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Aplikace matematiky, Vol. 30 (1985), No. 5, 375–392

Persistent URL: `http://dml.cz/dmlcz/104164`

ON THE OPTIMAL CONTROL PROBLEM GOVERNED BY THE EQUATIONS OF VON KÁRMÁN II. MIXED BOUNDARY CONDITIONS

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(Received November 22, 1984)

We consider a control problem for the system of nonlinear Kármán's equations for a thin elastic plate. In contrast to [2] we shall deal with mixed boundary conditions Chapters 1, 2 and 3 are devoted to a formulation and solution of the state problem.

Further, we prove the existence of an optimal control for the problem with a control variable on the right-hand side of the state equation, i.e. we control the transversal loading. Using the differentiability in the sense of Fréchet of the state function with respect to the control variable, we derive conditions for the uniqueness of the optimal control. In the last chapter the problem with a stress function as a control variable is considered. The previous results can be extended to this problem.

1. FORMULATION OF THE STATE PROBLEM

Let Ω be a bounded, simply connected region with Lipschitz boundary $\partial\Omega = \Gamma = \bigcup_{j=1}^{l} S_j$, where S_j are simple smooth arcs and the angles of the tangents at the corners (if any) between the adjacent arcs are positive. We define the following problem.

Problem I: to find function y, Φ which are solutions of the system of Kármán's equations

$$\Delta^2 y = [\Phi, y] + v \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\Delta^2 \Phi = -[y, y] \quad \text{in} \quad \Omega, \tag{1.2}$$

where

$$[\varphi, \psi] = \varphi_{11}\psi_{22} + \varphi_{22}\psi_{11} - 2\varphi_{12}\psi_{12},$$

$$\varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \, \partial x_j}, \quad i, j = 1, 2.$$

Here $y \equiv y(x_1, x_2)$ represents the (reduced) deflection of the plate, while $\Phi \equiv \equiv \Phi(x_1, x_2)$ is the (reduced) Airy stress function.

Let

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \quad \Gamma_i \cup \Gamma_j = \emptyset, \quad i \neq j, \tag{1.3}$$

where each Γ_i ($i = 1, 2, 3$) is either empty, or possesses a positive measure (length) and does not contain isolated points. We consider the following boundary conditions for y:

$$\begin{aligned} y = y_n &= 0 \quad \text{on} \quad \Gamma_1, \\ y = 0, \quad M(y) + k_2 y_n &= 0 \quad \text{on} \quad \Gamma_2, \\ M(y) + k_{31} y_n = 0, \quad T(y) + k_{32} y_n &= 0 \quad \text{on} \quad \Gamma_3, \end{aligned} \tag{1.4}$$

where

$$y_n = \frac{\partial y}{\partial n},$$

$$M(y) = \mu \Delta y + (1 - \mu)(y_{11} n_1^2 + 2 y_{12} n_1 n_2 + y_{22} n_2^2),$$

$$T(y) = -\frac{\partial}{\partial n} \Delta y + (1 - \mu) \frac{\partial}{\partial s} [y_{11} n_1 n_2 - y_{12}(n_1^2 - n_2^2) - y_{22} n_1 n_2] + X y_1 + Y y_2 .$$

Here $\boldsymbol{n} = (n_1, n_2)$ is the unit outward normal vector with respect to Γ, $\mu \in [0, \frac{1}{2})$ is the Poisson constant, X, Y are prescribed functions and the functions k_2, k_{31}, k_{32} satisfy the following conditions

$$\begin{gathered} k_2 \in L^p(\Gamma_2), \quad k_2 \geqq 0 \quad \text{a.e. on} \quad \Gamma_2, \\ k_{31} \in L^p(\Gamma_3), \quad k_{32} \in L^1(\Gamma_3), \quad k_{3j} \geqq 0 \quad \text{a.e. on} \quad \Gamma_3 \quad (j = 1, 2), \\ 1 < p < \infty . \end{gathered}$$

