is an expression which does not contain any derivatives of φ and which depends linearly on φ and $\partial\psi/\partial x_i$ $(i = 1, 2, 3)$. Using Suppositions 1–3 we can show that $(1/f)\,\chi_i$, $i = 1, 2, 3$, are bounded functions in $\mathbf{E}_3 \times (0, \infty) \times \Omega$ a.e. for any t so that Eq. (12) has just one solution $\partial\varphi/\partial x_i$ for any i (see again [9]) and $(1/f)\,(\partial\varphi/\partial x_i)$ is bounded on the set $\mathbf{E}_3 \times (0, \infty) \times \Omega$ a.e. for any t. Therefore the derivation of Eq. (12) from Eq. (11) is justified.

Now all conditions for the differentiation of Eq. (11) by t are fulfilled. By this differentiation Eq. (1) is obtained. Clearly, $\varphi \in C_2\{f; \mathbf{E}_3\}$ for any $t \in [0, \infty)$ fixed. Q.E.D.

Let us note that Theorem 1 may be extended also to the case of an independent neutron source $Q(\boldsymbol{x}, E, \boldsymbol{\omega}, t)$. Obviously, if Q satisfies the conditions of Theorem 1 (imposed on the initial distribution ψ) for any $t \in [0, \infty)$, the proof remains without changes.

In Theorem 1, the spatial domain considered is the whole space $\mathbf{E}_3$ but, in practice, the following special cases are important:

A) The material medium is contained in a bounded convex body $\mathbf{D} \subset \mathbf{E}_3$ surrounded by vacuum. As the densities $N_i(\boldsymbol{x})$, $i = 1, 2, \ldots$, identically vanish in vacuum, the domain of the respective integrals in (6) and (8) can be restricted to $\mathbf{D}$. In this way the operators $\boldsymbol{W}_1(\delta, t)$ and $\boldsymbol{W}(\eta, t)$ change into operators $\widetilde{\boldsymbol{W}}_1$ and $\widetilde{\boldsymbol{W}}$ which map the space $C_1\{f; \mathbf{D}\}$ into itself for all $t \in [0, \infty)$. Let us recall that then the problem a) assumes the standard form: To find solution φ of Eq. (1) which belongs to class $C_2\{f; \mathbf{D}\}$ for any $t \in (0, \infty)$ fixed and such that $\varphi(\boldsymbol{x}, E, \boldsymbol{\omega}, t \to 0) = \psi$ and $\varphi(\boldsymbol{x} \in \partial\mathbf{D}, E, \boldsymbol{\omega}, t) = 0$ for $\boldsymbol{n\omega} < 0$ where $\boldsymbol{n}$ is external normal to boundary $\partial\mathbf{D}$ of $\mathbf{D}$.

B) The material medium is spread over an infinite range $\widetilde{\mathbf{D}} \subset \mathbf{E}_3$ (surrounded possibly by vacuum in one or two dimensions) but there exist a bounded and relatively compact set $\mathbf{D} \subset \mathbf{D}$ and three bounded vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ such that

$$N_i(\boldsymbol{x} \pm k\boldsymbol{b}_j) = N_i(\boldsymbol{x})\,, \quad \widetilde{\mathbf{D}} \subseteqq \bigcup_{k,l,m} \{\mathbf{D} \pm k\boldsymbol{b}_1 \pm l\boldsymbol{b}_2 \pm m\boldsymbol{b}_3\}\,, \tag{13}$$
$$\boldsymbol{x} \in \mathbf{D}\,, \quad i, j, k, l, m = 1, 2, \ldots .$$

Then, in the problems a), b) and c), such functions $\Phi \in C_1\{f; \mathbf{E}_3\}$ will play the substantial role which have the property (13) with respect to the spatial variable. In such a case $\boldsymbol{R}_1(\delta)\,\Phi$ and $\boldsymbol{R}(\eta)\,\Phi$ have the form of a convergent sum of integrals the spatial domain of which is contained in $\mathbf{D}$. In this way the operators $\boldsymbol{W}_1(\delta, t)$ and $\boldsymbol{W}(\eta, t)$ change into operators $\tilde{\boldsymbol{W}}_1^*$ and $\tilde{\boldsymbol{W}}^*$, respectively, which map the class $C_1\{f; \mathbf{D}\}$ into itself.

From now on we restrict our considerations to the geometrical situation corresponding to case A) or B). We will be looking for solutions of the problems a), b) and c) in the space $C_1\{f; \mathbf{D}\}$ where $\mathbf{D} \subset \mathbf{E}_3$ is the compact modifying the sense of the operations $\boldsymbol{R}_1(\delta), \boldsymbol{R}(\eta), \boldsymbol{S}, \ldots$ in the respective manner. For the operators $\tilde{\boldsymbol{W}}_1(\tilde{\boldsymbol{W}})$ and $\tilde{\boldsymbol{W}}_1^*(\tilde{\boldsymbol{W}})^*$ we will use the common notation $\boldsymbol{W}_1(\delta, t)\,(\boldsymbol{W}(\eta, t))$.

Theorem 2. *For any $t \in (0, \infty)$, $\delta \in [0, 1)$, $k, l = 0, 1, 2, \ldots$ and any complex number η, $|\eta| < \infty$, the operator $\boldsymbol{G}^{kl}(\delta, \eta, t)$,*

$$\boldsymbol{G}^{kl}(\delta, \eta, t)\,\Phi \equiv ((\boldsymbol{R}(\eta) - \boldsymbol{R}_1(\delta))\,(\boldsymbol{R}_1^k(\delta)\,\boldsymbol{P})^l\,(\boldsymbol{R}(\eta) - \boldsymbol{R}_1(\delta))\,\Phi)\,(t)\,, \quad \Phi \in C_1\{f; \mathbf{D}\}\,,$$

is compact as an operator mapping $C_1\{f; \mathbf{D}\}$ into itself. In the case $\eta \in \mathbf{E}_1$, $\eta \geqq \delta$ this operator is positive with a positive spectral radius and has $KL(\mathbf{M})$ property for some $\mathbf{M} \subset \mathbf{M}_0 \equiv \mathbf{D} \times (0, \infty) \times \Omega$, $\mu(\mathbf{M}) > 0$.

