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# THE OPTIMIZATION OF THE STATIONARY HEAT EQUATION WITH A VARIABLE RIGHT-HAND SIDE

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*Summary*. Solving the stationary heat equation we optimize the temperature on part of the boundary of the domain under investigation. First the Poisson equation is solved; both the Neumann condition on part of the boundary and the Newton condition on the rest are prescribed, the distribution of the heat sources being variable. In the second case, the heat equation also contains a convective term, the distribution of heat sources is specified and the Neumann condition is variable on part of the boundary.

*Keywords:* stationary heat equation, boundary value problem, distribution of heat sources — optimization, Neumann boundary condition, Newton boundary condition, Poisson equation.

## 1. INTRODUCTION

Computing the Earth's temperature field we regard the terrestrial heat flow and temperature as available empirical data and we also have some notion about the heat conductivity of the rocks of the upper part of the Earth (lithosphere). If we confine ourselves to the investigation of the temperature field of this part, we can regard it as a stationary one. In order to be able to solve the heat equation, we need to know the detailed distribution of heat sources as well as the conditions on the whole boundary of the domain. We shall use two known boundary conditions on the Earth's surface by considering the Neumann condition there (or the Newton condition on part of the surface) and by optimizing the temperature.

In the first case (Problem 1 in Section 2), we are going to determine the interface $y$ which separates two subdomains with different densities of heat sources, i.e. with

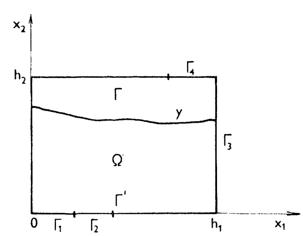


Fig. 1.

different densities of radioactive elements. The situation is sketched in Fig. 1. We shall solve this problem on a rectangle whose upper side, $\Gamma$, is the Earth's surface and the lower one, $\Gamma'$, is the bottom of the investigated domain. For simplicity, we shall suppose that the heat flow crossing the sides $\partial\Omega \setminus \Gamma$ is negligible. The problem is only suitable for regions in which no heat transfer by convection occurs. Since this is the inverse problem for Poisson's equation, it may be also useful in gravimetry.

In the second case (Problem 2 in Section 2), we expect to know the heat sources distribution (if necessary it may be supposed that they are negligible). A more general case is considered when the heat equation also contains the convective term. The problem is formulated for a region near the so-called mid-oceanic ridges where the rocks of the upper mantle are going up and forming the lithspheric plate which is moving horizontally with a velocity of about several cm/year, as the hypothesis of the global plate tectonics asserts (see [1, 4, 6, 9, 11, 14, 16, 17, 18]). Now $\Gamma$ represents the oceanic floor, $x_1$ is the horizontal distance from a mid-oceanic ridge and $\Gamma'$ is the bottom of the lithospheric plate. The horizontal component of the heat flow beneath the ridge is supposed to vanish owing to the symmery. The same is true for the component on the side $\Gamma_3$ because of a great distance from a thermally disturbed region near a mid-ocean ridge. In this problem we are going to determine the unknown boundary condition on the lithospheric bottom $\Gamma'$ so that the temperature on the oceanic floor may be optimized.

Hitherto we have tacitly supposed that the velocity pattern of the moving lithosphere appearing in the convective term is known. We shall use the model of the velocity pattern defined by Problem 3 in Section 2. In this model the velocity pattern is supposed to have the scalar potential $V$ and to satisfy the Laplace equation (for $V$), which is nothing else than the stationary equation of continuity. We suppose that the material enters the domain $\Omega$ only across the parts of the boundary $\Gamma_1$ as well as $\Gamma_2$ and flows out across the side $\Gamma_3$ as a solid plate moving horizontally. We do not know the function describing the flow on $\Gamma_1$; we are going to determine it in such a way that the motion of the rocks in the domain may "aproach" the motion of a solid plate. We shall use the corresponding Newton condition on $\Gamma_2$ in order that the horizontal component of the gradient $V$ may be equal to the velocity of the plate provided the vertical component of the gradient vanishes. A detailed numerical realization of Problems 2 and 3 is described in [8].

## 2. VARIATIONAL FORMULATION AND EXISTENCE OF SOLUTIONS

We shall use the following notation:

$\Omega, \partial\Omega$ – a domain in $E_n$ and its boundary,

$H^k(\Omega)$ – the Sobolev space of functions having square-integrable derivatives of all orders from 1 to $k$ in $\Omega$; we shall denote $H(\Omega) \equiv H^1(\Omega)$, $L^2(\Omega) \equiv H^0(\Omega)$,

$L^\infty(\Omega)$ – the space of bounded measurable functions,
$(\cdot, \cdot)_{k,\Omega}$, $(\cdot, \cdot)_\Omega$, $(\cdot, \cdot)_{\partial\Omega}$ – the inner product in $H^k(\Omega)$, $L^2(\Omega)$, $L^2(\partial\Omega)$, respectively,
$\|\cdot\|_{k,\Omega}$ – the norm in $H^k(\Omega)$, $k \geqq 0$,
$u \,.\, v$ – the inner product of vectors from $E_n$,
$\Omega \in C^k$, $\Omega \in C^{k,1}$ – $\Omega$ has a boundary of order $C^k$ or the derivatives of order $k$ satisfying the Lipschitz condition, respectively (for exact definition see [5, 12]).
If no misunderstanding may arise, we shall omit the symbol $\Omega$ in the notation of the function spaces, the inner products and the norms.