In the presence of corners s_i, $i = 1, \ldots, r$ in the interior of Γ_3, we have to add the conditions

$$H(s_i^+) - H(s_i^-) = 0, \quad i = 1, \ldots, r . \tag{1.5}$$

where

$$\begin{aligned} H(s) &= (1 - \mu)[y_{11} n_1 n_2 - y_{12}(n_1^2 - n_2^2) - y_{22} n_1 n_2], \\ H(s_i^+) &= \lim H(s) \quad \text{for} \quad s \to s_i^+, \\ H(s_i^-) &= \lim H(s) \quad \text{for} \quad s \to s_i^- . \end{aligned} \tag{1.6}$$

Finally, we prescribe the boundary conditions for the function Φ:

$$\Phi = \varphi_0, \quad \Phi_n = \varphi_1 \quad \text{on} \quad \Gamma, \tag{1.7}$$

$$\Phi_{22} n_1 - \Phi_{12} n_2 = X, \quad \Phi_{11} n_2 - \Phi_{12} n_1 = Y \quad \text{on} \quad \Gamma_3 . \tag{1.8}$$

Following the article [4] we obtain

$$(1.9)\qquad \Phi = A + Bx_1 + Cx_2 + \int_0^s dt \left[n_2 \int_0^t Y \, du + n_1 \int_0^t X \, du \right],$$

$$\Phi_n = Bn_1 + Cn_2 - n_1 \int_0^s Y \, du + n_2 \int_0^s X \, du ,$$

where A, B, C are arbitrary constants. Hence the functions φ_0, φ_1 from (1.7) depend on the given functions X, Y according to (1.9).

Note that the case $\Gamma_2 \cup \Gamma_3 = \emptyset$ has been analyzed in [2].

2. FORMULATION OF A WEAK SOLUTION

We denote by $L^p(\Omega)$ $(1 \leqq p < \infty)$ the space of all real measurable functions which are integrable with power p on Ω in the Lebesgue sense. In particular, $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(2.1)\qquad (u, v)_0 = \int_\Omega uv \, dx$$

and the associated norm

$$(2.2)\qquad |u|_0 = (u, u)_0^{1/2} .$$

For any integer $m \geqq 1$ we define the space

$$W^{m,p}(\Omega) = \{u \mid u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leqq m\}$$

where the derivatives

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2 ,$$

are to be understood in the sense of distributions. In particular, we denote by $H^2(\Omega) = W^{2,2}(\Omega)$ the Hilbert space with the scalar product

$$(2.3)\qquad (u, v)_2 = \int_\Omega \left(uv + \sum_{|\alpha|=2} D^\alpha u \, D^\alpha v\right) dx$$

and the norm

$$(2.4)\qquad \|u\|_2 = (u, u)_2^{1/2} .$$

Let $C^\infty(\bar{\Omega})$ be the space of all infinitely continuously differentiable functions in Ω which together with all their derivatives can be continuously extended onto $\bar{\Omega}$. We set

$$\mathscr{V} = \{u \mid u \in C^\infty(\bar{\Omega}),\ u = u_n = 0 \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_2\}$$

and denote

$$V = \overline{\mathscr{V}}$$

its closure in $H^2(\Omega)$. Further, we introduce two bilinear forms on $V \times V$:

$$A(u, v) = \int_\Omega [u_{11}v_{11} + 2(1 - \mu)\, u_{12}v_{12} + u_{22}v_{22} + \mu(u_{11}v_{22} + u_{22}v_{11})]\, \mathrm{d}x\,,$$

$$a(u, v) = \int_{\Gamma_2} k_2 u_n v_n\, \mathrm{d}s + \int_{\Gamma_3} (k_{31} u_n v_n + k_{32} uv)\, \mathrm{d}s\,.$$

If the boundary decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ satisfies some suitable conditions (see [4], Lemma 3.1), the bilinear form

$$((u, v)) = A(u, v) + a(u, v)\,, \quad u, v \in V$$

determines a scalar product on V. (For instance, the latter conditions are satisfied if (i) meas $\Gamma_1 > 0$ or (ii) $\Gamma = \Gamma_2$.) In this case, the corresponding norm $\|u\| = ((u, u))^{1/2}$ is equivalent to the original norm $\|u\|_2$. Hence V is a Hilbert space with the scalar product $((u, v))$ and the norm $\|u\|$.

