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# EXACT SOLUTIONS TO SOME EXTERNAL MIXED PROBLEMS IN POTENTIAL THEORY

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*Summary.* A new and elegant procedure is proposed for the solution of mixed potential problems in a half-space with a circular line of division of boundary conditions. The approach is based on a new type of integral operators with special properties. Two general external problems are solved: i) An arbitrary potential is specified at the boundary outside a circle, and its normal derivative is zero inside; ii) An arbitrary normal derivative is given outside the circle, and the potential is zero inside. Several illustrative examples are considered. Certain methods of application of the proposed technique to the solution of a few complex problems are also discussed.

## INTRODUCTION AND PRELIMINARIES

Various applications of the potential theory in electrostatics, fluid flow, heat transfes, linear elasticity, etc. are well known [1]. A majority of these solutions dealing with classical mixed problems is "constructed" rather than derived. The derivation, if it exists, is very complicated while the final result is simple and is often expressed in terms of elementary functions. It seems logical that since the solution is simple, there should exist an elementary and straightforward procedure for obtaining it. On the basis of this logic, this investigation presents a new method for obtaining such results.

Some preliminary considerations are necessary to understand the approach proposed. Introduce the following function

$$\lambda(K, \psi) = \frac{1 - K^2}{1 + K^2 - 2K\cos\psi} = \sum_{n=-\infty}^{\infty} K^{|n|}\, e^{in\psi}. \tag{1}$$

Define the integral $L$-operator

$$L(K) f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(K, \phi - \psi) f(\psi)\, d\psi = \tag{2}$$

$$= \sum_{n=-\infty}^{\infty} K^{|n|}\, e^{in\phi} \frac{1}{2\pi}\int_0^{2\pi} e^{-in\psi} f(\psi)\, d\psi = \sum_{n=-\infty}^{n} K^{|n|} f_n e^{in\phi}.$$

Here $f_n$ is the Fourier coefficient of the function $f$. The following properties of the $L$-operators are obvious from (2):

$$L(K)\, L(K_1) = L(KK_1)\,, \quad L(1) f = f\,. \tag{3}$$

The properties (3) allow a construction of the operator inverse to $L$ as

$$L^{-1}(K) = L(K^{-1})\,. \tag{4}$$

The following two integrals will be widely used in this paper:

$$\int \frac{\lambda\left(\dfrac{x^2}{pq}, \beta\right) \mathrm{d}x}{\sqrt{(p^2 - x^2)}\,\sqrt{(q^2 - x^2)}} = -\frac{1}{R_{pq}} \tan^{-1} \frac{y_1(x)}{R_{pq}}\,, \tag{5}$$

$$\int \frac{\lambda\left(\dfrac{pq}{x^2}, \beta\right) \mathrm{d}x}{\sqrt{(x^2 - p^2)}\,\sqrt{(x^2 - q^2)}} = \frac{1}{R_{pq}} \tan^{-1} \frac{y_2(x)}{R_{pq}} \tag{6}$$

where

$$R_{pq}^2 = p^2 + q^2 - 2pq \cos \beta\,. \tag{7}$$

Both integrals (5) and (6) are easily verifiable by the substitution

$$\begin{aligned} y_1(x) &= \sqrt{(p^2 - x^2)}\,\sqrt{(q^2 - x^2)}/x\,, \\ y_2(x) &= \sqrt{(x^2 - p^2)}\,\sqrt{(x^2 - q^2)}/x\,, \end{aligned} \tag{8}$$

respectively.

Two non-axisymmetric external mixed problems of the potential theory for a homogeneous half-space with a circular line of division of boundary conditions are solved in the next section. The problems are called external because nonzero boundary conditions are prescribed outside a circle while zero conditions are given inside. Various illustrative examples are considered in the third section. All the solutions obtained are exact and expressed in terms of elementary functions. The last section is devoted to the dicsussion of the results and possibilities of further applications of the technique introduced.

## SOLUTION OF EXTERNAL MIXED PROBLEMS

**Problem 1.** Consider a homogeneous half-space $z \geqq 0$. It is necessary to find a harmonic function $W$ subject to the boundary conditions at $z = 0$, namely

$$\begin{aligned} \frac{\partial W}{\partial z} &= 0 \quad \text{for} \quad \varrho < a\,, \quad 0 \leqq \phi < 2\pi\,, \\ W &= w(\varrho, \phi) \quad \text{for} \quad \varrho \geqq a\,, \quad 0 \leqq \phi < 2\pi\,. \end{aligned} \tag{9}$$

Here a set of cylindrical coordinates $\varrho$, $\phi$, $z$ is used. It is well known that the potential $W$ can be represented as

$$(10)\qquad W(\varrho, \phi, z) = \int_0^{2\pi}\int_0^a \frac{\sigma(r, \psi)}{R}\, r\, \mathrm{d}r\, \mathrm{d}\psi + \int_0^{2\pi}\int_a^\infty \frac{\sigma(r, \psi)}{R}\, r\, \mathrm{d}r\, \mathrm{d}\psi$$

where

$$(11)\qquad \sigma = -\frac{1}{2\pi}\frac{\partial W}{\partial z}\,, \; z = 0 \quad \text{and} \quad R^2 = \varrho^2 + r^2 - 2\varrho r \cos(\phi - \psi) + z^2\,.$$

Introducing the quantities

$$(12)\qquad l_i(r) = \tfrac{1}{2}[\sqrt{((r + \varrho)^2 + z^2)} + (-1)^i \sqrt{((r - \varrho)^2 + z^2)}] \quad \text{for} \quad i = 1, 2\,,$$

one can verify that

$$(13)\qquad l_1(r)\, l_2(r) = \varrho r\,, \quad l_1^2(r) + l_2^2(r) = \varrho^2 + r^2 + z^2$$

and

$$(14)\qquad \lim_{z\to 0} l_1(r) = \min(r, \varrho)\,, \quad \lim_{z\to 0} l_2(r) = \max(\varrho, r)\,.$$

Making use of (5) and (6), and substituting $p$ by $l_1$, $q$ by $l_2$ and $\beta$ by $\phi - \psi$, the following integral representations become possible:

$$(15)\qquad \frac{1}{R} = \frac{2}{\pi}\int_0^{l_1(r)} \frac{\lambda\left(\dfrac{x^2}{\varrho r}, \phi - \psi\right)\mathrm{d}x}{\sqrt{(l_1^2(r) - x^2)}\sqrt{(l_2^2(r) - x^2)}}\,,$$

$$(16)\qquad \frac{1}{R} = \frac{2}{\pi}\int_{l_2(r)}^\infty \frac{\lambda\left(\dfrac{\varrho r}{x^2}, \phi - \psi\right)\mathrm{d}x}{\sqrt{(x^2 - l_1^2(r))}\sqrt{(x^2 - l_2^2(r))}}\,.$$

Substituting (15) in the first term of (10), and (16) in the second, one obtains after changing the order of integration and making necessary transformations due to (13)

$$(17)\qquad W(\varrho, \phi, z) = 4\int_0^{l_1(a)} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}}\int_{g(x)}^a \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - g^2(x))}}\, L\left(\frac{x^2}{\varrho r}\right)\sigma(r, \phi) +$$
$$+ 4\int_{l_2}^\infty \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}}\int_a^{g(x)} \frac{r\,\mathrm{d}r}{\sqrt{(g^2(x) - r^2)}}\, L\left(\frac{\varrho r}{x^2}\right)\sigma(r, \phi)\,.$$

Hereafter, for the sake of simplicity $l_1$ is understood as $l_1(a)$ and $l_2$ as $l_2(a)$; the operator $L$ is understood according to its definition (2) as

$$L(k)\,\sigma(r, \psi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi - \psi)\,\sigma(r, \psi)\,\mathrm{d}\psi$$

and

$$(18) \qquad g(x) = x\sqrt{\left(1 + \frac{z^2}{\varrho^2 - x^2}\right)}.$$

One can notice that the function $g(x)$ is inverse to $l_1$ for $x < \varrho$, and is inverse to $l_2$ for $x^2 > \varrho^2 + z^2$.