Proof. For simplicity let us consider $l = k = 1$ and $\varphi \in C_1\{f; \mathbf{D}\}$. We can write

(14)

$$\boldsymbol{G}^{11}(\delta, \eta, t)\,\varphi = \int_0^t \mathrm{d}t_1 \exp\left(-\int_{t_1}^t \mathrm{d}t'\,\sqrt{(2E)}\sum_i N_i(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t'))\,\sigma_{ti}(E, \boldsymbol{\omega})\right) .$$
$$.\int_0^\infty \mathrm{d}E_1 \int_\Omega \mathrm{d}\boldsymbol{\omega}_1\,\sqrt{(2E)}\sum_i N_i(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_1))\,[(\delta + 1)\,\sigma_{si1} + (\delta + \eta)\,\sigma_{fi}]\,.$$
$$.\,(E_1 \to E, \boldsymbol{\omega}_1 \to \boldsymbol{\omega})\int_0^{t_1} \mathrm{d}t_2 \exp\left(-\int_{t_2}^{t_1} \mathrm{d}t'\,\sqrt{(2E_1)}\sum_i N_i(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_1) - \right.$$
$$\left. - \sqrt{(2E_1)}\,\boldsymbol{\omega}_1(t_1 - t'))\,\sigma_{ti}(E_1, \boldsymbol{\omega}_1)\right)\sqrt{(2E_1)}\int_0^\infty \mathrm{d}E_2 \int_\Omega \mathrm{d}\boldsymbol{x}_3 \left\{\sum_i N_i(\boldsymbol{x}_3 + \boldsymbol{y})\,.\right.$$
$$.\,[\sigma_{si2} + \delta\sigma_{si1} + \delta\sigma_{fi}]\left(E_2 \to E_1, \frac{\boldsymbol{y}}{y} \to \boldsymbol{\omega}_1\right)\exp\left[-\sqrt{(2E_2)}\left(\int_0^{t_2} + \int_{t_2 - y/\sqrt{2E_2}}^{t_2}\right)\mathrm{d}t'\,.\right.$$
$$\left..\,\sigma_{ti}\left(E_2, \frac{\boldsymbol{y}}{y}\right)\sum_i N_i\left(\boldsymbol{y} + \boldsymbol{x}_3 - \sqrt{(2E_2)}\,(t_2 - t')\,\frac{\boldsymbol{y}}{y}\right)\right].\int_0^\infty \mathrm{d}E_3 \int_\Omega \mathrm{d}\boldsymbol{\omega}_3\,\frac{\sqrt{(2E)}}{y^2}\,.$$
$$\left..\sum_i N_i(\boldsymbol{x}_3)\,[(\delta + 1)\,\sigma_{si1} + (\delta + \eta)\,\sigma_{fi}]\left(E_3 \to E_2, \boldsymbol{\omega}_3 \to \frac{\boldsymbol{y}}{y}\right)\varphi(\boldsymbol{x}_3, E_3, \boldsymbol{\omega}_3)\right\}$$

where we have used $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_3 - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_1) - \sqrt{(2E_1)}\,\boldsymbol{\omega}_1(t_1 - t_2)$ and $y = |\boldsymbol{y}|$ for brevity. In general, the region of the spatial integration in (14) is $\mathbf{E}_3$. It is understood that the spatial argument of the integrand which lies outside $\mathbf{D}$

is replaced by the appropriate one which belongs to $\mathbf{D}$ according to our agreement. For brevity, let us denote by

$$\iiint \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}E_3 \, \mathrm{d}\omega_3 \, K(\boldsymbol{x}, E, \omega; \boldsymbol{x}_3, E_3, \omega_3) \, \varphi(\boldsymbol{x}_3, E_3, \omega_3)$$

the right hand side of (14). Then, almost everywhere,

$$\left| \frac{\partial}{\partial x_i} \boldsymbol{G}^{11}(\delta, \eta, t) \, \varphi \right| \leqq \iiint \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}E_3 \, \mathrm{d}\omega_3 \left| \frac{\partial}{\partial x_i} K(\boldsymbol{x}, E, \omega; \boldsymbol{x}_3, E_3, \omega_3) \right| . \, |\varphi(\boldsymbol{x}_3, E_3, \omega_3)| \leqq$$

$$\leqq C \iiint \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}E_3 \, \mathrm{d}\omega_3 \left| \frac{\partial}{\partial x_i} K(\boldsymbol{x}, E, \omega; \boldsymbol{x}_3, E_3, \omega_3) \right| f(E_3) \, , \quad i = 1, 2, 3 \, ,$$

where $C \in (0, \infty)$ is a constant. But, due to Suppositions $1-3$, the last expression is obviously bounded. So there exists a constant $C_1 \in (0, \infty)$ such that

$$\left| \frac{\partial}{\partial x_i} \boldsymbol{G}^{11}(\delta, \eta, t) \, \varphi \right| \leqq C_1 f \quad \text{a.e.} \, , \quad i = 1, 2, 3 \, . \tag{15}$$

Now, let $\{\varphi_n\}$ be a sequence of elements belonging to $C_1\{f; \mathbf{D}\}$, $\|\varphi_n\| \leqq 1$, $n = 1, 2, \ldots$, and let us set

$$\psi_n = \boldsymbol{G}^{11}(\delta, \eta, t) \, \varphi_n \, , \quad n = 1, 2, \ldots \, .$$

Formula (14) and Suppositions 1, 2d) and 3 imply existence of positive finite constants C_2 and C_3 such that

$$\lim_{E \to 0} \frac{1}{f} \psi_n \leqq C_2 \tag{16}$$

on the set $\mathbf{D} \times \Omega$ a.e. and

$$\|\psi_n\| \leqq C_3 \tag{17}$$

for all $n = 1, 2, \ldots$.

Let us consider $R \in (0, \infty)$ arbitrary and put

$$\psi'_n = \psi_n \quad \text{for} \quad E \leqq R \quad \text{and}$$
$$\psi'_n = 0 \quad \text{otherwise} \, .$$

Clearly, Supposition 2d) implies

$$\|\psi_n - \psi'_n\| \xrightarrow[R \to \infty]{} 0 \tag{18}$$

uniformly with respect to n.