Let us now consider (see also Fig. 1): a domain $\Omega \subset E_2$, $\Omega \in C^{0,1}$; $\Gamma_i \subset \partial\Omega$, $i = 1, \ldots, 4$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$; $\Gamma \subset \partial\Omega$, $\Gamma' \subset \partial\Omega$; $\Gamma \cap \Gamma' = \emptyset$; $\Gamma, \Gamma'$ and $\Gamma_i$ are open sets of positive one-dimensional measure; $k \in L^\infty(\Omega)$, $k \geqq k_0 > 0$, $k_0 \in E_1$; $\varrho \in L^\infty(\Omega)$, $\varrho \geqq \varrho_0 > 0$, $\varrho_0 \in E_1$; $c \in L^\infty(\Omega)$, $c \geqq c_0 > 0$, $c_0 \in E_1$; $A \in L^2(\Omega)$, $A_d \in L^2(\Omega)$; $\sigma_1 \in L^\infty(\partial\Omega)$, $\sigma_1 = 0$ on $\partial\Omega \setminus \Gamma_4$, $\sigma_1 > 0$ on $\Gamma_4$; $\sigma_2 \in L^\infty(\partial\Omega)$, $\sigma_2 = 0$ on $\partial\Omega \setminus \Gamma_2$, $\sigma_2 > 0$ on $\Gamma_2$; $f \in L^2(\partial\Omega)$, $f = 0$ on $\partial\Omega \setminus \Gamma$; $q \in L^2(\partial\Omega)$, $q = 0$ on $\partial\Omega \setminus (\Gamma_2 \cup \Gamma_3)$.

**Problem 1.** *In addition to the above conditions, let $\Omega$ be the rectangle described in Fig. 1 and let the constant $d > 0$ be given. Let us denote*

$$U = \{y(x_1) \in C(\langle 0, h_1 \rangle) ;$$
$$0 \leqq y(x_1) \leqq h_2 , \quad |y(a) - y(b)| \leqq d|a - b| \quad \forall a, b \in \langle 0, h_1 \rangle\} ,$$

*where $h_1$ and $h_2$ are the lengths of the sides $\Gamma$ and $\Gamma_3$, respectively,*

$$a(u, v) = \int_\Omega k \nabla u \,.\, \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \sigma_1 uv \,\mathrm{d}S . \tag{2.1}$$

*Let us seek functions $T \in H$ and $y \in U$ such that*

$$a(T, \varphi) = (A, \varphi) + (f, \varphi)_{\partial\Omega} + \int_0^{h_1} \int_0^{y(x_1)} A_d \varphi \,\mathrm{d}x_1 \,\mathrm{d}x_2 \quad \forall \varphi \in H , \tag{2.2}$$

$$F(T) = \int_\Gamma T^2 \,\mathrm{d}S \tag{2.3}$$

*is minimal.*

**Problem 2.** *Moreover, let $V \in H^2$ be given. Let us denote*

$$o(u, v) = a(u, v) + \int_\Omega \varrho c v \nabla V \,.\, \nabla u \,\mathrm{d}x . \tag{2.4}$$

*Let us seek $h \in L^2(\partial\Omega)$, $h = 0$ on $\partial\Omega \setminus \Gamma'$, and $T \in H$ such that*

$$o(T, \varphi) = (A, \varphi) + (f, \varphi)_{\partial\Omega} + (h, \varphi)_{\partial\Omega} \quad \forall \varphi \in H \tag{2.5}$$

*and $F(T)$ defined by (2.3) is minimal.*

**Problem 3.** *Moreover, let $V_0 \in H$ be given. Let us denote*

$$b(u, v) = \int_\Omega \varrho \nabla u \,.\, \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \sigma_2 uv \,\mathrm{d}S \,. \tag{2.6}$$

*Let us seek $g \in L^2(\partial\Omega)$, $g = 0$ on $\partial\Omega \setminus \Gamma_1$ and $V \in H$ such that*

$$b(V, \varphi) = (q, \varphi)_{\partial\Omega} + (g, \varphi)_{\partial\Omega} \quad \forall \varphi \in H \,, \tag{2.7}$$

$$G(V) = \int_\Omega [\nabla(V - V_0)] \,.\, [\nabla(V - V_0)] \,\mathrm{d}x \tag{2.8}$$

*is minimal.*

**Lemma 2.1.** *Let us define the C-norm in the space $C(\langle 0, h_1 \rangle)$ by*

$$\|y\|_C = \max_{x_1 \in \langle 0, h_1 \rangle} |y(x_1)| \,. \tag{2.9}$$

*Then there exists a solution of Problem* 1.