Now the substitution of the boundary conditions (9) in (17) leads to the integral equation

$$(19) \qquad 4\int_\varrho^\infty \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}}\int_a^x \frac{r\,\mathrm{d}r}{\sqrt{(x^2 - r^2)}}\, L\left(\frac{\varrho r}{x^2}\right)\sigma(r, \phi) = w(\varrho, \varphi)\,.$$

Equations of the type (19) were treated in [2]. Here, a different type of solution is derived. Let the operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\varrho\,\mathrm{d}\varrho}{\sqrt{(\varrho^2 - t^2)}}\, L\left(\frac{1}{\varrho}\right)$$

be applied to both sides of (19). The use of properties of the Abel-type operators and properties of the $L$-operators (3) results in

$$-\,2\pi\int_a^t \frac{r\,\mathrm{d}r}{\sqrt{(t^2 - r^2)}}\, L\left(\frac{r}{t^2}\right)\sigma(r, \phi) = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\varrho\,\mathrm{d}\varrho}{\sqrt{(\varrho^2 - t^2)}}\, L\left(\frac{1}{\varrho}\right) w(\varrho, \phi)\,.$$

The next operator to apply is

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t\,\mathrm{d}t}{\sqrt{(y^2 - t^2)}}\, L(t^2)$$

and the result is

$$-\pi^2 y\, L(y)\,\sigma(y, \phi) = \frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t\,\mathrm{d}t}{\sqrt{(y^2 - t^2)}}\, L(t^2)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\varrho\,\mathrm{d}\varrho}{\sqrt{(\varrho^2 - t^2)}}\, L\left(\frac{1}{\varrho}\right) w(\varrho, \phi)\,.$$

Finally, using (4), the solution takes the form

$$(20) \quad \sigma(y, \phi) = -\,\frac{1}{\pi^2 y}\, L\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t\,\mathrm{d}t}{\sqrt{(y^2 - t^2)}}\, L(t^2)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\varrho\,\mathrm{d}\varrho}{\sqrt{(\varrho^2 - t^2)}}\, L\left(\frac{1}{\varrho}\right) w(\varrho, \phi)\,.$$

Differentiation under the integral sign gives another form of the solution, namely

$$(21) \qquad \sigma(y, \phi) = -\,\frac{1}{\pi^2}\left\{\frac{\chi(a, y, \phi)}{\sqrt{(y^2 - a^2)}} + \int_a^y \frac{\mathrm{d}r}{\sqrt{(y^2 - r^2)}}\,\frac{\partial}{\partial r}\,\chi(r, y, \phi)\right\}$$

where

$$(22) \qquad \chi(r, y, \phi) = r\int_r^\infty \frac{\mathrm{d}\varrho}{\sqrt{(\varrho^2 - r^2)}}\,\frac{\partial}{\partial\varrho}\left[L\left(\frac{r^2}{y\varrho}\right) w(\varrho, \phi)\right].$$

The following transformation can now be performed:

$$(23) \qquad \frac{\partial}{\partial r}\,\chi(r, y, \phi) = \frac{\partial}{\partial r}\left\{r\int_r^\infty \frac{\mathrm{d}\varrho}{\sqrt{(\varrho^2 - r^2)}}\,(Lw)'\right\} =$$

$$= \int_r^\infty \frac{d\varrho}{\sqrt{(\varrho^2 - r^2)}} [(Lw)' + \varrho(Lw)'' - 2(L'\varrho w)'] =$$

$$= \int_r^\infty \frac{\varrho \, d\varrho}{\sqrt{(\varrho^2 - r^2)}} \left[ L\left(w'' + \frac{1}{\varrho} w'\right) - \left(L'' + \frac{1}{\varrho} L'\right) w \right].$$

Here the primes (′) indicate partial derivatives with respect to $\varrho$, $L$ stands for $L(r^2/y\varrho)$, $w \equiv w(\varrho, \phi)$, and the following identity is used:

$$\frac{\partial}{\partial r} L\left(\frac{r^2}{y\varrho}\right) = -2 \frac{\varrho}{r} \frac{\partial}{\partial \varrho} L\left(\frac{r^2}{y\varrho}\right).$$

As

$$L \frac{1}{\varrho^2} \frac{\partial^2 w}{\partial \phi^2} = \frac{1}{\varrho^2} \frac{\partial^2 L}{\partial \phi^2} w,$$

an addition to and subtraction from (23) transforms it into

$$\int_r^\infty \frac{\varrho \, d\varrho}{\sqrt{(\varrho^2 - r^2)}} [L \Delta w - (\Delta L) w],$$

where $\Delta$ is the Laplace operator in the polar coordinates. Since $\lambda$ is harmonic $\Delta L = 0$, and (23) finally simplifies to

$$\frac{\partial}{\partial r} \chi(r, y, \phi) = \int_r^\infty \frac{\varrho \, d\varrho}{\sqrt{(\varrho^2 - r^2)}} L \Delta w. \tag{24}$$

Substitution of (24) in (21) gives

(25)

$$\sigma(y, \phi) = -\frac{1}{\pi^2} \left\{ \frac{\chi(a, y, \phi)}{\sqrt{(y^2 - a^2)}} \right\} + \int_a^y \frac{dr}{\sqrt{(y^2 - r^2)}} \int_r^\infty \frac{\varrho \, d\varrho}{\sqrt{(\varrho^2 - r^2)}} L\left(\frac{r^2}{y\varrho}\right) \Delta w(\varrho, \phi).$$

Expression (25) presents a new type of solution for the integral equation (19). It may be noticed that the first term in (25) becomes singular, and the second term tends to zero when $y \to a$. In the case of $w$ being a harmonic function, the second term in (25) vanishes, and the solution is represented by the first term only. Further, integration with respect to $r$ becomes possible in (25) after changing the order of integration and using (5). The result is

$$\sigma(y, \phi) = -\frac{1}{\pi^2} \left\{ \frac{\chi(a, y, \phi)}{\sqrt{(y^2 - a^2)}} + \right. \tag{26}$$

$$\left. + \frac{1}{2\pi} \int_0^{2\pi} \int_a^\infty \frac{\Delta w(\varrho, \psi) \, \varrho \, d\varrho \, d\psi}{\sqrt{(\varrho^2 + y^2 - 2y\varrho \cos(\phi - \psi))}} \tan^{-1} \frac{\sqrt{(\varrho^2 - a^2)} \sqrt{(y^2 - a^2)}}{\sqrt{(\varrho^2 + y^2 - 2y\varrho \cos(\phi - \psi))}} \right\}.$$

Solutions in the forms (20) or (25) are appropriate to use when exact evaluation of integrals is possible, while the solution in the form (26) has certain advantages when numerical integration is to be employed.

It is of interest to express the potential $W$ in the half-space directly through its boundary value $w$. As $\sigma$ inside the circle is zero, expression (17) takes the form

$$(27) \qquad W(\varrho, \phi, z) = 4\int_{l_2}^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}} \int_a^{g(x)} \frac{y \, \mathrm{d}y}{\sqrt{(g^2(x) - y^2)}} L\left(\frac{\varrho y}{x^2}\right) \sigma(y, \phi) .$$

Substitution of (20) in (27) and integration with respect to $y$ yields

(28)

$$W(\varrho, \phi, z) = -\frac{2}{\pi}\int_{l_2}^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}} L\left(\frac{\varrho\, g^2(x)}{x^2}\right) \frac{\partial}{\partial g(x)} \int_{g(x)}^{\infty} \frac{r \, \mathrm{d}r}{\sqrt{(r^2 - g^2(x))}} L\left(\frac{1}{r}\right) w(r, \phi) .$$

Here the properties (3) of the $L$-operators were used along with the following identity valid for the Abel-type operators:

$$(29) \qquad \int_a^g \frac{\mathrm{d}y}{\sqrt{(g^2 - y^2)}} \frac{\mathrm{d}}{\mathrm{d}y} \int_a^y \frac{f(t)\, t \, \mathrm{d}t}{\sqrt{(y^2 - t^2)}} = \frac{\pi}{2} f(g) .$$

A change of the order of integration in (28) and integration with respect to $x$ give

$$(30) \qquad W(\varrho, \phi, z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} \frac{z}{R^3}\left[\frac{R}{\xi} + \tan^{-1}\frac{\xi}{R}\right] w(r, \psi)\, r \, \mathrm{d}r \, \mathrm{d}\psi .$$

Here $R$ is defined by (11), and $\xi$ can be presented in several equivalent forms, namely,

$$(31) \qquad \xi = \frac{\sqrt{(r^2 - a^2)}\sqrt{(l_2^2 - a^2)}}{a} = \frac{\sqrt{(r^2 - a^2)}\sqrt{(\varrho^2 - l_1^2)}}{l_1} = \frac{\sqrt{(l_1^2(r) - l_1^2)}\sqrt{(l_2^2(r) - l_1^2)}}{l_1} = \frac{z\sqrt{(r^2 - a^2)}}{\sqrt{(a^2 - l_1^2)}} ,$$

each form being useful in different specific transformations. It must be noted that throughout this paper, $l_1$ and $l_2$ are understood as $l_1(a)$ and $l_2(a)$ according to the general definition (12). Details of the derivation of (30) are given in Appendix A. In the particular case of $z = 0$, expression (30) simplifies to

$$(32) \qquad W(\varrho, \phi, 0) = \begin{cases} \dfrac{1}{\pi^2}\sqrt{(a^2 - \varrho^2)} \displaystyle\int_0^{2\pi}\int_a^{\infty} \frac{w(r,\psi)\, r \, \mathrm{d}r \, \mathrm{d}\psi}{\sqrt{(r^2 - a^2)}[r^2 + \varrho^2 - 2r\varrho\cos(\phi - \psi)]} & \text{for } \varrho \leqq a , \\ w(\varrho, \phi) & \text{for } \varrho \geqq a . \end{cases}$$

Expression (32) corresponds to the result previously reported in [2].