Next, let us denote by Φ_n the complex function defined on the set $\mathbf{M}_R \equiv \mathbf{D} \times \times [0, R] \times \Omega$, given by

$$\Phi_n = \psi'_n \, , \quad E > 0$$
$$\Phi_n(\boldsymbol{x}, 0, \omega) = \lim_{E \to 0} \psi'_n \, , \quad n = 1, 2, \ldots \, .$$

Taking into account Suppositions 1–3 we easily see that Φ_n, $n = 1, 2, \ldots$ have $SPS(\mathbf{M}_R)$ property and that they are continuous in the variable $\boldsymbol{x}$ on $\mathbf{D}$ for all pairs $\{E, \omega\} \in [0, R] \times \Omega$. As $\mathbf{M}_R$ is a compact set, there exists a finite decomposition $\{\mathbf{V}_i\}_{i=0}^N$ of $\mathbf{M}_R$ corresponding to $PS(\mathbf{M}_R)$ property of the functions Φ_n and, clearly, this decomposition is common for all these functions.

Now, using appropriately Suppositions 1–3 and inequalities (15)–(17), the relation

$$\lim_{\delta \to 0^+} \sup_{\substack{z, z' \in \mathbf{V}_i \\ |z - z'| < \delta \\ n \geqq 1}} |\Phi_n(z) - \Phi_n(z')| = 0\,, \quad i = 1, 2, \ldots, N \tag{19}$$

can be easily verified. Here, the notation $z = \{\boldsymbol{x}, E, \omega\}$ is used for brevity.

In what follows we will proceed in a way similar to the proof of Ascoli-Arzela's theorem ([10], p. 125):

By the properties of the decomposition $\{\mathbf{V}_i\}_{i=0}^N$ there exists a countable set $\mathbf{I} \subset \bigcup_{i=1}^N \mathbf{V}_i$ which is dense in $\mathbf{M}_R$. Then the inequalities (16) and (17) imply existence of $\{\Phi_{n_1}\}$, a subsequence of $\{\Phi_n\}$ which is convergent on $\mathbf{I}$. Let us consider $i \geqq 1$ fixed and $\varepsilon \in (0, \infty)$. By Rel. (19) there exists $\delta > 0$ such that

$$|\Phi_{n_1}(z) - \Phi_{n_1}(z')| < \varepsilon$$

for all n, and any $z, z' \in \mathbf{V}_i$, $|z - z'| < \delta$. Next, as $\mathbf{M}_R$ is a compact set, there exists a finite set $\mathbf{I}_\delta \subset \mathbf{I}$, $\mathbf{I}_\delta = \bigcup_n z_j$ such that

$$\min_{z_j \in \mathbf{I}_\delta} |z - z_j| < \delta$$

for any $z \in \mathbf{M}_R$. So there exists n_0 such that

$$|\Phi_{n_1}(z_j) - \Phi_{m_1}(z_j)| < \varepsilon$$

for any $z_j \in \mathbf{I}_\delta$ and any n_1, $m_1 > n_0$. Therefore, for any $z \in \mathbf{V}_i$, there exists $z_{j_0} \in \mathbf{I}_\delta$ such that

$$|\Phi_{n_1}(z) - \Phi_{m_1}(z)| \leqq |\Phi_{n_1}(z) - \Phi_{n_1}(z_{j0})| + \\ + |\Phi_{n_1}(z_{j0}) - \Phi_{m_1}(z_{j0})| + |\Phi_{m_1}(z_{j0}) - \Phi_{m_1}(z)| < 3\varepsilon\,.$$

So we have proved that $\{\Phi_{n_1}\}$ converges to a function Φ on any set $\mathbf{V}_i$, $i \geqq 1$. Since Φ_{n_1} has $SPS(\mathbf{M}_R)$ property for any n_1 the function Φ has $PS(\mathbf{M}_R)$ property. Furthermore, Φ is continuous in the variable $\boldsymbol{x}$ in $\mathbf{D}$ as is seen from (14) and from the basic Suppositions 1–3. Then compactness of the operator $\boldsymbol{G}^{11}(\delta, \eta, t)$ is proved by (18). Compactness of the operator $\boldsymbol{G}^{kl}(\delta, \eta, t)$ for the other k, l can be proved in the same way. Finally, positivity of the operator $\boldsymbol{G}^{kl}(\delta, \eta, t)$ for $\eta \geqq \delta$ is obvious while positivity of its spectral radius is a consequence of Suppositions 2a) and 3. Theorem 2 is proved.

Theorem 3. *Let a linear operator* $A: C_1\{f; \mathbf{D}\} \to C_1\{f; \mathbf{D}\}$ *be bounded and positive with a positive spectral radius* $r(A)$ *and let it have property* $KL(\mathbf{M})$, $\mathbf{M} \subset \mathbf{M}_0$, $\mu(\mathbf{M}) > 0$. *Next, let there exist* $r_1 \in (0, r(A))$ *such that the set*

$$\sigma_{\text{per}} \equiv \mathscr{E}\{\lambda \in \sigma(A);\ |\lambda| > r_1\}$$

consists only of isolated points at which the operators $(\lambda - A)^{-1}$ *and* $(\lambda - A^*)^{-1}$ *have poles* (A^* *being the adjoint operator to* A). *Then*:

a) $r(A)$ *is an eigenvalue of the operators* A *and* A^*. *The eigenfunction* φ_0 *and the eigenfunctional* ψ_0 *corresponding to it are nonnegative and positive, respectively.*

b) *The function* φ_0 *is positive on the set* $\mathbf{M}$ *a.e. and it is the only linearly independent eigenfunction of* A *corresponding to the eigenvalue* $r(A)$. *Furthermore,*

$$\langle \psi_0, \varphi_0 \rangle > 0\,.$$

c) $r(A) > \sup\limits_{\lambda \in \sigma(A) \div r(A)} |\lambda|$.

d) *There is no other nonnegative nontrivial eigenfunction corresponding to a nonzero eigenvalue of operator* A.