Proof. According to the Arzelà theorem, $U$ is a compact set in $C(\langle 0, h_1 \rangle)$ with the $C$-norm specified by (2.9). The bilinear form $a$ is elliptic and the right-hand side of (2.2) is a continuous linear form for each fixed $y \in U$. It follows from the Lax-Milgram theorem that there exists one and only one solution $T(y)$ of (2.2) for each $y \in U$. Let us denote $Y(y)(x_1, x_2) = A_d(x_1, x_2)$ for $x_2 \leqq y(x_1)$, $Y(y)(x_1, x_2) = 0$ for $x_2 > y(x_1)$. The trace theorem, the Lax-Milgram theorem (see [12, 15]) and the Schwarz inequality imply that there exist constants $\alpha > 0$, $\beta > 0$ such that for $y_i \in U$, $y_0 \in U$,

$$\|T(y_0) - T(y_i)\|_{L^2(\partial\Omega)} \leqq \beta \|T(y_0) - T(y_i)\|_1 \leqq \beta/\alpha \sup_{\|\varphi\|_1 \leqq 1} (Y(y_i) - Y(y_0), \varphi) \leqq$$
$$\leqq \beta/\alpha \sup_{\|\varphi\|_1 \leqq 1} \|Y(y_i) - Y(y_0)\|_0 \,.\, \|\varphi\|_0 \leqq \beta/\alpha \|Y(y_i) - Y(y_0)\|_0 \,.$$

If $y_i \to y_0$ in the $C$-norm, then evidently $Y(y_i) \to Y(y_0)$ in $L^2(\Omega)$. $F$ is continuous on $L^2(\partial\Omega)$, so the mapping $y \mapsto F(T(y))$ is continuous on $U$, q.e.d.

**Lemma 2.2.** *Let $M_g$ be a non-empty bounded convex and closed set from $L^2(\partial\Omega)$ so that all functions from $M_g$ are equal to zero on $\partial\Omega \setminus \Gamma_1$. Then the functional $G$ given by* (2.7), (2.8) *attains its minimum on $M_g$.*

Proof. The mapping $g \mapsto G(V(g))$ is continuous on $L^2(\partial\Omega)$. $G$ is a convex positive functional and so it attains its minimum on $M_g$ [3], q.e.d.

If we want to use the solution $V$ of Problem 3 in Problem 2, we need also some regularity of the function $V$. Now we shall turn to this problem. We shall use the following two theorems.

**Theorem 2.1** (see [12]): *Let $\Omega \in C^{k,1}$. Then there exists a linear continuous mapping*

$$T_k \colon \prod_{l=0}^{k-1} H^{k-l}(\partial\Omega) \mapsto H^k(\Omega)$$

*such that for all*

$$(u_0, u_1, \ldots, u_{k-1}) \in \prod_{l=0}^{k-1} H^{k-l}(\partial\Omega)\,, \quad T_k(u_0, u_1, \ldots, u_{k-1}) = v\,,$$

$$\frac{\partial^l v}{\partial v^l} = u_l$$

*holds on $\partial\Omega$ for $l = 0, 1, \ldots, k-1$.*

(The symbol $\partial^l/\partial v^l$ is the $l$-th derivative in the direction of the outer normal to the boundary $\partial\Omega$.)

**Theorem 2.2** (see [10]): *Let $\Omega \in C^2$, $\Omega \subset E_n$, $A \in L^2(\Omega)$, $a(x) \equiv (a_1(x), \ldots, a_n(x))$, $a_i(x) \in C^1(\bar{\Omega})$, $i = 1, \ldots, n$, $\sigma \in C^1(\partial\Omega)$. Let $V \in H$ satisfy*

$$(2.10) \quad \int_\Omega \nabla V \,.\, \nabla\varphi \,\mathrm{d}x + \int_\Omega \varphi\, \nabla V \,.\, a \,\mathrm{d}x + \int_{\partial\Omega} \sigma V \varphi \,\mathrm{d}S = \int_\Omega A\varphi \,\mathrm{d}x \quad \forall \varphi \in H(\Omega)\,.$$

*Then $V \in H^2(\Omega)$.*

**Lemma 2.3.** *If $\Omega \in C^{2,1}$, $\varrho \in C^2(\bar{\Omega})$, $\sigma_2 \in C^1(\partial\Omega)$, $g/\varrho \in H(\partial\Omega)$, $q/\varrho \in H(\partial\Omega)$, then the solution $V$ of (2.7) is from $H^2(\Omega)$.*

Proof. Let $\tilde{g} \in H^2(\Omega)$ satisfy

$$(2.11) \quad \left.\frac{\partial \tilde{g}}{\partial v}\right|_{\partial\Omega} = (q + g)/\varrho\,,$$

$$(2.12) \quad \tilde{g}|_{\partial\Omega} = 0\,.$$

According to Theorem 2.1 this function exists. Let us find the solution $V$ of the equation (2.7) in the form $V = \tilde{V} + \tilde{g}$. Then $\tilde{V}$ satisfies

$$(2.13) \quad b(\tilde{V}, \varphi) = (q + g, \varphi)_{\partial\Omega} - \int_\Omega \varrho\, \nabla\tilde{g} \,.\, \nabla\varphi \,\mathrm{d}x - \int_{\partial\Omega} \sigma_2 \tilde{g} \varphi \,\mathrm{d}S\,.$$