The solution can now be interpreted. The charge density $\sigma$ is given by the two equivalent expressions (20) and (26); the potential is given by (28) and (30), the former being more convenient for exact evaluation of the integrals while the latter is better suited for numerical integration.

**Problem 2.** Consider the following external mixed problem: to find a harmonic function $W$, satisfying the boundary conditions at $z = 0$

$$W = 0 \quad \text{for} \quad \varrho \leqq a\,, \quad 0 \leqq \phi < 2\pi\,,$$

$$\frac{\partial W}{\partial z} = -2\pi\sigma(\varrho, \phi) \quad \text{for} \quad \varrho > a\,, \quad 0 \leqq \phi < 2\pi\,. \tag{33}$$

Substitution of the boundary conditions (33) in (17) leads to the integral equation

$$\int_0^\varrho \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_x^a \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - x^2)}}\, L\left(\frac{x^2}{\varrho r}\right) \sigma(r, \phi) = \tag{34}$$

$$= - \int_a^\infty \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}} \int_a^x \frac{r\,\mathrm{d}r}{\sqrt{(x^2 - r^2)}}\, L\left(\frac{\varrho r}{x^2}\right) \sigma(r, \phi)\,.$$

One should notice that $\sigma$ on the right side of (34) is known from (33), while the value of $\sigma$ on the left side of (34) is yet unknown.

Using (15) and changing the order of integration, the right side of (34) can be transformed so that Eq. (34) takes the form

$$\int_0^\varrho \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_x^a \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - x^2)}}\, L\left(\frac{x^2}{\varrho r}\right) \sigma(r, \phi) =$$

$$= - \int_0^\varrho \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_a^\infty \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - x^2)}}\, L\left(\frac{x^2}{\varrho r}\right) \sigma(r, \phi)\,,$$

with the immediate result

$$\int_x^a \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - x^2)}}\, L\left(\frac{1}{r}\right) \sigma(r, \phi) = - \int_a^\infty \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - x^2)}}\, L\left(\frac{1}{r}\right) \sigma(r, \phi)\,. \tag{35}$$

The application of the operator

$$L(\varrho) \frac{\partial}{\partial \varrho} \int_\varrho^a \frac{x\,\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}}$$

to both sides of (35) gives, after necessary transformations,

$$\sigma(\varrho, \phi) = - \frac{2}{\pi\sqrt{(a^2 - \varrho^2)}} \int_a^\infty \frac{\sqrt{(r^2 - a^2)}}{r^2 - \varrho^2}\, L\left(\frac{\varrho}{r}\right) \sigma(r, \phi)\, r\,\mathrm{d}r \quad \text{for} \quad \varrho < a\,, \tag{36}$$

or, interpreting the $L$-operator, one obtains

$$\sigma(\varrho, \phi) = - \frac{1}{\pi^2 \sqrt{(a^2 - \varrho^2)}} \int_0^{2\pi} \int_a^\infty \frac{\sqrt{(r^2 - a^2)}\, \sigma(r, \psi)\, r\,\mathrm{d}r\,\mathrm{d}\psi}{r^2 + \varrho^2 - 2r\varrho \cos(\phi - \psi)}\,. \tag{37}$$

Now the value of $\sigma$ is known all over the plane $z = 0$, and (17) can be used for expressing the potential $W$ directly through the given value of $\sigma$. Substitution of (36) in the first term of (17) gives, after integration with respect to $r$,

$$(38)\qquad W(\varrho, \phi, z) = 4\int_0^{l_1} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_a^{\infty} \frac{y\,\mathrm{d}y}{\sqrt{(y^2 - g^2(x))}}\, L\left(\frac{x^2}{\varrho y}\right) \sigma(y, \phi) +$$

$$+ 4\int_{l_2}^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}} \int_a^{g(x)} \frac{r\,\mathrm{d}r}{\sqrt{(g^2(x) - r^2)}}\, L\left(\frac{\varrho r}{x^2}\right) \sigma(r, \phi)\,.$$

The second term in (38) is equivalent to the second term in (10) which, in turn, can be represented, using (15), as

$$4\int_a^{\infty} r\,\mathrm{d}r \left\{\int_0^{l_1(r)} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}\sqrt{(r^2 - g^2(x))}}\, L\left(\frac{x^2}{\varrho r}\right)\right\} \sigma(r, \phi)\,.$$

Now the following scheme of change of the order of integration can be used:

$$\int_a^{\infty} \mathrm{d}r \int_0^{l_1(r)} \mathrm{d}x = \int_0^{l_1(a)} \mathrm{d}x \int_a^{\infty} \mathrm{d}r + \int_{l_1(a)}^{l_1(\infty)} \mathrm{d}x \int_{g(x)}^{\infty} \mathrm{d}r\,,$$

and the second term in (38) can be rewritten as

$$(39)\qquad 4\int_0^{l_1} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_a^{\infty} \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - g^2(x))}}\, L\left(\frac{x^2}{\varrho r}\right) \sigma(r, \phi) +$$

$$+ 4\int_{l_1}^{l_1(\infty)} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_{g(x)}^{\infty} \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - g^2(x))}}\, L\left(\frac{x^2}{\varrho r}\right) \sigma(r, \phi)\,.$$

Substitution of (39) in (38) gives, by virtue of $l_1(\infty) = \varrho$,

$$(40)\qquad W(\varrho, \phi, z) = 4\int_{l_1}^{\varrho} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}} \int_{g(x)}^{\infty} \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - g^2(x))}}\, L\left(\frac{x^2}{\varrho r}\right) \sigma(r, \phi)\,.$$

A change of the order of integration in (40) and integration with respect to $x$, using (5), results in

$$(41)\qquad W(\varrho, \phi, z) = \frac{2}{\pi} \int_0^{2\pi} \int_a^{\infty} \frac{\sigma(r, \psi)}{R} \tan^{-1} \frac{\xi}{R}\, r\,\mathrm{d}r\,\mathrm{d}\psi$$

where $R$ is defined by (11) and $\xi$ is any one of the expressions in (31).

The solution can now be explained. Expression (37) defines the charge density $\sigma$ inside a circle directly through its values outside, the potential $W$ is given by the two equivalent expressions (40) and (41), the first one being recommended for an exact evaluation of the integrals involved while the second has certain advantages for numerical integration.

## ILLUSTRATIVE EXAMPLES

Several particular cases of general solutions, obtained in the previous section, are considered here in order to illustrate the effectiveness of the proposed technique.

**Example 1.** Consider an external mixed problem with the boundary conditions at $z = 0$

$$W = w_0/\varrho^n \quad \text{for} \quad \varrho \geqq a\,, \qquad \frac{\partial W}{\partial z} = 0 \quad \text{for} \quad \varrho < a\,. \tag{42}$$

The conditions (42) correspond to those of problem 1 given in (9). The solution is given by (28) and (21).

Substitution of (42) in (28) gives, after integration with respect to $r$,

$$W(\varrho, \phi, z) = \frac{2w_0}{\sqrt{\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)} \int_{l_2}^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2 - \varrho^2)}\, g^n(x)} \tag{43}$$

where $g(x)$ is defined by (18).

Here the following integral was employed [3]:

$$\int_g^{\infty} \frac{\mathrm{d}r}{r^n \sqrt{(r^2 - g^2)}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n/2)}{\Gamma[(n+1)/2]} \frac{1}{g^n}\,. \tag{44}$$

For any integer $n$, the integral in (43) can be evaluated in terms of elementary functions, but the procedure is slightly different for even values of $n$ from those for odd $n$. For example, for $n = 2k$, the problem reduces to the evaluation of the integral

$$\int_{l_1}^{\infty} \frac{(x^2 - \varrho^2)^{k-1/2}\, \mathrm{d}x}{x^{2k}(x^2 - \varrho^2 - z^2)^k}\,,$$

which can be solved by introduction of a new variable $t = x/\sqrt{(x^2 - \varrho^2)}$. The final result is

$$W(\varrho, \phi, z) = \frac{2w_0}{\sqrt{\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\, z^n} \left\{ \sum_{m=1}^{k} \frac{A_m}{2m-1} \left[1 - Q_0^{2m-1}\right] + \right.$$
$$\left. + 2B_1 \ln Q + \sum_{m=2}^{k} \frac{B_m}{1-m} \left[(Q_1^{m-1} - Q_2^{m-1}) - (Q_3^{m-1} - Q_4^{m-1})\right] \right\} \tag{45}$$

where

$$A_{k-m+1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\eta^{m-1}} \left[\frac{(\eta - 1)^{k-1}}{(r_1^2 - \eta)^k}\right] \quad \text{for} \quad \eta = 0,\ r_1^2 = 1 + \varrho^2/z^2\,, \tag{46}$$

$$B_{k-m+1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{(t^2 - 1)^{k-1}}{t^{2k}(r_1 + t)^k}\right] \quad \text{for} \quad t = r_1 = \sqrt{(1 + \varrho^2/z^2)}\,,$$

$$Q_0 = \frac{\sqrt{l_2^2 - \varrho^2)}}{l_2}\,, \quad Q = \frac{(\sqrt{(\varrho^2 + z^2)} + \sqrt{(l_2^2 - a^2)})\, l_2}{(\sqrt{(\varrho^2 + z^2)} + z)\, a}\,,$$

$$Q_1 = \frac{z(\sqrt{(\varrho^2 + z^2)} + \sqrt{(l_2^2 - a^2)})}{l_1^2}\,, \quad Q_2 = \frac{z(\sqrt{(\varrho^2 + z^2)} - \sqrt{(l_2^2 - a^2)})}{l_1^2}\,,$$

$$Q_3 = \frac{z(\sqrt{(\varrho^2 + z^2)} + z)}{\varrho^2}, \quad Q_4 = \frac{z(\sqrt{(\varrho^2 + z^2)} - z)}{\varrho^2}.$$