Proof. Assertion a) is a direct consequence of Theorem 6.1 of paper [4] (though this theorem is formulated for compact operators its proof remains true also in our case). Next, since $C_1\{f; \mathbf{D}\} \subset L_\infty(\mathbf{M}_0)$, there exists a continuous functional $\psi \in L^*_\infty(\mathbf{M}_0)$ such that

$$\langle \psi, \Phi \rangle = \langle \psi_0, \Phi \rangle$$

holds for all $\Phi \in C_1\{f; \mathbf{D}\}$ ([10], IV, §5, Theorem 1). Due to the representation of $L^*_\infty(\mathbf{M}_0)$, there exists a complex function $\tilde{\psi}$ of the set which is finitely additive and μ – absolutely continuous while

$$\langle \psi, \tilde{\phi} \rangle = \int_{\mathbf{M}_0} \tilde{\phi}\tilde{\psi}(d\mu)$$

for all $\tilde{\phi} \in L_\infty(\mathbf{M}_0)$ ([10], IV, §9).

Let Φ be an element of $C_1\{f; \mathbf{D}\}$ and set

$$\Phi = \Phi_1 + \Phi_2$$

where $\Phi_1(\Phi_2)$ vanishes on the set $\mathbf{M}_0 \dot{-} \mathbf{M}(\mathbf{M})$ a.e. By $KL(\mathbf{M})$ property of the operator A we have

$$(20) \qquad (r(A))^m \langle \psi_0, \Phi \rangle = (r(A))^m \{\langle \psi_0, \Phi_1 \rangle + \langle \psi_0, \Phi_2 \rangle\} =$$
$$= \langle \psi_0, A^m\Phi_1 \rangle + \langle \psi_0, A^m\Phi_2 \rangle = \int_{\mathbf{M}_0} \tilde{\psi} A^m \Phi_1 = \int_{\mathbf{M}} \tilde{\psi} A^m \Phi_1 = \langle \psi_0, \Phi_1 \rangle (r(A))^m$$

for some integer $m > 0$. But ψ_0 is a nontrivial positive functional and $\Phi \in C_1\{f; \mathbf{D}\}$ is arbitrary. Therefore, by (20), there exists a set $\mathbf{P} \subset \mathbf{M}$, $\mu(\mathbf{P}) > 0$ such that $\tilde{\psi} > 0$ on $\mathbf{P}$. Obviously $\tilde{\psi} \geqq 0$ on the set $\mathbf{M}_0$.

Now, since $\varphi_0 \geqq 0$ is nontrivial and the operator A has $KL(\mathbf{M})$ property, the equation

$$r(A)\,\varphi_0 = A\varphi_0 \tag{21}$$

implies that $\varphi_0 > 0$ on $\mathbf{M}$ a.e. Therefore

$$\langle \psi_0, \varphi_0 \rangle > 0\,. \tag{22}$$

Let us suppose that besides φ_0 Eq. (21) has another independent solution $\varphi_1 \in \in C_1\{f; \mathbf{D}\}$. We take $\varphi = a\varphi_0 + \varphi_1$ where, by (22), the constant a is chosen so that $\langle \psi_0, \varphi \rangle = 0$. By Rel. (20), taking into account $KL(\mathbf{M})$ property of A, there exist disjoint sets $\mathbf{P}_1 \subset \mathbf{M}$, $\mathbf{P}_2 \subset \mathbf{M}$, $\mu(\mathbf{P}_1) \neq 0$, $\mu(\mathbf{P}_2) \neq 0$ and a decomposition

$$\varphi = \varphi^+ - \varphi^-$$

where $\varphi^+ > 0$ ($\varphi^- > 0$) on the set $\mathbf{P}_1(\mathbf{P}_2)$ a.e. and $\varphi^+ = 0$ ($\varphi^- = 0$) on the set $\mathbf{M} \dot{-} \mathbf{P}_1$ ($\mathbf{M} \dot{-} \mathbf{P}_2$). We have

$$|\varphi| = \varphi^+ + \varphi^- \quad \text{on } \mathbf{M} \text{ a.e.}$$

and

$$|A\varphi| < A|\varphi|$$

on a set $\mathbf{P}_3 \subset \mathbf{M}$, $\mu(\mathbf{P}_3) > 0$. (Clearly, we obtain a contradiction in the case of equality because then $\varphi^+(\varphi^-)$ satisfies (21) so that $\varphi^+ > 0$ ($\varphi^- > 0$) on the set $\mathbf{M}$ a.e.) Then, by Rel. (20),

$$\langle \psi_0, |A\varphi| \rangle < \langle \psi_0, A|\varphi| \rangle = r(A)\,\langle \psi_0, |\varphi| \rangle = \langle \psi_0, |A\varphi| \rangle$$

which is a contradiction. Assertion b) is proved.

To prove assertion c), we will suppose on the contrary that there exists an eigenvalue ν to A which corresponds to the eigenfunction φ,

$$\nu = r(A)\,e^{i\zeta}\,, \quad \zeta \in \mathbf{E}_1\,, \quad \zeta \neq \pm 2k\pi\,, \quad k = 0, 1, 2, \ldots\,.$$

Then

$$|\nu|\,|\varphi| = |A\varphi| \leqq A|\varphi|$$

and

$$|\nu|\,\langle \psi_0\, |\varphi| \rangle \leqq \langle \psi_0, A|\varphi| \rangle = |\nu|\,\langle \psi_0, |\varphi| \rangle\,.$$

This inequality and $KL(\mathbf{M})$ property of A yield

$$|\nu|\,|\varphi| = A|\varphi|$$

so that $\varphi = \varphi_0 e^{i\Theta}$ where Θ is a real function of the variables $\boldsymbol{x}$, E and $\boldsymbol{\omega}$. Further,

$$Ae^{i\Theta}\varphi_0 = |\nu|\,e^{i(\zeta+\Theta)}\varphi_0 = e^{i(\zeta+\Theta)}A\varphi_0$$

and, therefore

$$\left(A - e^{-i(\zeta+\Theta)}Ae^{i\Theta}\right)\varphi_0 = 0\,.$$

Considering the real part of the last equation and taking into account $KL(\mathbf{M})$ property of the operator A we obtain $\zeta = 0$ and $\Theta = \text{const}$, which show that $r(A)$ is the only eigenvalue of A on the circle $|v| = r(A)$ (see also [6]).