It follows from (2.12) that the third term on the right-hand side of (2.13) vanishes. We use the Green theorem [12, 13] for the second term and by means of (2.11), (2.12) we conclude

$$(2.14) \quad b(\tilde{V}, \varphi) = \int_\Omega \nabla(\varrho\, \nabla\tilde{g})\, \varphi \,\mathrm{d}x\,.$$

Since $C^1(\bar{\Omega})$ is dense in $H(\Omega)$, it is sufficient to take the trial function $\varphi \in C^1(\bar{\Omega})$. Now we can rewrite the first term of the bilinear form $b$ (see (2.6)) as

$$\int_\Omega \varrho \nabla \tilde{V} . \nabla \varphi \, dx = \int_\Omega \nabla(\varrho\varphi) . \nabla \tilde{V} dx - \int_\Omega \varphi \nabla \varphi . \nabla \tilde{V} dx .$$

Putting $\tilde{\varphi} = \varrho\varphi$, it is evident that (2.14) is equivalent with

$$\int_\Omega \nabla \tilde{V} . \nabla \tilde{\varphi} \, dx - \int_\Omega \frac{\tilde{\varphi}}{\varrho} \nabla \varrho . \nabla \tilde{V} dx + \int_{\partial\Omega} \sigma_2 \tilde{V} \frac{\tilde{\varphi}}{\varrho} dS = \tag{2.15}$$

$$= \int_\Omega \frac{\tilde{\varphi}}{\varrho} \nabla(\varrho \nabla \tilde{g}) \, dx \quad \forall \tilde{\varphi} \in C^1(\bar{\Omega}) .$$

Since $(1/\varrho) \nabla(\varrho \nabla \tilde{g}) \in L^2(\Omega)$, it follows from Theorem 2.2 that $\tilde{V} \in H^2(\Omega)$, and so $V \in H^2(\Omega)$, q.e.d.

Consider now Problem 2, which is more complicated. First, we have to show the convergence of the integral on the right-hand side of (2.4) for $u \in H$, $v \in H$. We shall use the following theorem.

**Theorem 2.3.** (see [5]): *Let $\Omega \subset E_2, \Omega \in C^{0,1}$. Then $H(\Omega) \hookrightarrow L^p$ for all $p \in \langle 1, \infty)$.* (The symbol $\hookrightarrow$ denotes the embedding.)

Using this theorem we get $\partial V/\partial x_i \in L^4$, $i = 1, 2$, $v \in L^4$; since $\partial u/\partial x_i \in L^2$, $i = 1, 2$, and $\varrho \in L^\infty$, the integral in (2.4) converges and the bilinear form $o$ is bounded.

In general, there may exist no solution of Problem 2. We shall use part of the Fredholm alternative:

**Theorem 2.4** (see [10]): *Let $\Omega \in C^2$, $\Omega \subset E_2$, $a \equiv (a_1, a_2)$, $a_i \in C^1(\bar{\Omega})$, $i = 1, 2$, $\sigma \in C^1(\partial\Omega)$, $A \in L^2(\Omega)$. Let us consider the equation*

$$\int_\Omega \nabla T . \nabla \varphi \, dx + \int_\Omega \varphi a . \nabla T dx + \int_{\partial\Omega} \sigma T \varphi \, dS = \int_\Omega A \varphi \, dx \quad \forall \varphi \in H(\Omega) . \tag{2.16}$$

*If there is no non-trivial solution of* (2.16) *for $A = 0$, then there exists one and only one solution for all $A \in L^2(\Omega)$. Moreover, we can find a constant $\alpha > 0$ such that*

$$\|T\|_1 \leqq \alpha \|A\|_0 . \tag{2.17}$$

Remark. In order to use this theorem, we need to rewrite the equation (2.5) into the form of (2.16) with coefficients belonging to the spaces as those described in Theorem 2.4. Let us consider $\Omega \in C^{2,1}$, $k \in C^2(\bar{\Omega})$, $\sigma_1 \in C^1(\partial\Omega)$, $\varrho \in C^1(\bar{\Omega})$, $c \in C^1(\bar{\Omega})$, $V \in C^2(\bar{\Omega})$, $h \in H(\partial\Omega)$, $f \in H(\partial\Omega)$. We can find a solution of (2.5) in the form $T =$