In the end we recall the space

$$H_0^2(\Omega) = \{u \mid u \in H^2(\Omega),\ u = u_n = 0 \text{ on } \Gamma \text{ in the sense of traces}\}\,.$$

It is well known that $H_0^2(\Omega)$ is a Hilbert space with the scalar product

$$((u, v))_0 = \int_\Omega \Delta u\, \Delta v\, \mathrm{d}x$$

and the norm

$$\|u\|_0 = ((u, u))_0^{1/2}\,.$$

Next we define the following trilinear form on $[H^2(\Omega)]^3$:

$$B(u, v, w) = \int_\Omega [u_{12}(v_2 w_1 + v_1 w_2) - u_{22} v_1 w_1 - u_{11} v_2 w_2]\, \mathrm{d}x\,. \tag{2.5}$$

If at least one function of the triple u, v, w belongs to $H_0^2(\Omega)$, then $B(u, v, w)$ can be expressed (see [3], Lemma 2.2.2) in the form

$$B(u, v, w) = \int_\Omega [u, v]\, w\, \mathrm{d}x\,. \tag{2.6}$$

Let us assume that the data of Problem I with the boundary conditions (1.4), (1.5), (1.7), (1.8) satisfy the conditions

$$v \in L^2(\Omega)\,, \tag{2.7}$$

$$X, Y \in L^p(\Gamma_3)\,, \quad 1 < p < \infty\,. \tag{2.8}$$

Definition 2.1. *A couple (y, Φ) is a weak solution of Problem I, if*

1° $y \in V$,

2° $\Phi \in H^2(\Omega)$, $\Phi = \varphi_0$, $\Phi_n = \varphi_1$ on Γ,

3° *the following equations hold:*

$$((y, \varphi)) = B(\Phi, y, \varphi) + (v, \varphi)_0 \quad \text{for all} \quad \varphi \in V, \tag{2.9}$$

$$((\Phi, \psi))_0 = -B(y, y, \psi) \quad \text{for all} \quad \psi \in H_0^2(\Omega). \tag{2.10}$$

It is convenient to introduce another definition of a weak solution with homogeneous boundary conditions. If the functions φ_0, φ_1 satisfy some smoothness conditions (see [4] – eqs. (4.1)), then there exists a function $g \in W^{2,2}(\Omega)$ such that

$$g = \varphi_0, \quad g_n = \varphi_1 \quad \text{on} \quad \Gamma \quad \text{(in the sense of traces)}. \tag{2.11}$$

Moreover, there exists a function $F \in H^2(\Omega)$ which fulfils the relations

$$F - g \in H_0^2(\Omega), \tag{2.12}$$

$$((F, \psi))_0 = 0 \quad \text{for all} \quad \psi \in H_0^2(\Omega). \tag{2.13}$$

It is readily seen that F satisfies the conditions

$$F = \varphi_0, \quad F_n = \varphi_1 \quad \text{on} \quad \Gamma. \tag{2.14}$$

Putting $\Phi = f + F$, where $f \in H_0^2(\Omega)$, we arrive at a new definition of a weak solution.

Definition 2.2. *The couple $[y, f] \in V \times H_0^2(\Omega)$ is an excess weak solution of Problem I if*

$$((y, \varphi)) = B(f, y, \varphi) + B(F, y, \varphi) + (v, \varphi)_0 \quad \textit{holds for all} \quad \varphi \in V \quad \textit{and} \tag{2.15}$$

$$((f, \psi))_0 = -B(y, y, \psi) \quad \textit{holds for all} \quad \psi \in H_0^2(\Omega). \tag{2.16}$$

3. EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION

By the method of Berger ([1], [3], [4]) the existence and the uniqueness of a weak solution can be verified. We transform the system (2.15), (2.16) into the form of an operator equation in the space V.

We first introduce some auxiliary operators.