For the case of an odd $n = 2k + 1$ the integration can be performed by using the substitution
$t^2 = x^2 - \varrho^2 - z^2$, and the final result is

$$(47) \qquad W(\varrho, \phi, z) = \frac{w_0}{\sqrt{\pi}} \frac{\Gamma[(n + 1)/2]}{\Gamma(n/2)} \left\{ \sum_{m=1}^{k} \frac{C_m}{(2m - 1)(a^2 - l_1^2)^{m-1/2}} + \right.$$

$$\left. + \sum_{m=1}^{k+1} D_m \frac{(-1)^m}{(m - 1)!} \frac{\partial^{m-1}}{\partial \eta^{m-1}} \left[ \frac{1}{\sqrt{\eta}} \cot^{-1} \frac{\sqrt{(a^2 - l_1^2)}}{\sqrt{\eta}} \right] \right\} \quad \text{for} \quad \eta = \varrho^2 + z^2$$

where

$$C_m = \frac{1}{(k - m)!} \frac{\mathrm{d}^{k-m}}{\mathrm{d}t^{k-m}} \left[ \frac{(t + z^2)^k}{(t + \varrho^2 + z^2)^{k+1}} \right] \quad \text{for} \quad t = 0,$$

$$(48) \qquad D_m = \frac{1}{(k + 1 - m)!} \frac{\mathrm{d}^{k+1-m}}{\mathrm{d}t^{k+1-m}} \left[ \frac{(t + z^2)^k}{t^k} \right] \quad \text{for} \quad t = -\varrho^2 - z^2.$$

Substitution of (42) in (21) yields, after integration [3],

$$(49) \qquad \sigma(\varrho, \phi) = \frac{\Gamma[(n + 1)/2]\, w_0}{\pi^{3/2} \Gamma(n/2)} \left\{ \frac{1}{a^n \sqrt{(\varrho^2 - a^2)}} - \right.$$

$$\left. - \frac{n \sqrt{(\varrho^2 - a^2)}}{\varrho^{n+2}} F\left( \frac{n}{2} + 1, \frac{1}{2}; \frac{3}{2}; 1 - \frac{a^2}{\varrho^2} \right) \right\}$$

and the Gauss hypergeometric function can be expressed through elementary functions [4], namely, for even $n = 2k$, $k = 1, 2, 3, \ldots$,

$$F\left( 1 + k, \frac{1}{2}; \frac{3}{2}; t \right) = \frac{1}{2k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[ t^{k-1/2} \ln \frac{1 + \sqrt{t}}{1 - \sqrt{t}} \right],$$

and for odd $n = 2k + 1$, $k = 0, 1, 2, \ldots$,

$$(50) \qquad F\left( k + \frac{3}{2}, \frac{1}{2}; \frac{3}{2}; t \right) = \frac{\sqrt{\pi}}{2\Gamma(3/2 + k) \sqrt{t}} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[ \frac{t^{k+1/2}}{\sqrt{(1 - t)}} \right].$$

The solution can now be presented for several specific values of $n$:

$n = 1$

$$(51) \qquad W(\varrho, \phi, z) = \frac{2}{\pi} \frac{w_0}{\sqrt{(\varrho^2 + z^2)}} \sin^{-1} \frac{\sqrt{(\varrho^2 + z^2)}}{l_2},$$

$$(52) \qquad \sigma(\varrho, \phi) = \frac{w_0 a}{\pi^2 \varrho^2 \sqrt{(\varrho^2 - a^2)}};$$

$n = 2$

$$W(\varrho, \phi, z) = \frac{w_0}{\varrho^2 + z^2}\left[1 - Q_0 + \frac{z}{\sqrt{(\varrho^2 + z^2)}} \ln Q\right]. \tag{53}$$

At the plane $z = 0$,

$$W(\varrho, \phi, 0) = \begin{cases} \dfrac{w_0}{\varrho^2}\left[1 - \dfrac{\sqrt{(a^2 - \varrho^2)}}{a}\right] & \text{for} \quad \varrho \leqq a\,, \\ \dfrac{w_0}{\varrho^2} & \text{for} \quad \varrho \geqq a\,, \end{cases}$$

$$\sigma(\varrho, \phi) = \frac{w_0}{2\pi\varrho^2}\left\{\frac{1}{\sqrt{(\varrho^2 - a^2)}} - \frac{1}{\varrho} \ln \frac{\varrho + \sqrt{(\varrho^2 - a^2)}}{a}\right\}; \tag{54}$$

$n = 3$

$$W(\varrho, \phi, z) = \frac{4w_0}{\pi(\varrho^2 + z^2)^2}\left\{\frac{z^2}{\sqrt{(a^2 - l_1^2)}} - \frac{l_1^2\sqrt{(a^2 - l_1^2)}}{2a^2} + \right. \tag{55}$$

$$\left. + \frac{\varrho^2 - 2z^2}{2\sqrt{(\varrho^2 + z^2)}} \sin^{-1} \frac{\sqrt{(\varrho^2 + z^2)}}{l_2}\right\}.$$

At the plane $z = 0$,

$$W(\varrho, \phi, 0) = \begin{cases} \dfrac{2w_0}{\pi\varrho^3}\left[\sin^{-1}\dfrac{\varrho}{a} - \dfrac{\varrho}{a^2}\sqrt{(a^2 - \varrho^2)}\right] & \text{for} \quad \varrho \leqq a\,, \\ \dfrac{w_0}{\varrho^3} & \text{for} \quad \varrho \geqq a\,, \end{cases}$$

$W(0, 0, 0) = 4w_0/3\pi a^3$,

$$\sigma(\varrho, \phi) = \frac{2w_0}{\pi^2\varrho^4} \frac{2a^2 - \varrho^2}{a\sqrt{(\varrho^2 - a^2)}}; \tag{56}$$

$n = 4$

$$W(\varrho, \phi, z) = \frac{3w_0}{2(\varrho^2 + z^2)^2}\left\{\frac{\varrho^2 - z^2}{\varrho^2 + z^2}(1 - Q_0) - \frac{1}{3}(1 - Q_0{}^3) + \right. \tag{57}$$

$$\left. + \frac{z}{2(\varrho^2 + z^2)}\left[\frac{l_2^2}{a^2}\sqrt{(l_2^2 - a^2)} - z\right] - \frac{z(2z^2 - 3\varrho^2)}{2(\varrho^2 + z^2)^{3/2}} \ln Q\right\}.$$

At the plane $z = 0$,

$$W(\varrho, \phi, 0) = \begin{cases} \dfrac{w_0}{\varrho^4}\left[1 - \dfrac{\sqrt{(a^2 - \varrho^2)}}{a} - \dfrac{\varrho^2\sqrt{(a^2 - \varrho^2)}}{2a^3}\right] & \text{for} \quad \varrho \leqq a, \\ \dfrac{w_0}{\varrho^4} & \text{for} \quad \varrho \geqq a\,, \end{cases}$$

$W(0, 0, 0) = 3w_0/8a^4$,

$$(58) \qquad \sigma(\varrho, \phi) = \frac{3w_0}{8\pi\varrho^4}\left\{\frac{3a^2 - \varrho^2}{a^2\sqrt{(\varrho^2 - a^2)}} - \frac{3}{\varrho}\ln\frac{\varrho + \sqrt{(\varrho^2 - a^2)}}{a}\right\}.$$

The total charge at the plane $z = 0$ is equal to $w_0$ for $n = 1$, and it is zero for $n \geqq 2$.

Figure 1 shows the dimensionless charge density $\sigma a^{n+1}/w_0$ versus $\varrho/a$ for $n = 1, 2, 3, 4$ (formulae 52, 54, 56, 58). The charge density is nonnegative for $n = 1$, and changes sign when $n \geqq 2$, its negative maximum increases with $n$ while the total charge stays at zero.