Finally, let $\tilde{v} \neq r(A)$ be another eigenvalue of the operator A, $\tilde{v} \neq 0$, and let $\tilde{\varphi} \geqq 0$ be the eigenfunction corresponding to it. Due to $KL(\mathbf{M})$ property of the operator A, it is clear that $\tilde{\varphi} > 0$ on a set $\tilde{\mathbf{P}} \subset \mathbf{M}$, $\mu(\tilde{\mathbf{P}}) > 0$. On the other hand,

$$\tilde{v}\langle \psi_0, \tilde{\varphi} \rangle = \langle \psi_0, A\tilde{\varphi} \rangle = r(A) \langle \psi_0, \tilde{\varphi} \rangle$$

so that

$$\langle \psi_0, \tilde{\varphi} \rangle = 0$$

which is a contradiction. Theorem 3 is proved.

Theorem 4. *Let the spectral radii of the operators* $\boldsymbol{W}(1, 1)$ *and* $\boldsymbol{W}_1(0, 1)$ *satisfy the inequality*

$$\lambda_0 \equiv \lg r(\boldsymbol{W}(1, 1)) > \lg r(\boldsymbol{W}_1(0, 1)) .$$

Then

A) *For* $\lambda = \lambda_0$ *there exists just one linearly independent nontrivial solution* φ_0 *of the problem* (2) *in the class* $C_2\{f; \mathbf{D}\}$, *and the nontrivial solution* ψ_0 *of the problem adjoint to* (2) *in the class* $C_2^*\{f; \mathbf{D}\}$. *The function* φ_0 *is nonnegative and there exists a set* $\mathbf{M} \subset \mathbf{M}_0$, $\mu(\mathbf{M}) > 0$ *such that* $\varphi_0 > 0$ *on* $\mathbf{M}$ *a.e. The eigenfunctional* ψ_0 *is positive.*

B) *For a complex number* λ, $\lambda \neq \lambda_0$, $Re\,\lambda > lg(r(\boldsymbol{W}_1(0, 1))$ *there exists no nonnegative eigensolution* $\varphi \in C_2\{f; \mathbf{D}\}$ *of the problem* (2) *which is nontrivial on the set* $\mathbf{M}$. *Furthermore, there is only the trivial solution to the problem* (2) *in the class* $C_2\{f; \mathbf{D}\}$ *for any* $\lambda \neq \lambda_0$, $\text{Re}\,\lambda \geqq \lambda_0$.

Proof. Let $\mathbf{C}$ denote the complex plane and

$$\mathbf{Z}(t) \equiv \mathscr{E}\{v \in \mathbf{C};\ |v| > r(\boldsymbol{W}_1(0, t)\} \cap \sigma(\boldsymbol{W}(1, t)) , \quad t \in (0, \infty) ,$$
$$\mathbf{\Lambda} \equiv \mathscr{E}\{\lambda;\ \lambda = \lg v,\ v \in \mathbf{Z}(1)\} .$$

Using the semigroup properties (7) of the operators $\boldsymbol{W}(1, t)$ and $\boldsymbol{W}_1(0, t)$, $t \in (0, \infty)$, and the spectral mapping theorem, we immediately see that

$$\mathbf{Z}(t) = \mathbf{Z}(1))^t = \exp(\mathbf{\Lambda} t) . \tag{23}$$

Let $v \in \mathbf{C}$, $|v| > r(\boldsymbol{W}_1(0, t))$ be arbitrary and $t \in (0, \infty)$ fixed.

We can write

$$(v - \boldsymbol{W}(1, t))^{-1} = [(v - \boldsymbol{W}_1(0, t))(\boldsymbol{I} - \boldsymbol{B}(v, t))]^{-1}$$

where

$$\boldsymbol{B}(v, t) = (v - \boldsymbol{W}_1(0, t))^{-1}(\boldsymbol{W}(1, t) - \boldsymbol{W}_1(0, t)) .$$

If the parameter t is sufficiently small the operator $\boldsymbol{B}^2(v, t)$ can be expressed as a series (convergent in norm) of operators having the form of the product of a linear bounded operator with a power of operator $\boldsymbol{G}^{kl}(0, 1, t)$. Therefore, by Theorem 2, $\boldsymbol{B}^2(v, t)$ is a compact operator for any t sufficiently small and $|v| > r(\boldsymbol{W}_1(0, t))$. Clearly this operator is holomorphic in the variable v for $|v| > r(\boldsymbol{W}_1(0, t))$.

Since $\|\boldsymbol{B}(v, t)\| \xrightarrow[|v| \to \infty]{} 0$ we infer ([11], VII, § 6, Lemma 13):

The operator $(\boldsymbol{I} - \boldsymbol{B}^2(v, t))^{-1}$ is bounded in the complex set $|v| > r(\boldsymbol{W}_1(0, 1))$ except for a set $\mathbf{H}(t)$ of isolated points. As $\boldsymbol{B}^2$ is a compact operator, 1 is an eigenvalue of $\boldsymbol{B}^2$ with a finitedimensional eigenspace for any $v \in \mathbf{H}(t)$.

By the spectral mapping theorem, the same is true for the operator $\boldsymbol{B}(v, t)$ and we have $\mathbf{H}(t) \equiv \mathbf{Z}(t) \not\equiv \emptyset$ by the assumpion of Theorem 4.

Now all necessary conditions are satisfied so that Theorem 3 holds for the operator $\boldsymbol{W}(1, t)$. So to prove Theorem 4, we are to show that the problems

$$\lambda\varphi = (\boldsymbol{T} + \boldsymbol{S} + \boldsymbol{F})\,\varphi\,, \quad \varphi \in C_2\{f; \mathbf{D}\} \tag{24}$$

and

$$e^{\lambda t}\varphi = \boldsymbol{W}(1, t)\,\varphi\,, \quad \varphi \in C_1\{f; \mathbf{D}\} \tag{25}$$

are equivalent for $\operatorname{Re} \lambda > \lg r(\boldsymbol{W}_1(0, 1))$.