The operator $M : L^2(\Omega) \to V$ is defined by

$$((Mv, \varphi)) = (v, \varphi)_0 \quad \text{for all} \quad \varphi \in V, \tag{3.1}$$

$C_1 : H_0^2(\Omega) \times V \to V$ by

$$((C_1(u, y), \varphi)) = B(u, y, \varphi) \quad \text{for all} \quad \varphi \in V \tag{3.2}$$

and $C_2: V \times V \to H_0^2(\Omega)$ by

$$((C_2(y, w), \psi))_0 = B(y, w, \psi) \quad \text{for all} \quad \psi \in H_0^2(\Omega)\,. \tag{3.3}$$

The operators M, C_1, C_2 are uniquely determined by virtue of the Riesz theorem. In fact, we have

$$|(v, \varphi)_0| \leqq c_0 |v|_0 \, \|\varphi\| \quad \forall v \in L^2(\Omega)\,, \quad \varphi \in H_0^2(\Omega)\,, \tag{3.4}$$

$$|B(u, y, \varphi)| \leqq c_1 \|u\|_0 \, \|y\| \, \|\varphi\| \quad \forall u \in H_0^2(\Omega)\,, \quad y, \varphi \in V\,, \tag{3.5}$$

$$|B(y, w, \psi)| \leqq c_2 \|y\| \, \|w\| \, \|\psi\|_0 \quad \forall y, w \in V\,, \quad \psi \in H_0^2(\Omega) \tag{3.6}$$

(see [4] – (5.5), (5.9) and the theorem on continuous imbedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ – – [5]).

The last inequalities imply that the linear operator M and the bilinear forms C_1, C_2 are bounded and their norms can be estimated as follows:

$$\|M\| = \sup_{\substack{v \in L^2(\Omega) \\ v \neq 0}} \frac{\|Mv\|}{|v|_0} \leqq c_0\,, \tag{3.7}$$

$$\|C_1\| = \sup_{\substack{u \in H_0{}^2(\Omega), y \in V \\ u \neq 0, y \neq 0}} \frac{\|C_1(u, y)\|}{\|u\|_0 \, \|y\|} \leqq c_1\,, \tag{3.8}$$

$$\|C_2\| = \sup_{\substack{y \in V, w \in V \\ y \neq 0, w \neq 0}} \frac{\|C_2(y, w)\|_0}{\|y\| \, \|w\|} \leqq c_2\,. \tag{3.9}$$

Finally, we define the operator $L: V \to V$ by the relation

$$((Ly, \varphi)) = B(F, y, \varphi) \quad \forall \varphi \in V\,, \quad \forall y \in V\,. \tag{3.10}$$

Lemma 3.1. *The operator $L: V \to V$, defined by* (3.10), *is linear, selfadjoint and compact.*

Proof. The Riesz theorem assures the existence of L. The linearity and selfadjointness of L are direct consequences of the definition (2.5) of the form $B(F, y, \varphi)$.

It remains to verify the compactness. We have (see [4], formula (5.3)) the estimate

$$|B(F, y, \varphi)| \leqq c_3 \|F\|_2 \, \|y\|_{W^{1,4}(\Omega)} \, \|\varphi\|\,, \tag{3.11}$$

and making use of (3.10) we obtain

$$\|Ly\| \leqq c_3 \|F\| \, \|y\|_{W^{1,4}(\Omega)} \quad \text{for all} \quad y \in V\,. \tag{3.12}$$

Let $\{y_n\}$ be a bounded sequence in V. As the imbedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ is compact (see [5]), there exists a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y_0$ in $W^{1,4}(\Omega)$. Then the sequence $\{Ly_{n_k}\}$ is convergent in V as a consequence of (3.12) and the compactness of L follows.

We proceed now to the existence and uniqueness theorem for a weak solution of Problem I.