$= \tilde{T} + \tilde{h}$ where $\tilde{h} \in H^2$ satisfies

$$\left.\frac{\partial \tilde{h}}{\partial \nu}\right|_{\partial\Omega} = \frac{f + h}{k},$$

$$\tilde{h}|_{\partial\Omega} = 0 .$$

Since $k|_{\partial\Omega} \in H(\partial\Omega)$ [12], the existence of $\tilde{h}$ follows from Theorem 2.1. In the same way as in the proof of Lemma 2.3 we get

$$(2.18) \qquad \int_\Omega \nabla\tilde{T} . \nabla\tilde{\varphi} \, dx + \int_\Omega \tilde{\varphi}\left(\frac{\varrho c}{k}\nabla V - \frac{\nabla k}{k}\right) . \nabla\tilde{T} \, dx + \int_{\partial\Omega} \frac{\sigma_1 \tilde{T}}{k} \tilde{\varphi} \, dS =$$

$$= \int_\Omega \frac{A}{k} \tilde{\varphi} \, dx + \int_\Omega \frac{\tilde{\varphi}}{k} \nabla(k \nabla\tilde{h}) \, dx - \int_\Omega \frac{\varrho c}{k} \tilde{\varphi} \nabla V . \nabla\tilde{h} \, dx \quad \forall\tilde{\varphi} \in H(\Omega) .$$

**Lemma 2.4.** *Let $M_h$ be a non-empty bounded convex and closed set from $H(\partial\Omega)$ so that all functions from $M_h$ are equal to zero on $\partial\Omega \setminus \Gamma'$. If $\Omega \in C^{2,1}$, $k \in C^2(\bar{\Omega})$, $\sigma_1 \in C^1(\partial\Omega)$, $\varrho \in C^1(\bar{\Omega})$, $c \in C^1(\bar{\Omega})$, $h \in H(\partial\Omega)$, $f \in H(\partial\Omega)$, $V \in C^2(\bar{\Omega})$ and the equation* (2.18) *has no non-trivial solution for the zero right-hand side, then there exists a solution of Problem* 2.

Proof. In view of (2.17) and the convexity of the cost functional $F: L^2(\partial\Omega) \mapsto E_1$, it is sufficient to prove that the second and third terms on the right-hand side of (2.18) are continuous linear functionals on $L^2(\Omega)$ for $h \to 0$ in $H(\partial\Omega)$. This is true, because $\tilde{h} \to 0$ in $H^2(\Omega)$ according to Theorem 2.1, q.e.d.

Remark. In the lemma we require $V$ to be from $C^2(\bar{\Omega})$. For $V$ as a solution of Problem 3 we only discussed such situation when $V \in H^2(\Omega)$. But, in general, it is true that the smoother $\partial\Omega$ and $\varrho$, $\sigma_2$, $q$, $g$ can be given, the more regular $V$ can be obtained. Since for $\Omega \subset E_2$, $\Omega \in C^{0,1}$ we have $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ [5], it is sufficient for $V$ to be from $H^4(\Omega)$.

## 3. SOLUTION

In this section we shall look for the Fréchet differential of the cost functionals $F$ and $G$. For finding solutions of the problems we shall present an algorithm which is the analogue of that described in [7].

**Lemma 3.1.** *The mapping $T \mapsto F(T)$ from $H(\Omega)$ to $E_1$ has the Fréchet differential $F'(T)$ and*

$$(3.1) \qquad F'(T)(\delta T) = 2\int_\Gamma T\delta T \, dS .$$

Proof. The relation (3.1) directly follows from the trace theorem and the definition of the Fréchet differential.

Now we shall examine the general dependence of the solution $T(y)$ of (2.2) on $y$. Let $y$ be an arbitrary function on $\langle 0, h_1 \rangle$. Then let us denote

$$P_y = \{(x_1, x_2);\ 0 \leqq x_1 \leqq h_1,\ x_2 \leqq y(x_1)\}\,, \quad \Omega_y = \Omega \cap P_y\,,$$

$$\tilde{y}(x_1) = \min(h_2, y^+(x_1)) \quad \text{where} \quad y^+(x_1) = \max(0, y(x_1))\,,$$

$$l(\varphi) = (A, \varphi) + (f, \varphi)_{\partial\Omega}\,.$$

If, moreover, $A_d \in C(\bar{\Omega})$, $u$ and $v$ are measurable functions on $\langle 0, h_1 \rangle$, let us define the linear functionals on $H(\Omega)$,

(3.2)
$$l(v)(\varphi) = \int_{\Omega_v} A_d \varphi \, dx\,, \quad l(u)(\varphi) = \int_{\Omega_u} A_d \varphi \, dx\,,$$

$$l'(u, v)(\varphi) = \int_0^{h_1} A_d(x_1, \tilde{u}(x_1)) \left[ \varphi(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta) \, d\eta \right] (\tilde{v}(x_1) - \tilde{u}(x_1)) \, dx_1\,,$$

where $\varphi(x_1, 0)$ is the trace of the function $\varphi$. Then

$$|l'(u, v)(\varphi)| \leqq \text{const} \int_0^{h_1} \left| \varphi(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta) \, d\eta \right| dx_1 \leqq \text{const} \, \|\varphi\|_{1,\Omega}\,.$$

According to the Lax-Milgram theorem, there exists $\tilde{T}'(u, v) \in H(\Omega)$ such that

$$a(\tilde{T}'(u, v), \varphi) = l'(u, v)(\varphi) \quad \forall \varphi \in H(\Omega)\,.$$