First, let φ be a solution to the problem (25) and put $\Phi = e^{\lambda t}\varphi$, $\boldsymbol{P}_1 = \boldsymbol{P}e^{-\lambda t}$. Let us confine ourselves to the case $t \in (0, t_0)$, t_0 being sufficiently small. Clearly

$$\Phi = \boldsymbol{R}(1)\,\Phi + \boldsymbol{P}_1\Phi\,. \tag{26}$$

On the other hand, we have

$$\varphi = \boldsymbol{B}(e^{\lambda t}, t)\,\varphi$$

and it is seen that $\boldsymbol{B}^2(e^{\lambda t}, t)$ is a compact integral operator:

$$\boldsymbol{B}^2(e^{\lambda t}, t)\,\tilde{\Phi} \equiv \int_{\mathbf{M}_0} \mathrm{d}\boldsymbol{x}'\,\mathrm{d}E'\,\mathrm{d}\boldsymbol{\omega}'\,K_B(\boldsymbol{x}, E, \boldsymbol{\omega}; \boldsymbol{x}', E', \boldsymbol{\omega}'; \lambda, t)\,\tilde{\Phi}(\boldsymbol{x}', E, \boldsymbol{\omega}')\,, \quad \tilde{\Phi} \in C_1\{f; \mathbf{D}\}\,. \tag{27}$$

In a similar way as in the case of inequality (15), we find

$$\left|\frac{\partial}{\partial x_i}\,\boldsymbol{B}^2\tilde{\Phi}\right| \leqq \int_{\mathbf{M}_0} \mathrm{d}\boldsymbol{x}'\,\mathrm{d}E'\,\mathrm{d}\boldsymbol{\omega}\left|\frac{\partial}{\partial x_i}\,K_B(\boldsymbol{x}, E, \boldsymbol{\omega}; \boldsymbol{x}', E', \boldsymbol{\omega}'; \lambda, t)\right| . \left|\tilde{\Phi}(\boldsymbol{x}', E', \boldsymbol{\omega}'\right| \leqq \leqq C\,.\,f\,.\,\|\tilde{\Phi}\| \quad \text{a.e.} \tag{28}$$

($i = 1, 2, 3$) in virtue of Suppositions 1 – 3. Here $C \in (0, \infty)$ is a constant.

Clearly $\varphi \in C_2\{f; \mathbf{D}\}$ for any $t \in (0, \infty)$ fixed on the basis of inequality (28) and of Suppositions 1 – 3.

Now, differentiating Eq. (26) by t, we obtain

$$\frac{\partial\Phi}{\partial t} = \lambda\Phi = (\boldsymbol{T} + \boldsymbol{S} + \boldsymbol{F})\,\Phi$$

so that φ is a solution to the problem (24).

Conversely, let φ be a solution to problem (24). Then Eq. (26) is satisfied by $\Phi = e^{+\lambda t}\varphi$. The solution of this equation by iterations has the form $\Phi = \boldsymbol{W}(1, t)\,\varphi$ so that φ is the solution to the problem (25). Theorem 4 is proved.

Corollary. *Let $\psi \in C_2\{f; \mathbf{D}\}$ be a function for which $1/f(\partial\psi/\partial x_i)$ $(i = 1, 2, 3)$ are bounded on the set $\mathbf{M}_0$ a.e., and let the assumptions of Theorem 4 be satisfied. Then*

a) *Equation (1) has just one solution φ for which $\varphi(\boldsymbol{x}, E, \boldsymbol{\omega}, t \to 0) = \psi$ and $\varphi \in C_2\{f; \mathbf{D}\}$ for any $t \in (0, \infty)$ fixed.*

b) *For $\lambda = \lambda_0$ there exists just one linearly independent nontrivial solution $\varphi_0 \in C_2\{f; \mathbf{D}\}$ of the problem (2) and a nontrivial solution ψ_0 of the problem adjoint to (2). The function φ_0 is nonnegative and $\varphi_0 > 0$ on some set $\mathbf{M} \subset \mathbf{M}_0$, $\mu(\mathbf{M}) > 0$. The functional ψ_0 is positive. Furthermore,*

$$\lim_{t\to\infty} e^{-\lambda_0 t}\varphi(\boldsymbol{x}, E, \boldsymbol{\omega}, t) = \langle\psi_0, \psi\rangle\, \varphi_0(\boldsymbol{x}, E, \boldsymbol{\omega})\,.$$

Proof. Assertion a) is a consequence of Theorem 1 while the first part of assertion b) is a consequence of Theorem 4. Let us denote by $\boldsymbol{P}_0$ a linear operator, $\boldsymbol{P}_0\colon C_1\{f; \mathbf{D}\} \to C_1\{f; \mathbf{D}\}$,

$$\boldsymbol{P}_0\chi \equiv \chi - \varphi_0\langle\psi_0, \chi\rangle\,.$$

Without any restriction we set

$$\langle\psi_0, \varphi_0\rangle = 1$$

(see (22)) so that

$$\boldsymbol{P}_0^2 = \boldsymbol{P}_0\,,$$

$$\boldsymbol{P}_0\boldsymbol{W}(1, t) = \boldsymbol{W}(1, t)\,\boldsymbol{P}_0\,, \quad t \in [0, \infty)\,.$$

Furthermore, it is possible to verify that the point spectra of the operators $\boldsymbol{W}(1, t)$ and $\boldsymbol{W}(1, t)\,\boldsymbol{P}_0$ coincide except the eigenvalue $e^{\lambda_0 t}$ (see (23) and e.g. [4], § 6). Then, for $t > 0$,

$$1 > \frac{r(\boldsymbol{W}(1, t)\,\boldsymbol{P}_0)}{r(\boldsymbol{W}(1, t))} = \lim_{n\to\infty}\left(\frac{\|(\boldsymbol{W}(1, t)\,\boldsymbol{P}_0)^n\|}{e^{\lambda_0 n t}}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\|\boldsymbol{W}(1, tn)\,\boldsymbol{P}_0\|}{e^{n\lambda_0 t}}\right)^{1/n}$$

and, therefore,

$$\lim_{t\to\infty} e^{-\lambda_0 t}\|\boldsymbol{W}(1, t)\,\boldsymbol{P}_0\| = 0\,. \tag{29}$$

We have

$$\varphi = \boldsymbol{W}(1, t)\,\psi$$

and

$$\psi = \boldsymbol{P}_0\psi + \varphi_0\langle\psi_0, \psi\rangle$$

which, together with (29), prove assertion b).