Theorem 3.1. *Let there exist a constant γ such that*

$$(3.13) \qquad ((Ly, y)) \leqq \gamma\|y\|^2 \quad \text{for all} \quad y \in V,$$

$$(3.14) \qquad 0 < \gamma < 1 - (3\|C_1\| \, \|C_2\| \, \|Mv\|^2)^{1/3},$$

or

$$(3.13') \qquad ((Ly, y)) \leqq \gamma\|y\|_0^2 \quad \textit{for all} \quad y \in H_0^2(\Omega),$$

$$(3.14') \qquad 0 < \gamma < 1 - (\|C_1\| \, \|Mv\|)^{2/3}$$

in the case $\Gamma = \Gamma_1$, $V = H_0^2(\Omega)$, $C_1 = C_2$; then there exists a unique weak solution $y \equiv y(v) \in V$ of Problem I. Moreover, the estimate

$$(3.15) \qquad \|y(v)\| \leqq (1 - \gamma)^{-1} \, \|Mv\|$$

holds.

Remark 3.1. In Section 8 (see Theorem 8.1) some possibilities of satisfying the assumptions (3.13), (3.14) will be shown. Another example has been presented in [2] – Section 2 for the case $\partial\Omega = \Gamma_1$.

Proof. Using the expression (3.1)–(3.3) we can replace the system (2.15), (2.16) by the operator equation in the space V:

$$(3.16) \qquad y - Ly + C(y) = Mv, \quad y \in V,$$

where $C : V \to V$ is defined by

$$(3.17) \qquad C(y) = C_1(C_2(y, y), y), \quad y \in V.$$

Hence the couple $[y, \Phi]$ is a weak solution of Problem I if and only if y is a solution of the equation (3.16) and $\Phi = C_2(y, y) + F$.

We shall investigate only the equation (3.16).

1° **Existence.** We can replace the equation (3.16) by

$$(3.18) \qquad y + C_0(y) = Mv, \quad y \in V,$$

where

$$(3.19) \qquad C_0(y) = -Ly + C(y), \quad y \in V.$$

On the basis of the existence theorem for the equation (3.18) ([4], Ch. 5) it suffices to verify that the operator C_0 is completely continuous and the operator $I + C_0$ is coercive, i.e.

$$(3.20) \qquad \lim_{\|y\| \to \infty} \frac{((y + C_0(y), y))}{\|y\|} = +\infty.$$

The first property was verified in the paper ([4], Ch. 5.). We proceed to the proof of (3.20). Using the symmetry of the form B we obtain

$$((C(y), y)) = ((C_1(C_2(y, y), y), y)) = B(C_2(y, y), y, y) =$$
$$= B(y, y, C_2(y, y)) = \|C_2(y, y\|_0^2 \geqq 0 \quad \text{for all} \quad y \in V.$$

Then the assumptions (3.13), (3.14) imply

$$(3.21) \qquad ((y + C_0(y), y)) \geqq (1 - \gamma) \|y\|^2 \quad \text{for all} \quad y \in V,$$

where

$$(3.22) \qquad 1 - \gamma > (3\|C_1\| \, \|C_2\| \, \|Mv\|^2)^{1/3} \geqq 0$$

and the condition (3.20) is verified. Hence there exists a weak solution $y \in V$ of Problem I. Moreover, the estimate (3.15) follows from (3.18), (3.21).

2° **Uniqueness.** Let y_1, y_2 be two solutions of (3.16). Then we have

$$(I - L)(y_1 - y_2) = C(y_2) - C(y_1)$$

and from (3.13), (3.17),

$$(1 - \gamma) \|y_1 - y_2\| \leqq \|C(y_1) - C(y_2)\| =$$
$$= \|C_1(C_2(y_1, y_1 - y_2), y_1) + C_1(C_2(y_2, y_1 - y_2), y_2) + C_1(C_2(y_1, y_2), y_1 - y_2\| \leqq$$
$$\leqq \tfrac{3}{2}\|C_1\| \, \|C_2\| \, (\|y_1\|^2 + \|y_2\|^2) \, \|y_1 - y_2\| \, .$$

Using the estimate (3.15) we arrive at the inequality

$$\|y_1 - y_2\| \leqq (1 - \gamma)^{-3} (3\|C_1\| \, \|C_2\| \, \|Mv\|^2) \, \|y_1 - y_2\| \, ,$$

which can be satisfied only for $y_1 = y_2$, as follows from (3.22).