Let us put $R(u, v) = T(v) - T(u) - \tilde{T}'(u, v)$. Then $R(u, v)$ is the solution of the equation

$$a(R(u, v), \varphi) = L(u, v)(\varphi) \quad \forall \varphi \in H(\Omega)\,,$$

where

$$L(u, v) = l(v) - l(u) - l'(u, v)\,.$$

According to the Lax-Milgram theorem

$$\|R(u, v)\|_1 \leqq \frac{1}{\alpha} \|L(u, v)\|_{H'}\,,$$

where $H'$ is the space of linear continuous functionals on $H$ and $\alpha > 0$ is the constant of ellipticity of the bilinear form $a$. Now

$$L(u, v)(\varphi) = \int_{\Omega_v} A_d \varphi \, dx - \int_{\Omega_u} A_d \varphi \, dx - \int_0^{h_1} A_d(x_1, \tilde{u}(x_1)) \Big[ \varphi(x_1, 0) +$$

$$+\int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta \Big] (\tilde{v}(x_1) - \tilde{u}(x_1))\, \mathrm{d}x_1 = \int_0^{h_1} \Big( \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} \Big[ A_{\mathrm{d}}(x_1, x_2)\, \varphi(x_1, x_2) -$$

$$- A_{\mathrm{d}}(x_1, \tilde{u}(x_1)) \Big( \varphi(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta \Big) \Big] \mathrm{d}x_2 \Big)\, \mathrm{d}x_1 .$$

We shall use the formula

$$\varphi(x_1, x_2) = \varphi(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta + \int_{\tilde{u}(x_1)}^{x_2} \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta ,$$

which holds almost everywhere (see [5]).

Now we can write

$$L(u, v)(\varphi) = \int_0^{h_1} \Big( \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} \Big[ A_{\mathrm{d}}(x_1, x_2) \int_{\tilde{u}(x_1)}^{x_2} \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta \Big] \mathrm{d}x_2 \Big)\, \mathrm{d}x_1 +$$

$$+ \int_0^{h_1} \Big[ \varphi(x_1, 0) \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} (A_{\mathrm{d}}(x_1, x_2) - A_{\mathrm{d}}(x_1, \tilde{u}(x_1)))\, \mathrm{d}x_2 \Big]\, \mathrm{d}x_1 +$$

$$+ \int_0^{h_1} \Big( \Big[ \int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta \Big] \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} [A_{\mathrm{d}}(x_1, x_2) - A_{\mathrm{d}}(x_1, \tilde{u}(x_1))]\, \mathrm{d}x_2 \Big)\, \mathrm{d}x_1 \equiv$$

$$\equiv L_1(\varphi) + L_2(\varphi) + L_3(\varphi) .$$

We shall need the formula

$$\int_a^b \Big( \int_a^x f(y)\, \mathrm{d}y \Big)\, \mathrm{d}x = \int_a^b (b - \eta)\, f(\eta)\, \mathrm{d}\eta .$$

We have

$$|L_1(\varphi)| \leqq \mathrm{const} \int_0^{h_1} \Big| \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} \Big( \int_{\tilde{u}(x_1)}^{x_2} \Big| \frac{\partial \varphi}{\partial x_2}(x_1, \eta) \Big|\, \mathrm{d}\eta \Big)\, \mathrm{d}x_2 \Big|\, \mathrm{d}x_1 =$$

$$= \mathrm{const} \int_0^{h_1} \Big| \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} (\tilde{v}(x_1) - \eta) \Big| \frac{\partial \varphi}{\partial x_2}(x_1, \eta) \Big|\, \mathrm{d}\eta \Big|\, \mathrm{d}x_1 \leqq$$

$$\leqq \mathrm{const}\, \|v - u\|_{L^\infty(0,h_1)} \int_0^{h_1} \Big| \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} \Big| \frac{\partial \varphi}{\partial x_2}(x_1, \eta)\, \Big|\mathrm{d}\eta \Big|\, \mathrm{d}x_1 \leqq$$

$$\leqq \mathrm{const}\, \|v - u\|_{L^\infty(0,h_1)} \Big( \int_0^{h_1} |\tilde{v}(x_1) - \tilde{u}(x_1)|\, \mathrm{d}x_1 \Big)^{1/2} \|\varphi\|_1 \leqq$$

$$\leqq \mathrm{const}\, \|v - u\|_{L^\infty(0,h_1)}^{3/2} \|\varphi\|_1 .$$

Moreover, let us suppose that $A_{\mathrm{d}} \in H(\Omega)$, then

$$|L_2(\varphi)|^2 = \Big| \int_0^{h_1} \Big( \varphi(x_1, 0) \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} \Big( \int_{\tilde{u}(x_1)}^{x_2} \frac{\partial A_{\mathrm{d}}}{\partial x_2}(x_1, \eta)\, \mathrm{d}\eta \Big)\, \mathrm{d}x_2 \Big)\, \mathrm{d}x_1 \Big|^2 =$$

$$= \left| \int_0^{h_1} \left( \varphi(x_1, 0) \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} (\tilde{v}(x_1) - \eta) \frac{\partial A_d}{\partial x_2}(x_1, \eta) \, d\eta \right) dx_1 \right|^2 \leqq$$