Theorem 5. *Let the density N_f of the fission medium be positive on a set $\mathbf{D}_f \subset \mathbf{D}$, $\mu(\mathbf{D}_f) > 0$. Then there exists a real value η_0 of the parameter η for which Eq.* (3) *has just one nonnegative solution $\varphi_0 \in C_2\{f; \mathbf{D}\}$, $\|\varphi_0\| = 1$, which is positive on a set of nonzero measure with respect to $\mathbf{M}_0$. There exists no other complex number η for which Eq.* (3) *would have a solution with such properties.*

Proof. Let us consider a complex number η, real number $\delta \in (0, \min(\delta_1, 1/\gamma, 1)$ (see (4) and Supposition 2d)) and introduce an operator $W_2(\eta, t)$,

$$W_2(\eta, t) \equiv \sum_{n=0}^{\infty} W_1^n(\delta, t)\,(W(\eta, t) - W_1(\delta, t))\,, \quad t \in (0, \infty)\,. \tag{30}$$

We easily find

$$W_2(\eta, t) = \sum_{n=0}^{\infty} W_1^n(\delta, t)\,[(R(\eta) - R_1(\delta))\,W(\eta, t) + \sum_{m=1}^{\infty} R_1^m(\delta)\,(R(\eta) - R_1(\delta))\,W(\eta, t)] \tag{31}$$

so that this operator is bounded and maps the class $C_1\{f; \mathbf{D}\}$ into itself for $|\eta| < \infty$ and t fixed.

Let $\eta \in \mathbf{E}_1$, $\eta \geqq \delta$. By Supposition 2d) there exists a constant $C(\eta) \in (0, \infty)$ such that

$$R(\eta)\,|\varphi| < C(\eta)\,R_1(\delta)\,|\varphi| \tag{32}$$

for any $\varphi \in C_1\{f; \mathbf{D}\}$. The operator $W_2(\eta, t)$ is positive and has a positive spectral radius, as follows from (30)–(32) and Supposition 2a). Furthermore, using (31) and Theorem 2, we find that the operator $W_2^2(\eta, t)$ is compact and there exists a set $\mathbf{M}(\eta, t) \subset \mathbf{M}_0$, $\mu(\mathbf{M}) > 0$ such that this operator has $KL(\mathbf{M})$ property. So Theorem 3 may be applied to the operator $W_2(\eta, t)$ provided $\eta \geqq \delta$, $t > 0$. Again

$$W_2^2(\eta, t)\,\Phi \in C_2\{f; \mathbf{D}\} \tag{33}$$

for any $\Phi \in C_1\{f; \mathbf{D}\}$ (the operator $W_2^2(\eta, t)$ can be written in the form (27) and Ineqs. (28) hold similarly to the case of the operator $B^2(v, t)$).

Now let φ_η be a nonnegative eigensolution to the problem

$$r(W_2(\eta, t))\,\varphi = W_2(\eta, t)\,\varphi\,, \quad \varphi \in C_1\{f; \mathbf{D}\}\,, \quad t \in (0, \infty)\,. \tag{34}$$

Obviously φ_η is positive on a set $\mathbf{M} \subset \mathbf{M}_0$, $\mu(\mathbf{M}) > 0$. Furthermore,

$$r(W_2(\eta_1, t)) = \lim_{n\to\infty} \|W_2^n(\eta_1, t)\|^{1/n} \geqq \lim_{n\to\infty} \frac{\|W_2^n(\eta_1, t)\,\varphi_{\eta_2}\|^{1/n}}{\|\varphi_{\eta_2}\|^{1/n}} \geqq \geqq \lim_{n\to\infty} \frac{\|W_2^n(\eta_2, t)\,\varphi_{\eta_2}\|^{1/n}}{\|\varphi_{\eta_2}\|^{1/n}} = r(W_2(\eta_2, t)) \tag{35}$$

for any pair $\eta_1, \eta_2 \in \mathbf{E}_1$, $\eta_1 > \eta_2 \geqq \delta$ in virtue of (31) and

$$\lim_{\eta\to+\infty} r(W_2(\eta, t)) = \infty\,. \tag{36}$$

We will show that

(37) $$r(W_2(\delta, t)) < 1\,.$$

Let φ_δ be a nonnegative eigensolution to the problem (34) for $\eta \simeq \delta$. Clearly this function is positive on a set $\mathbf{M} \subset \mathbf{M}_0$, $\mu(\mathbf{M}) > 0$ and, in virtue of (33), (34) and (28), the function $W(\delta, t)\,\varphi_\delta$ is a solution to the initial-value problem

$$\frac{\partial \varphi}{\partial t} = (T + S + \delta F)\,\varphi\,, \quad \varphi(x, E, \omega, t \to 0) = \varphi_\delta\,,$$

$\varphi \in C_2\{f; \mathbf{D}\}$ for any $t \geqq 0$ fixed.

Therefore

$$\frac{\partial}{\partial t}\int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\, W(\delta, t)\,\varphi_\delta = -\int_{\partial\mathbf{D}} \mathrm{d}x \int_0^\infty \mathrm{d}E \int_\Omega \mathrm{d}\omega\,\sqrt{(2E)}\,\omega n(x)\,W(\delta, t)\,\varphi_\delta(x, E, \omega) -$$

$$- \int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\,\sqrt{(2E)}\sum_i N_i(x)\,\sigma_{ti}(E, \omega)\,W(\delta, t)\,\varphi_\delta(x, E, \omega) +$$

$$+ \int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega (S + \delta F)\,W(\delta, t)\,\varphi_\delta(x, E, \omega)$$

where $n(x)$ means the outer normal to the surface $\partial\mathbf{D}$ of the region $\mathbf{D}$ at the point x.