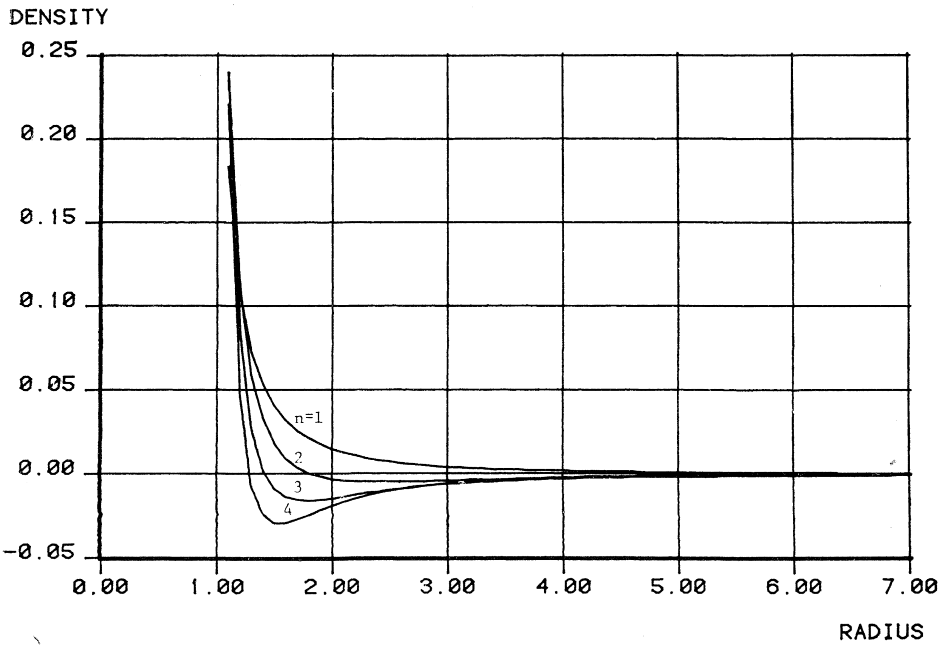


Fig. 1. Charge density for n = 1, 2, 3, 4

Equipotential lines for the case $n = 2$ (formula 53) are presented in Fig. 2.

The range of values of the dimensionless potential $W_0 a^n/w$ is taken as 0·05 to 0·6. Because of symmetry of the problem, only a quarter of each equipotential line is presented.

**Example 2.** Consider the boundary conditions

$$(59) \qquad W = \frac{w_n}{\varrho_n} e^{in\phi} \quad \text{for} \quad \varrho \geqq a,$$

$$\frac{\partial W}{\partial z} = 0 \qquad \text{for} \quad \varrho < a$$

where $w_n$ is a constant. The solution is given by (28) and (20). Substitution of (59) in (28) gives after first integration [3]

$$W(\varrho, \phi, z) = \frac{2\Gamma(n + 1/2)}{\sqrt{\pi}\,\Gamma(n)} \varrho^n e^{in\phi} \int_{l_2}^{\infty} \frac{dx}{x^{2n} \sqrt{(x^2 - \varrho^2)}}.$$

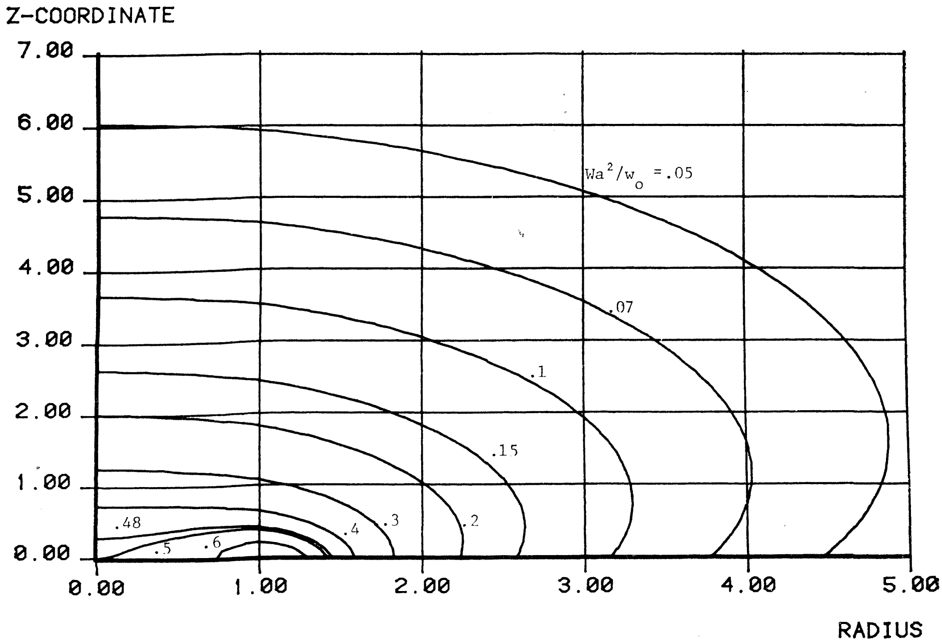


Fig. 2. Equipotential lines for n = 2

The second integration yields

$$(60) \quad W(\varrho, \phi, z) = \frac{2}{\sqrt{\pi}} \frac{w_n}{\varrho^n} e^{in\phi} \sum_{m=1}^{n} \frac{(-1)^{m-1}\,\Gamma(n + 1/2)}{\Gamma(m)\,\Gamma(n + 1 - m)\,(2m - 1)} (1 - Q_0^{2m-1})$$

and $Q_0$ is defined by (46). Substituting (59) in (20), one gets, after integration [3],

$$(61) \qquad \sigma(\varrho, \phi) = \frac{\Gamma(n + 1/2)}{\pi^{3/2}\,\Gamma(n)} \frac{w_n e^{in\phi}}{\varrho^n \sqrt{(\varrho^2 - a^2)}}.$$

Evidently, it can be noticed that (61) can be obtained by differentiation of (60) with respect to $z$ for $z = 0$, instead of integration of (20).

Here are some explicit expressions for several particular values of $n$.

$n = 1$

$$W(\varrho, \phi, z) = \frac{w_1}{\varrho} e^{i\phi} \left[ 1 - \frac{\sqrt{(l_2^2 - \varrho^2)}}{l_2} \right];$$

$n = 2$

$$W(\varrho, \phi, z) = \frac{3}{2} \frac{w_2}{\varrho^2} e^{2i\phi} \left\{ \left[ 1 - \frac{\sqrt{(l_2^2 - \varrho^2)}}{l_2} \right] - \frac{1}{3} \left[ 1 - \left( \frac{\sqrt{(l_2^2 - \varrho^2)}}{l_2} \right)^3 \right] \right\}; \tag{62}$$

$n = 3$

$$W(\varrho, \phi, z) = \frac{15}{4} \frac{w_3}{\varrho^3} e^{3i\phi} \left\{ \frac{1}{2} \left[ 1 - \frac{\sqrt{(l_2^2 - \varrho^2)}}{l_2} \right] - \frac{1}{3} \left[ 1 - \left( \frac{\sqrt{(l_2^2 - \varrho^2)}}{l_2} \right)^3 \right] + \right.$$
$$\left. + \frac{1}{10} \left[ 1 - \left( \frac{\sqrt{(l_2^2 - \varrho^2)}}{l_2} \right)^5 \right] \right\}.$$

Equipotential lines at the plane $\phi = 0$ for $n = 2$ due to (62) are presented in Fig. 3. The curves correspond to a set of values of the dimensionless potential $(Wa^2/w_2)$ in the range of 0·05 to 0·4.

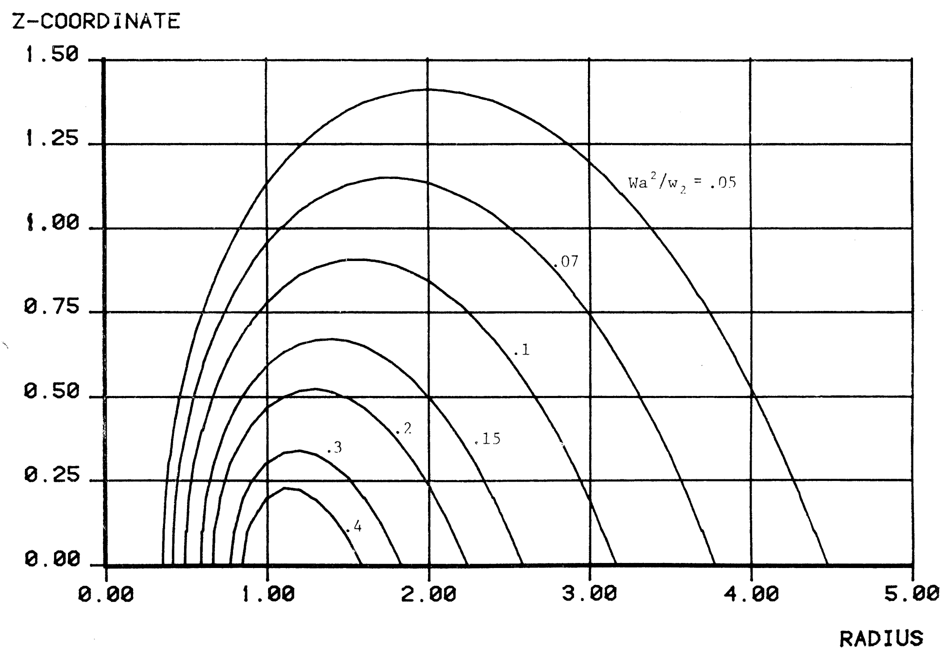


Fig. 3. Equipotential lines for $n = 2$

**Example 3.** Consider several cases related to problem 2, with the boundary conditions

$$(63)\qquad W = 0 \qquad \text{for} \quad \varrho \leqq a\,,$$
$$\frac{\partial W}{\partial z} = -2\pi \frac{\sigma_0}{\varrho^n} \qquad \text{for} \quad \varrho > a\,.$$