$$\leqq \int_0^{h_1} \varphi^2(x_1, 0) \, dx_1 \int_0^{h_1} \left[ \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} (\tilde{v}(x_1) - \eta) \frac{\partial A_d}{\partial x_2}(x_1, \eta) \, d\eta \right]^2 dx_1 \leqq$$

$$\leqq \text{const} \, \|\varphi\|_1^2 \int_0^{h_1} \left[ \left( \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} (\tilde{v}(x_1) - \eta)^2 \, d\eta \right) \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} \left( \frac{\partial A_d}{\partial x_2}(x_1, \eta) \right)^2 d\eta \right] dx_1 \leqq$$

$$\leqq \text{const} \, \|\varphi\|_1^2 \, \|v - u\|^3_{L^\infty(0,h_1)} \, .$$

Analogously,

$$|L_3(\varphi)|^2 \leqq \int_0^{h_1} \left[ \int_0^{\tilde{u}(x_1)} \frac{\partial \varphi}{\partial x_2}(x_1, \eta) \, d\eta \right]^2 dx_1 \int_0^{h_1} \left( \int_{\tilde{u}(x_1)}^{\tilde{v}(x_1)} (\tilde{v}(x_1) - \eta) \, . \right.$$

$$\left. . \frac{\partial A_d}{\partial x_2}(x_1, \eta) \, d\eta \right)^2 dx_1 \leqq \text{const} \, \|\varphi\|_1^2 \, \|v - u\|^3_{L^\infty(0,h_1)} \, ,$$

and consequently

$$|L(\varphi)| \leqq \text{const} \, \|\varphi\|_1 \, \|v - u\|^{3/2}_{L^\infty(0,h_1)} \, .$$

According to the Lax-Milgram theorem,

$$\frac{\|R(u, v)\|_1}{\|v - u\|_{L^\infty(0,h_1)}} \leqq \text{const} \, \|v - u\|^{1/2}_{L^\infty(0,h_1)} \, . \tag{3.3}$$

In the space $C(\langle 0, h_1 \rangle)$, there is the norm $L^\infty$ equivalent to the $C$-norm (1.9). Let us denote $\delta v = v - u$. $\tilde{T}'(u, u + \delta v)$ is not the Fréchet differential of the mapping $y \mapsto T(y)$ because $\delta v \mapsto \tilde{T}'(u, u + \delta v)$ is not linear, as one can see from (3.2). Let $u \in C(\langle 0, h_1 \rangle)$ be such that $0 < u(x_1) < h_2$. For arbitrary $\delta v \in C(\langle 0, h_1 \rangle)$ there exists $\varepsilon > 0$ such that $0 < u(x_1) + \varepsilon \delta v(x_1) < h_2$. Let us define

$$T'(u)(\delta v) = \frac{1}{\varepsilon} \tilde{T}'(u, u + \varepsilon \delta v) \, . \tag{3.4}$$

If $u$ lies on the boundary of the set $U$, we can analogously define the Fréchet differential for an increment $\delta v$ directing only into the interior of the set $U$.

Let us suppose $A_d \in C(\bar{\Omega}) \cap H(\Omega)$ in Problem 1. If we omit the terms of higher orders, we can get $\delta T = T(\tilde{v}) - T(\tilde{u})$ for $u \to v$ in the $C$-norm (2.9) as a solution of the equation

$$a(\delta T, \varphi) = l'(u, v)(\varphi) \, . \tag{2.5}$$

Let us define $P \in H$ as a solution of the adjoint equation

$$a(P, \varphi) = -2 \int_\Gamma T(\tilde{u}) \, \varphi \, dS \, . \tag{3.6}$$

As the trace theorem implies, the right-hand side of (3.6) is a continuous linear functional, therefore $P$ is correctly defined. (3.5) and (3.6) must be satisfied for all

$\varphi \in H$. Let us put $\varphi = P$ in (3.5) and $\varphi = \delta T$ in (3.6). Since $a$ is the symmetrical bilinear form, we find by means of (3.1) and (3.2), that

$$(3.7)\ \delta F = -\int_0^{h_1} A_d(x_1, \tilde{u}(x_1)) \left( P(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial P}{\partial x_2}(x_1, \eta)\, d\eta \right) (\tilde{v}(x_1) - \tilde{u}(x_1))\, dx_1 .$$

If we take

$$(3.8)\ \ \delta\tilde{v} \equiv \tilde{v}(x_1) - \tilde{u}(x_1) = \beta(x_1)\, A_d(x_1, \tilde{u}(x_1)) \left( P(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial P}{\partial x_2}(x_1, \eta)\, d\eta \right)$$

for $\beta(x_1) \geqq 0$ "sufficiently small", we obtain $\delta F < 0$ (in the case that $\delta\tilde{v}$ is not identically equal to zero). If the functions $k$, $\sigma$ and the boundary $\partial\Omega$ are sufficiently smooth, $P$ is so regular that

$$P(x_1, 0) + \int_0^{\tilde{u}(x_1)} \frac{\partial P}{\partial x_2}(x_1, \eta)\, d\eta \quad \text{is continuous} .$$

By an appropriate choice of $\beta(x_1)$ it is now easy to guarantee that $\tilde{u} + \delta\tilde{v}$ lies in $U$.