The first term on the right hand side of this equation is nonpositive due to our agreement A) and B). Next, for any $\delta > 0$ sufficiently small, the remaining part of the right hand side is negative in virtue of Supposition 2d). So

$$\int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\, W(\delta, t)\,\varphi_\delta(x, E, \omega) < \int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\,\varphi_\delta(x, E, \omega)$$

for any $t > 0$. Similarly it can be shown that

$$\int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\, W_1(\delta, t)\,\varphi_\delta(x, E, \omega) \leqq \int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\,\varphi_\delta(x, E, \omega)$$

for small $\delta > 0$ and $t \geqq 0$.

Now, from (34) we have

$$r(W_2(\delta, t))\,\varphi_\delta = W(\delta, t)\,\varphi_\delta + (r(W_2(\delta, t)) - 1)\,W_1(\delta, t)\,\varphi_\delta$$

which, provided $r(W_2) \geqq 1$, yields

$$r(W_2)\int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\,\varphi_\delta < \int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega[\varphi_\delta + (r(W_2) - 1)\,\varphi_\delta] =$$

$$= r(W_2)\int_{\mathbf{M}_0} \mathrm{d}x\,\mathrm{d}E\,\mathrm{d}\omega\,\varphi_\delta\,.$$

This is a contradiction and, therefore, Ineq. (37) holds.

On the other hand, $r(W_2^2(\eta, t)$ is continuous in the interval $[\delta, \infty)$ ([11], VII, § 6, Theorem 9). Therefore, by virtue of (36) and (37), there exists a point $\eta_0 \in [\delta, \infty)$ such that

$$r(W_2^2(\eta_0, t)) = 1 , \quad t > 0 . \tag{38}$$

We will show that there is only one value of the parameter $\eta \in [\delta, \infty)$ which satisfies (38). Indeed, if (38) holds for $\eta_1 \neq \eta_0$ then, by (35), this equation holds on the whole interval $[\eta_0, \eta_1]$ (or $[\eta_1, \eta_0]$ if $\eta_1 < \eta_0$). But, in virtue of (31), the operator $W_2^2(\zeta, t)$ is holomorphic in the variable ζ on the set $\operatorname{Re} \zeta > \delta$ which together with Ineq. (37) leads to contradiction ([11], VII, § 6, Lemma 13).

Let us denote by φ_0 a nontrivial nonnegative eigensolution of the problem (34), and by ψ_0 the positive eigenfunctional corresponding to the problem adjoint to (34) for $\eta = \eta_0$. We can write

$$(\boldsymbol{I} - W_1(\delta, t))(\boldsymbol{I} - W_2(\eta_0, t)) = \boldsymbol{I} - W(\eta_0, t)$$

so that φ_0 is a solution of the problem

$$\varphi = W(\eta_0, t)\,\varphi , \quad \varphi \in C_1\{f; \mathbf{D}\} . \tag{39}$$

We have $\varphi_0 \in C_2\{f; \mathbf{D}\}$ and, as Theorem 3 holds for $W(\eta_0, t)$, this is the only nontrivial and linearly independent solution to the problem (39).

Next, let us suppose that there exists a solution φ to the problem

$$\varphi = W(\eta, t)\,\varphi , \quad 0 \leqq \varphi \in C_1\{f; \mathbf{D}\} \tag{40}$$

and $\varphi > 0$ on some set $\mathbf{M}_\eta \subset \mathbf{M}_0$, $\mu(\mathbf{M}_\eta) > 0$ and $\eta \neq \eta_0$. Clearly, without any restrictions, we can confine ourselves to the case $\eta \in \mathbf{E}_1$. We have

$$\varphi - \varphi_0 = W(\eta_0, t)(\varphi - \varphi_0) + (W(\eta, t) - W(\eta_0, t))\,\varphi$$

and the condition of solvability of this problem with respect to the function $\varphi - \varphi_0$ gives

$$0 = \langle \psi_0, W^n(\eta_0, t)(W(\eta, t) - W(\eta_0, t))\,\varphi \rangle , \quad n = 0, 1, \ldots .$$

But, by our assumption, $(W(\eta, t) - W(\eta_0, t))\,\varphi \neq 0$ on a set $\mathbf{M}_1 \subset \mathbf{M}$, $\mu(\mathbf{M}_1) > 0$ and, since the operator $W(\eta_1, t) - W(\eta_2, t)$ is positive for any $\eta_1 > \eta_2 \geqq \delta$, we necessarily have

$$\langle \psi_0, W^n(\eta_0, t)(W(\eta, t) - W(\eta_0, t))\,\varphi \rangle \neq 0 , \quad n = 0, 1, \ldots$$

which is a contradiction. Theorem 5 is proved because any solution $\varphi \in C_2\{f; \mathbf{D}\}$ of the problem (3) obviously fulfils (40).

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Souhrn

O TŘECH PROBLÉMECH Z TEORIE PŘENOSU NEUTRONŮ

JAN KYNCL

Jde o úlohu s počáteční podmínkou, problém asymptotického chování řešení úlohy s počáteční podmínkou v čase a problém kritičnosti. Zmíněné úlohy jsou studovány pro případ lineární Boltzmannovy rovnice a konečného i nekonečného prostředí. Prostor funkcí, ve kterém jsou úlohy řešeny, je vybrán tak, aby byla zahrnuta co nejširší oblast fyzikálních situací. Používaným matematickým aparátem jsou teorie pozitivních ohraničených operátorů a teorie semigrup. Hlavní výsledky jsou shrnuty do tří základních tvrzení.

Резюме

О ТРЕХ ПРОБЛЕМАХ ТЕОРИИ ПЕРЕНОСА НЕЙТРОНОВ

JAN KYNCL

В данной работе рассмотрены проблема с начальным условием, проблема асимптотического поведения ее решения во времени и проблема критичности. Эти проблемы изучены для случая линейного уравнения Больцмана и для ограниченной или неограниченной среды. Пространство функций, в котором эти проблемы решены, выбрано таким образом, чтобы включить как можно наиболее широкий класс физических ситуаций. Математическим средством служат теория положительных ограниченных операторов и теория полугрупп. Главные результаты подведены в трех основных теоремах.

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