The solution is given by (40) and (36). Substitution of (63) in (40) yields, after first integration, using (44),

$$(64)\qquad W(\varrho, \phi, z) = 2\sqrt{\pi}\,\sigma_0 \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \int_{l_1}^{\varrho} \frac{\mathrm{d}x}{\sqrt{(\varrho^2 - x^2)}\, g^{n-1}(x)}$$

where $g(x)$ is defined by (18). The technique, used in the previous examples, can be employed here for further integration. The final result depends on the value of $n$ being even or odd. For $n = 2k$, $k = 1, 2, 3, \ldots$, the potential is

$$(65)\qquad W(\varrho, \phi, z) = 2\sqrt{\pi}\,\sigma_0 \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \Bigg\{ 2B_1 \ln Q + \sum_{m=1}^{k-1} \frac{A_m}{(2m-1)\, Z^{2m-1}} \left[ 1 - \left( \frac{\sqrt{(a^2 - l_1^2)}}{a} \right)^{2m-1} \right] + \sum_{m=1}^{k-1} \frac{B_{m+1}}{m z^m} \left[ Q_1^m - Q_2^m - (Q_3^m - Q_4^m) \right] \Bigg\}\,.$$

Here $Q$, $Q_1$, $Q_2$, $Q_3$, $Q_4$, are defined by (46), and

$$(66)\qquad A_{k-m} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\eta^{m-1}} \left[ \frac{(\eta - z^2)^{k-1}}{(\varrho^2 + z^2 - \eta)^k} \right] \qquad \text{for} \quad \eta = 0\,,$$

$$B_{k-m+1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[ \frac{(t^2 - z^2)^{k-1}}{t^{2k-2}(\sqrt{(\varrho^2 + z^2)} - t)^k} \right] \qquad \text{for} \quad t = -\sqrt{(\varrho^2 + z^2)}\,.$$

For $n = 2k + 1$, the result is

$$(67)\qquad W(\varrho, \phi, z) = 2\sqrt{\pi} \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \frac{\sigma_0}{z^{n-1}} \sum_{m=1}^{k} \Bigg\{ \frac{G_m}{2m-1} \left( \frac{\sqrt{(l_2^2 - a^2)}}{a} \right)^{2m-1} - \frac{(-1)^m H_m}{\Gamma(m)} \frac{\partial^{m-1}}{\partial \eta^{m-1}} \left[ \frac{1}{\sqrt{\eta}} \tan^{-1} \frac{\sqrt{(l_2^2 - a^2)}\sqrt{\eta}}{a} \right] \qquad \text{for} \quad \eta = \frac{\varrho^2 + z^2}{z^2}\,.$$

Here,

$$(68)\qquad G_{k-m+1} = \frac{1}{\Gamma(m)} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[ \frac{(1+t)^{k-1}}{(\eta + t)^k} \right] \qquad \text{for} \quad t = 0\,, \eta = \frac{\varrho^2 + z^2}{z^2}\,,$$

$$H_{k-m+1} = \frac{1}{\Gamma(m)} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[ \frac{(1+t)^{k-1}}{t^k} \right] \qquad \text{for} \quad t = -\frac{\varrho^2 + z^2}{z^2}\,.$$

The solutions for several specific values of $n$ can now be given.

$n = 2$

$$W(\varrho, \phi, z) = \frac{2\pi\sigma_0}{\sqrt{(\varrho^2 + z^2)}} \ln Q \tag{69}$$

and $Q$ is defined by (46). The charge density can be found by differentiation

$$\sigma = -\frac{1}{2\pi} \frac{\partial W}{\partial z}\Bigg/_{z=0} \tag{70}$$

and as

$$\frac{\partial}{\partial z} \ln Q = \mathrm{Re}\left\{-\frac{1}{\varrho} + \frac{a}{\varrho\sqrt{(a^2 - \varrho^2)}}\right\} \quad \text{for} \quad z = 0\,, \tag{71}$$

we have

$$\sigma(\varrho, \phi) = \frac{\sigma_0}{\varrho^2} \mathrm{Re}\left\{1 - \frac{a}{\sqrt{(a^2 - \varrho^2)}}\right\}, \quad \sigma(0, 0) = -\frac{\sigma_0}{2a^2} \tag{72}$$

and Re means that only the real part of the expression inside the brackets is taken.

$n = 3$

$$W(\varrho, \phi, z) = \frac{4\sigma_0}{\varrho^2 + z^2}\left[\frac{\sqrt{(l_2^2 - a^2)}}{a} - \frac{z}{\sqrt{(\varrho^2 + z^2)}} \sin^{-1}\frac{\sqrt{(\varrho^2 + z^2)}}{l_2}\right]. \tag{73}$$

Differentiating (73) and using (70)

$$\sigma(\varrho, \phi) = \begin{cases} \dfrac{2\sigma_0}{\pi\varrho^2}\left[\dfrac{1}{\varrho}\sin^{-1}\dfrac{\varrho}{a} - \dfrac{1}{\sqrt{(a^2 - \varrho^2)}}\right] & \text{for} \quad \varrho < a\,, \\ \dfrac{\sigma_0}{\varrho^3} & \text{for} \quad \varrho \geqq a \end{cases} \tag{74}$$

and $\sigma(0, 0) = -2\sigma_0/3\pi a^3$. The following formula was used in the derivation of (74):

$$\frac{\partial}{\partial z}\sqrt{(l^2 - a^2)} = \mathrm{Re}\left\{\frac{a}{\sqrt{(a^2 - \varrho^2)}}\right\} \quad \text{for} \quad z = 0\,. \tag{75}$$

$n = 4$

$$W(\varrho, \phi, z) = \frac{\pi\sigma_0}{2(\varrho^2 + z^2)^2}\left\{\frac{\varrho^2 - 2z^2}{\sqrt{(\varrho^2 + z^2)}} \ln Q + \right. \tag{76}$$

$$\left. + \frac{2z}{a}\sqrt{(a^2 - l_1^2)} - 3z + \frac{l_2^2}{a^2}\sqrt{(l_2^2 - a^2)}\right\}.$$

At the plane $z = 0$ we have

$$W(\varrho, \phi, o) = \frac{\pi\sigma_0}{2\varrho^3}\left\{\ln\frac{\varrho + \sqrt{(\varrho^2 - a^2)}}{a} + \frac{\varrho\sqrt{(\varrho^2 - a^2)}}{a^2}\right\} \quad \text{for} \quad \varrho \geqq a$$

and $W(\varrho, \phi, o) = 0$ for $\varrho \leqq a$. Further,

$$W(0, 0, z) = \frac{\pi\sigma_0}{z^3}\left[\frac{z^2}{2a^2} - \ln\frac{\sqrt{(a^2 + z^2)}}{a}\right].$$

The charge density $\sigma$ can be found from (76) using (70), (71) and (75):

$$(77) \qquad \sigma(\varrho, \phi) = \frac{\sigma_0}{\varrho^4} \operatorname{Re}\left\{1 - \frac{2a^2 - \varrho^2}{2a\sqrt{(a^2 - \varrho^2)}}\right\}, \quad \sigma(0, 0) = -\frac{\sigma_0}{8a^4}.$$

**Example 4.** Consider the boundary conditions at the plane $z = 0$

$$(78) \qquad W = 0 \qquad \text{for} \quad \varrho \leqq a\,,$$

$$\frac{\partial W}{\partial z} = -2\pi\frac{\sigma_n}{\varrho^n}\,e^{in\phi} \qquad \text{for} \quad \varrho > a\,.$$

Substitution of (78) in (40) gives after in:egration

$$(79) \qquad W(\varrho, \phi, z) = 2\sqrt{\pi}\,\frac{\Gamma(n - 1/2)}{\Gamma(n)}\,\frac{\sigma_n}{\varrho^n}\,e^{in\phi}\left\{\sqrt{(l_2^2 - a^2)} - z + \right.$$

$$\left. + z\sum_{m=1}^{n-1}\frac{(-1)^m\,\Gamma(n)}{\Gamma(m + 1)\,\Gamma(n - m)\,(2m - 1)}\left[1 - (1 - l_1^2/a^2)^{m-1/2}\right]\right\},$$

and at the plane $z = 0$,

$$W(\varrho, \phi, o) = 2\sqrt{\pi}\,\frac{\Gamma(n - 1/2)}{\Gamma(n)}\,\frac{\sigma_n}{\varrho^n}\,e^{in\phi}\operatorname{Re}\sqrt{(\varrho^2 - a^2)}$$

with the charge density

$$(80) \qquad \sigma(\varrho, \phi) = \frac{\Gamma(n - 1/2)}{\sqrt{\pi}\,\Gamma(n)}\,\frac{\sigma_n}{\varrho^n}\,e^{in\phi}\operatorname{Re}\left\{1 - \frac{a}{\sqrt{(a^2 - \varrho^2)}} - \right.$$

$$\left. - \sum_{m=1}^{n-1}\frac{(-1)^m\,\Gamma(n)}{\Gamma(m + 1)\,\Gamma(n - m)\,(2m - 1)}\left[1 - (1 - \varrho^2/a^2)^{m-1/2}\right]\right\}.$$