We get the following algorithm:

1) Choose $y_1 \in U$, put $i = 1$.
2) Compute $T_i$ from (2.2).
3) Compute $P_i$ from (3.6).
4) Compute $\delta y_i$ from (3.8).
5) Put $y_{i+1} = y_i + \delta y_i$.
6) Add one to $i$ and go to point 2).

The situation is easier in the case of Problems 2 and 3. The functionals $F$ and $G$ have the Fréchet differentials because the functions $h \mapsto T(h)$ and $g \mapsto V(g)$ are continuous affine mappings. Therefore, we can use the analogue of the algorithm presented.

The numerical realization of Problems 2 and 3 is described in (8) for the East Pacific Rise region.

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*References*

[1] *D. J. Andrews:* Numerical Simulation of Sea-floor Spreading. J. Geoph. Res. 77 (1972), 6470–6481.

[2] *S. Fučík, A. Kufner:* Nonlinear Differential Equations. Elsevier, Amsterdam, 1980.

[3] *S. Fučík, J. Nečas, V. Souček:* Einführung in die Variationsrechnung. Teubner, Leipzig 1977.

[4] *Y. Ida:* Thermal Circulation of Partial Melt and Volcanism behind Trenches. Oceanol. Acta, 1981. Proc. 26th Intern. Geolog. Congress, Geology of Continental Margins Symposium, Paris 1980, July 7–17, p 241–244.

[5] *A. Kufner, O. John, S. Fučík:* Function Spaces. Academia, Praha 1977.

[6] *X. LePichon, J. Francheteau, J. Bonnin:* Plate Tectonics. Elsevier, Amsterdam etc. 1973.

[7] *A. Marocco, O. Pironneau:* Optimum Design with Lagrangian Finite Elements: Design of an Electromagnet. Comp. Meth. in Appl. Math. 15 (1978), 277—308.
[8] *C. Matyska:* The thermal Field of the Lithosphere in the Region of Mid-ocean Ridges Modelled on the Basis of Known Surface Temperature and Heat Flows. Studia geoph. et geodet. 28 (1984), 407—417.
[9] *D. P. McKenzie, J. M. Roberts, N. O. Weiss:* Convection in the Earth's Mantle: towards a Numerical Simulation. J. Fluid Mech. 62 (1974), part 3, 465—538.
[10] *S. Mizohata:* Теория уравнений с частными производными. Mir, Moskva 1977 (translated from Japanese).
[11] *D. R. Moore, N. O. Weiss:* Two-dimensional Rayleigh-Bénard Convection. J. Fluid. Mech. 58 (1973), part 2, 289—312.
[12] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.
[13] *J. Nečas, I. Hlaváček:* Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction. Elsevier, Amsterdam etc. 1981.
[14] *P. Olson, K. M. Corcos:* A Boundary Layer Model for Mantle Convection with Surface Plates. Geoph. J. R. Astr. Soc. 62 (1980), 195—219.
[15] *K. Rektorys:* Variational Methods. Reidel C., Dordrecht—Boston 1977.
[16] *G. Schubert, C. Froidevaux, D. A. Yuen:* Oceanic Lithosphere and Asthenosphere: Thermal and Mechanical Structure. J. Geoph. Res. 81 (1976), 3525—3540.
[17] *J. G. Sclater, J. Francheteau:* The Implications of Terrestrial Heat Flow Observations on Current Tectonic and Geochemical Models of the Crust and Upper Mantle of the Earth. Geoph. J. R. Astr. Soc. 20 (1970), 509—542.
[18] *L. D. Turcotte, E. R. Oxburgh:* Mantle Convection and the New Global Tectonics. Ann. Rev. Fluid. Mech. 4 (1972), 33—68.

## Souhrn

### OPTIMALIZACE STACIONÁRNÍ ROVNICE ŠÍŘENÍ TEPLA PODLE PRAVÉ STRANY

CTIRAD MATYSKA

Při řešení stacionární rovnice šíření tepla je ve dvou případech optimalizována teplota na části hranice zkoumané oblasti. V prvním případě je pro Poissonovu rovnici předepsána na části hranice Neumannova podmínka, na zbytku hranice Newtonova podmínka a mění se rozložení zdrojů tepla. V druhém případě obsahuje rovnice šíření tepla navíc konvekční člen, rozložení zdrojů tepla je dáno a proměnná je Neumannova okrajová podmínka na části hranice.

## Резюме

### ОПТИМАЛИЗАЦИЯ СТАЦИОНАРНОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С ПЕРЕМЕННОЙ ПРАВОЙ ЧАСТЬЮ

CTIRAD MATYSKA

Решая стационарное уравнение теплопроводности, автор оптимизирует распределение источников тепла или тепловой поток на части границы рассматриваемой области таким образом, чтобы температура на другой части границы сходилась к нулю.

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