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NON-POLYCONVEXITY OF THE STORED ENERGY FUNCTION OF A SAINT VENANT-KIRCHHOFF MATERIAL

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Summary. A direct proof of the non-polyconvexity of the stored energy function of a Saint Venant-Kirchhoff material is given by means of a simple counter-example.

Keywords: polyconvexity, stored energy function, Saint Venant-Kirchhoff material.

In his famous paper [1] dealing with existence theorems in nonlinear elasticity, John Ball introduced the notion of polyconvexity and proved the existence of an equilibrium state – understood as a minimizer of the total energy function – for hyperelastic materials whose stored energy function is polyconvex, subjected to conservative applied forces. It is well known, for instance, that Ogden's materials are polyconvex materials [2], but as far as we know there has been no direct proof of the non-polyconvexity of the usual Saint Venant-Kirchhoff model. The purpose of this note is to provide such a direct proof by constructing an easy counter-example. However, there exists an "indirect" proof, where the non-polyconvexity is a consequence of the non weak lower semi-continuity of the associated functional (cf. Nečas [4]).

Let M^3 be the set of real matrices of order 3 and let M^3_+ be the subset of matrices with determinant > 0. Let us recall [1], [2] that a real-valued function W defined on M^3_+ is polyconvex if and only if there exists a convex function g defined on $M^3 \times M^3 \times \mathbb{R}^{+*}$ such that

$$(1) \qquad \forall F \in M^3_+ , \quad W(F) = g(F, \operatorname{adj} F, \det F)$$

where $\operatorname{adj} F = \det F(F^{-1})$. Notice [1], [3] that $M^3 \times M^3 \times \mathbb{R}^{+*}$ coincides with the convex hull of $\{(F. \operatorname{adj} F, \det F), F \in M^3_+\}$. The stored energy function of a Saint Venant-Kirchhoff material with Lamé's coefficients λ and μ is

$$(2) \qquad W(F) = a_1 \operatorname{tr}(F^{\mathrm{T}}F) + a_2 \operatorname{tr}(F^{\mathrm{T}}F)^2 + b_1 \operatorname{tr} \operatorname{adj}(F^{\mathrm{T}}F)$$

where

$$(3) \qquad a_1 = -\frac{3\lambda + 2\mu}{4}, \quad a_2 = \frac{\lambda + 2\mu}{8}, \quad b_1 = \frac{\lambda}{4}.$$

We want to decide whether W is polyconvex or not. It is well-known that for physical reasons λ and μ are positive; therefore the first coefficient in W is negative and this is the first indication that W need not be polyconvex (note that if all coefficients were nonnegative polyconvexity would be immediate [1], [3]).

Theorem. *W is not polyconvex.*

Proof. Let us construct a counter-example. Let ε be a positive number, and let F and F' be the following elements of M^3_+:

$$F = \varepsilon I, \quad F' = \varepsilon \operatorname{diag}(1, 1, 3).$$

One immediately obtains

$$\det F = \varepsilon^3, \quad \operatorname{adj} F = \varepsilon^2 I, \quad \det F' = 3\varepsilon^3, \quad \operatorname{adj} F' = \varepsilon^2 \operatorname{diag}(3, 3, 1),$$

$$\frac{F + F'}{2} = \varepsilon \operatorname{diag}(1, 1, 2),$$

$$\det \frac{F + F'}{2} = 2\varepsilon^3, \quad \operatorname{adj} \frac{F + F'}{2} = \varepsilon^2 \operatorname{diag}(2, 2, 1),$$

so that the following relations are satisfied, (of course, they do not hold for arbitrary F and F' in M^3_+):

$$(4) \qquad \frac{F + F'}{2} \in M^3_+, \quad \operatorname{adj} \frac{F + F'}{2} = \frac{\operatorname{adj} F + \operatorname{adj} F'}{2}, \quad \det \frac{F + F'}{2} = \frac{\det F + \det F'}{2}.$$

If W were polyconvex, equations (1) and (4) would lead to

$$(5) \qquad W\left(\frac{F + F'}{2}\right) \leqq \tfrac{1}{2}(W(F) + W(F')).$$

For the sake of brevity, let us write

$$F^{\mathrm{T}}F = \varepsilon^2 I, \quad F'^{\mathrm{T}}F' = \varepsilon^2 J, \quad \left(\frac{F + F'}{2}\right)^{\mathrm{T}} \left(\frac{F + F'}{2}\right) = \varepsilon^2 K$$

with $J = \operatorname{diag}(1, 1, 9)$, $K = \operatorname{diag}(1, 1, 4)$.

Then using expression (2), where the first term is homogeneous of degree 1 and the remaining terms are homogeneous of degree 2 with respect to $F^{\mathrm{T}}F$, we derive from inequality (5)

$$a_1 \operatorname{tr} K\varepsilon^2 + (a_2 \operatorname{tr} K^2 + b_1 \operatorname{tr} \operatorname{adj} K)\, \varepsilon^4 \leqq$$

$$\tfrac{1}{2}(a_1(\operatorname{tr} I + \operatorname{tr} J)\, \varepsilon^2 + (a_2(\operatorname{tr} I + \operatorname{tr} J^2) + b_1(\operatorname{tr} I + \operatorname{tr} \operatorname{adj} J))\, \varepsilon^4)$$

and this inequality amounts to

$$a_1 + (25a_2 + 2b_1)\, \varepsilon^2 \geqq 0$$

which (recall that a_1 is negative) cannot be true for ε small enough. □

References

[1] *J. Ball:* Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rat. Mech. Anal. 63 (1977), p. 337—403.

[2] *P. G. Ciarlet:* Lectures on three-dimensional elasticity. Tata Institute Lecture Notes, Springer-Verlag, 1983.

[3] *P. G. Ciarlet:* Topics in mathematical elasticity, vol. I. North-Holland, Amsterdam, 1985.

[4] *J. Nečas:* Introduction to the theory of nonlinear equations. Teubner Texte fr Mathematik, Band 52, Leipzig.

Souhrn

NEPOLYKONVEXITA FUNKCE VNITŘNÍ ENERGIE SAINT VENANTOVA-KIRCHHOFFOVA MATERIÁLU

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Je podán protipříklad dokazující, že funkce vnitřní energie Saint Venantova-Kirchhoffova materiálu není polykonvexní.

Резюме

НЕ-ПОЛИВЫПУКЛОСТЬ ВНУТРЕННЕЙ ФУНКЦИИ МАТЕРИАЛА СЕН ВЭНАН-КИРХГОФА

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Дается пример материала Сен Вэнан-Кирхгофа, функция внутренней энергии которого не является поливыпуклой.

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