The results for several particular values of $n$ are

$n = 1$

$$(81) \qquad W(\varrho, \phi, z) = 2\pi\frac{\sigma_1}{\varrho}\,e^{i\phi}\left[\sqrt{(l_2^2 - a^2)} - z\right],$$

$$\sigma(\varrho, \phi) = \frac{\sigma_1}{\varrho}\,e^{i\phi}\operatorname{Re}\left[1 - \frac{a}{\sqrt{(a^2 - \varrho^2)}}\right];$$

$n = 2$

$$(82) \qquad W(\varrho, \phi, z) = \pi\frac{\sigma_2}{\varrho^2}\,e^{2i\phi}\left[\sqrt{(l_2^2 - a^2)} - 2z + \frac{z}{a}\sqrt{(a^2 - l_1^2)}\right]$$

$$\sigma(\varrho, \phi) = \frac{\sigma_2}{\varrho^2} e^{2i\phi} \operatorname{Re}\left[1 - \frac{2a^2 - \varrho^2}{2a\sqrt{(a^2 - \varrho^2)}}\right];$$

$n = 3$

(83) $$W(\varrho, \phi, z) = \frac{3\pi}{4} \frac{\sigma_3}{\varrho^3} e^{3i\phi} \left\{\sqrt{(l_2^2 - a^2)} - \frac{8}{3} z + 2\frac{z}{a}\sqrt{(a^2 - l_1^2)} - \right.$$
$$\left. - \frac{z(a^2 - l_1^2)^{3/2}}{3a^3}\right\};$$

(84) $$\sigma(\varrho, \phi) = \frac{3}{8} \frac{\sigma_3}{\varrho^3} e^{3i\phi} \operatorname{Re}\left\{\frac{8}{3} - \frac{a}{\sqrt{(a^2 - \varrho^2)}} - 2\frac{\sqrt{(a^2 - \varrho^2)}}{a} + \frac{(a^2 - \varrho^2)^{3/2}}{3a^3}\right\}.$$

A more general case of boundary conditions, namely,

(85) $$W = 0 \qquad \text{for} \quad \varrho \leqq a,$$
$$\frac{\partial W}{\partial z} = -2\pi \frac{\sigma_{mn}}{\varrho^n} e^{im\phi} \quad \text{for} \quad \varrho \geqq a$$

can also be considered using the same technique as in the previous examples, and the final result can always be expressed in elementary functions. The form of the result will be different for $(m + n)$ even and for $(m + n)$ odd. As an example, the following expression can be obtained by substituting (85) in (36), namely for $m + n = 2k$,

(86) $$\sigma(\varrho, \phi) = \frac{\sigma_{mn}}{\varrho^n} e^{im\phi} \operatorname{Re}\left\{1 - \frac{a}{\sqrt{(a^2 - \varrho^2)}}\left[1 - \sum_{j=2}^{k} \frac{\Gamma(j - 3/2)}{2\sqrt{\pi}\,\Gamma(j)} \left(\frac{\varrho}{a}\right)^{2j-2}\right]\right\},$$

and for $m + n = 2k + 1$ and $\varrho < a$,

(87) $$\sigma(\varrho, \phi) = \frac{2}{\pi} \frac{\sigma_{mn}}{\varrho^n} e^{im\phi} \left\{\sin^{-1}\frac{\varrho}{a} - \frac{\varrho}{\sqrt{(a^2 - \varrho^2)}}\left[1 - \sum_{j=2}^{k} \frac{\sqrt{\pi}\,\Gamma(j - 1)}{4\Gamma(j + 1/2)}\left(\frac{\varrho}{a}\right)^{2j-2}\right]\right\}.$$

Expressions (86) and (87) represent general formulae which cover all the particular cases considered in Example 3 and Example 4. The form of the solution may sometimes be different, for example, for $m = n = 3$ formula (86) gives

(88) $$\sigma(\varrho, \phi) = \frac{\sigma_{33}}{\varrho^3} e^{3i\phi} \operatorname{Re}\left\{1 - \frac{a}{\sqrt{(a^2 - \varrho^2)}} + \frac{\varrho^2}{2a\sqrt{(a^2 - \varrho^2)}} + \frac{\varrho^4}{8a^3\sqrt{(a^2 - \varrho^2)}}\right\}$$

which looks different from (84), but as one can easily verify expressions (84) and (88) are equivalent.

Using (36) one can obtain the following expression for the total charge $\Sigma$ directly through the given charge density $\sigma$:

(89) $$\Sigma = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty} \sigma(r, \phi) \cos^{-1}\frac{a}{r}\, r\, dr\, d\phi.$$

In the particular case of $\sigma = \sigma_0 r^{-n}$, (89) results in

$$\Sigma = \frac{2\sigma_0 \sqrt{\pi}\, \Gamma[(n-1)/2]}{(n-2)\, \Gamma(n/2)\, a^{n-2}}. \tag{90}$$

The total charge for $n = 3$ is $4\sigma_0/a$, and for $n = 4$, $\Sigma = \pi\sigma_0/2a^2$.

Figure 4 presents the charge density versus radius defined by formulae (72), (74) and (77). The equipotential lines, defined by expression (73), are presented in Fig. 5.

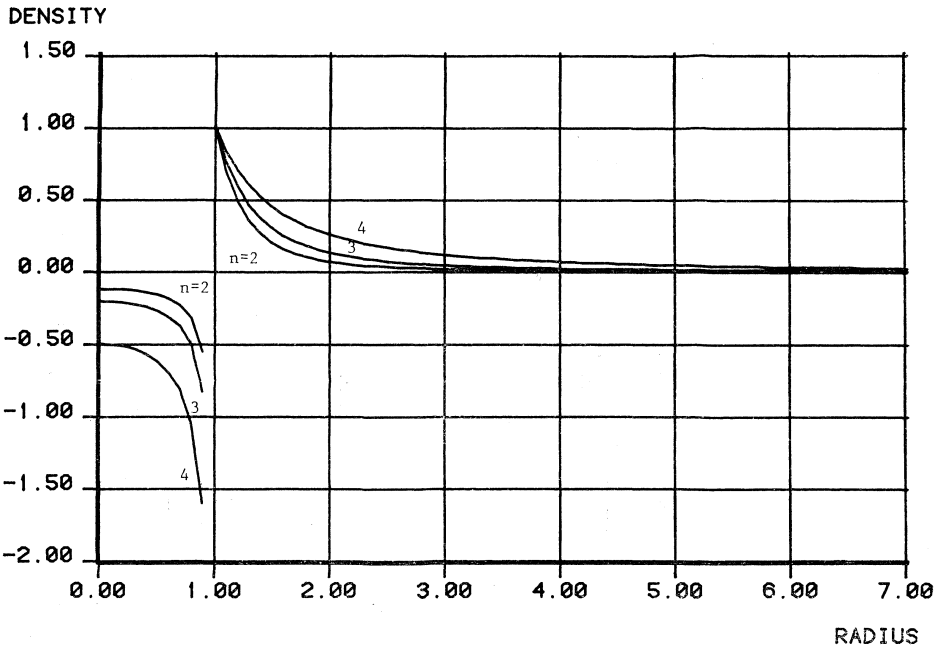


Fig. 4. Charge density for n = 2, 3, 4,

The dimensionless potential $w^* = Wa^2/\sigma_0$ was varied from 0·5 to 1·3. One can notice that the equipotential lines for $w^* < .92$ have two branches. The corresponding equipotential surfaces can be seen in Fig. 6.

## DISCUSSION

No new attention has been paid in literature to the basics of the potential theory after classical contributions in this field by Green, Lord Kelvin, Hobson and others. This paper presents a novel approach to the problem and the advantages of this new approach become evident when compared with the previous classical results.

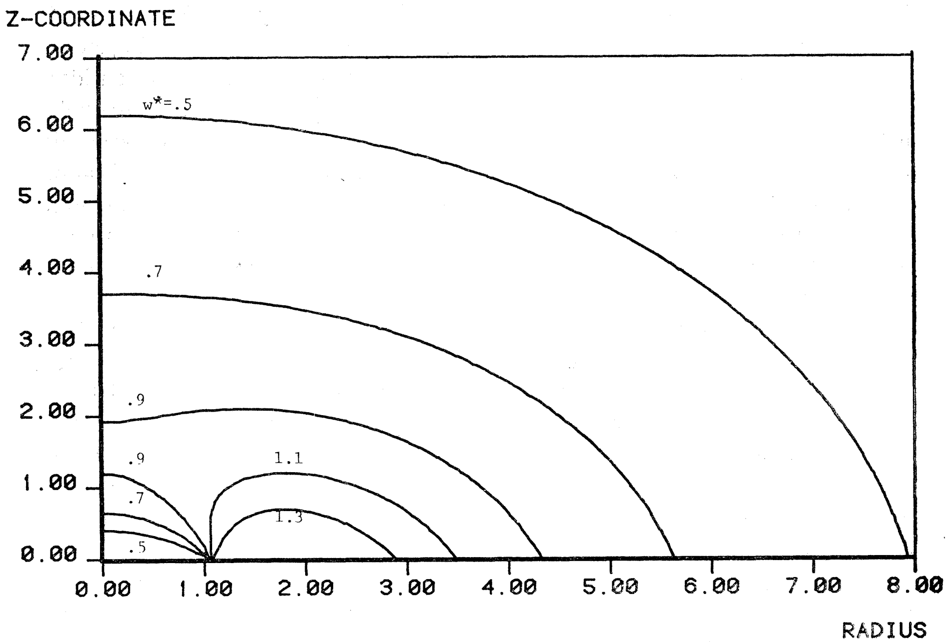


Fig. 5. Equipotential lines for n = 3

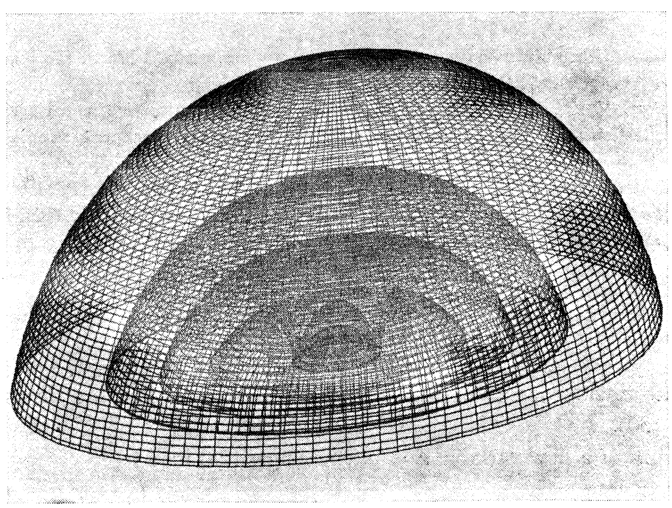

Fig. 6. Equipotential surfaces for n = 3

A formula similar to (30) was first published without proof by Lord Kelvin in 1847, and 22 years later he proved it in his "Papers on Electrostatics and Magnetism" using the method of images which requires a lot of ingenuity in geometric considerations. Later the same formula was derived by Hobson [5] who used Sommerfeld's method of a potential in a Riemannian space. Later several solutions were also published using various integral transforms. Unfortunately, practical applications of these results are rather difficult due to the complexity of the integrals involved. The main advantage of the new approach is its simplicity where the generation of the solution is reduced to a straightforward procedure, and the integrals involved are elementary as demonstrated in the previous section.

The technique of this paper can be used for the solution of many different problems, and, among various immediate applications, one can indicate the punch and crack problems in linear elasticity. For example, expression (25) is proportional to the normal stress distribution for the external punch problem when the normal displacements $w$ are given outside a circle, and the normal stresses are zero inside; expression (37) can give the normal stress distribution in the crack neck through its values on the crack faces; the total force at infinity in external crack problems can be defined by (89), etc.

This new approach can be generalized for more complicated boundary conditions. It can be used not only for a disk, but also for a spherical bowl, and possibly for an arbitrary surface of revolution.

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## APPENDIX A

Here some details of the derivation are presented which allow transformation of (28) into (30).

Introduction of a new variable $u = g(x)$ changes (28) into

$$\text{(A1)} \quad W(\varrho, \phi, z) = -\frac{2}{\pi}\int_a^\infty \frac{\mathrm{d}l_2(u)}{\sqrt{(l_2^2(u) - \varrho^2)}}\, L\left(\frac{l_1^2(u)}{\varrho}\right) \frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty \frac{r\,\mathrm{d}r}{\sqrt{(r^2 - u^2)}}\, L\left(\frac{1}{r}\right) w(r, \phi).$$

Change of the order of integration in (A1) yields

$$\text{(A2)}\quad W(\varrho, \phi, z) = \frac{2}{\pi}\int_a^\infty L\left(\frac{1}{r}\right) w(r, \phi)\, \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\int_a^r \frac{u\, \mathrm{d}l_2(u)}{\sqrt{(r^2 - u^2)}\sqrt{(l_2^2(u) - \varrho^2)}} L\left(\frac{l_1^2(u)}{\varrho}\right).$$

Here the following general formula was used:

$$\text{(A3)}\qquad \int_a^\infty F(u)\, \mathrm{d}u \frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty \frac{r f(r)\, \mathrm{d}r}{\sqrt{(r^2 - u^2)}} = -\int_a^\infty f(r)\, \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\int_a^r \frac{u\, F(u)\, \mathrm{d}u}{\sqrt{(r^2 - u^2)}}.$$

Differentiation under the integral sign in (A2) gives

$$\text{(A4)}\qquad W(\varrho, \phi, z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty \left\{ \frac{l'(a)\, \lambda\left(\dfrac{l_1^2(a)}{\varrho r}, \phi - \psi\right)}{\sqrt{(r^2 - a^2)}\sqrt{(l_2^2(a) - \varrho^2)}} + \right.$$

$$\left. + \int_a^r \frac{\mathrm{d}u}{\sqrt{(r^2 - u^2)}} \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{l_2'(a)\, \lambda\left(\dfrac{l_1^2(u)}{\varrho r}, \phi - \psi\right)}{\sqrt{(l_2^2(u) - \varrho^2)}}\right]\right\} w(r, \psi)\, r\, \mathrm{d}r\, \mathrm{d}\psi\, .$$

Formula (2) was used here along with the following rule of differentiation under the integral sign:

$$\text{(A5)}\qquad \frac{\mathrm{d}}{\mathrm{d}r}\int_a^r \frac{u\, f(u)\, \mathrm{d}u}{\sqrt{(r^2 - u^2)}} = \frac{f(a)\, r}{\sqrt{(r^2 - a^2)}} + r\int_a^r \frac{\mathrm{d}f(u)}{\sqrt{(r^2 - u^2)}}.$$

Introducing the notation

$$\text{(A6)}\quad F(u) = \frac{z}{R^3}\left[\frac{R}{\xi(u)} + \tan^{-1}\frac{\xi(u)}{R}\right], \quad \xi(u) = \frac{\sqrt{(r^2 - u^2)}\sqrt{(l_2^2(u) - u^2)}}{u}.$$

expression (A4) can be rewritten as

$$\text{(A7)}\qquad W(\varrho, \phi, z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty \left\{\frac{r^2 - a^2}{a}\frac{\mathrm{d}F(a)}{\mathrm{d}a} + \right.$$

$$\left. + \int_a^r \frac{\mathrm{d}u}{\sqrt{(r^2 - u^2)}}\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{(r^2 - u^2)^{3/2}}{u}\frac{\mathrm{d}F(u)}{\mathrm{d}u}\right]\right\} w(r, \psi)\, r\, \mathrm{d}r\, \mathrm{d}\psi\, .$$

Now integration with respect to $u$ in (A7) can be performed elementarily by parts, and the result is

$$\text{(A8)}\qquad W(\varrho, \phi, z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty F(a)\, w(r, \psi)\, r\, \mathrm{d}r\, \mathrm{d}\psi\, .$$

Taking into consideration (A6), one can see that expression (A8) is equivalent to (30).

Souhrn

## EXAKTNÍ ŘEŠENÍ NĚKTERÝCH VNĚJŠÍCH SMÍŠENÝCH PROBLÉMŮ V TEORII POTENCIÁLU

VALERY I. FABRIKANT

Je navržena nová metoda řešení smíšeného problému potenciálu v poloprostoru s okrajovými podmínkami na částech hranice rozdělených kruhovým obloukem. Postup je založen na novém typu integrálních operátorů se speciálními vlastnostmi.

Řeší se dva obecné vnější problémy: 1) libovolný potenciál je specifikován na hranici vně kružnice, a jeho normálová derivace uvnitř je nulová; 2) vně kružnice je dána libovolná normálová derivace, uvnitř je potenciál roven nule.

Je uvedeno několik ilustrativních příkladů a diskutují se některé metody aplikace navržených postupů na řešení několika složitých problémů.

Резюме

## ТОЧНОЕ РЕШЕНИЕ НЕКОТОРЫХ ВНЕШНИХ СМЕШАННЫХ ЗАДАЧ ТЕОРИИ ПОТЕНЦИАЛА

VALERY I. FABRIKANT

Предлагается новый метод решения смешанных задач теории потенциала в полупространстве с круговой линией раздела граничных условий. Метод основан на новом типе интегральных операторов со специальными свойствами. Рассмотрены несколько иллюстративных примеров и обсуждены возможности применения нового метода к решению сложных задач теории упругости.

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