In the case $\Gamma = \Gamma_1$ it suffices to consider the conditions (3.13'), (3.14'), because $V = H_0^2(\Omega)$, $C_1 = C_2 : H_0^2(\Omega) \times H_0^2(\Omega) \to H_0^2(\Omega)$ and we can use the estimate

$$((C(y_1) - C(y_2), y_1 - y_2)) \leqq \|C_1\|^2 \max \{\|y_1\|^2, \|y_2\|^2\} \, \|y_1 - y_2\|^2$$

(see [3], Lemma 2.2.5).

4. PROBLEM OF THE OPTIMAL CONTROL BY TRANSVERSAL LOAD

Henceforth we shall assume that there exists a constant $\gamma \in (0, 1)$ such that the estimate (3.13) or (3.13′) holds. We shall consider the following admissible set of controls

$$(4.1) \qquad U_{ad} = \left\{ v \mid v \in L^2(\Omega) \, , \, |v|_0 \leqq \frac{\alpha}{c_0} (3\|C_1\| \, \|C_2\|)^{-1/2} (1 - \gamma)^{3/2} \right\},$$

or

$$(4.1') \qquad U'_{ad} = \left\{ v \mid v \in L^2(\Omega),\ |v|_0 \leqq \frac{\alpha}{c_0} \|C_1\|^{-1} (1-\gamma)^{3/2} \right\},$$

if $\Gamma = \Gamma_1$,

where $\alpha \in (0, 1)$ is an arbitrary constant and c_0 is the constant from (3.4), (3.7), i.e.

$$(4.2) \qquad |\varphi|_0 \leqq c_0 \|\varphi\| \quad \text{for all} \quad \varphi \in V.$$

If $v \in U_{ad}$, then thete exists a unique weak solution of **Problem I.** Indeed, we have from the definition of M:

$$\|Mv\|^2 = (v, Mv)_0 \leqq |v|_0 \, |Mv|_0 \leqq c_0 |v|_0 \, \|Mv\|$$

and hence

$$(4.3) \qquad \|Mv\| \leqq c_0 |v|_0 \quad \text{for all} \quad v \in L^2(\Omega)$$

and

$$(4.4) \qquad \|Mv\| \leqq \alpha(3\|C_1\| \, \|C_2\|)^{-1/2} (1-\gamma)^{3/2}.$$

Consequently, the condition (3.14) from Theorem 3.1 is satisfied. In the same way we obtain the inequality (3.14') for $v \in U'_{ad}$, $\Gamma = \Gamma_1$.

Next we can introduce a cost functional

$$(4.5) \qquad J(v) = \mathscr{J}(y(v)) + j(v), \quad v \in U_{ad},$$

where $y \equiv y(v)$ is a solution of the equation (3.16) and $\mathscr{J} : V \to R$, $j : L^2(\Omega) \to R$ are any functionals. The definition of J is correct due to the unique solvability of (3.16) for every $v \in U_{ad}$.

We define the following optimal control problem:

Optimal Control Problem P: to find $u \in U_{ad}$ such that

$$(4.6) \qquad J(u) = \min_{v \in U_{ad}} J(v),$$

$$(4.7) \qquad y(u) - L(y(u)) + C(y(u)) = Mu.$$

Theorem 4.1. *If the functionals $\mathscr{J}$, j are weakly lower semicontinuous on V and $L^2(\Omega)$ respectively, then there exists a solution $u \in U_{ad}$ of Optimal Control Problem P.*

Proof. There exists a minimizing sequence $\{u_n\} \subset U_{ad}$,

$$(4.8) \qquad \lim_{n\to\infty} J(u_n) = \inf_{v \in U_{ad}} J(v).$$

Since the admissible set U_{ad} is bounded in $L^2(\Omega)$, it is weakly closed. Then there exists a subsequence $\{u_m\}$ such that

$$(4.9) \qquad u_m \to u \quad \text{(weakly) in} \quad L^2(\Omega), \quad u \in U_{ad}.$$

The set $\{y(v)\}$, $v \in U_{ad}$, is bounded in V. In fact, the es:imates (3.15), (4.4) imply

$$\|y(v)\| \leqq \alpha(1-\gamma)^{1/2}\,(3\|C_1\|\,\|C_2\|)^{-1/2} \quad \text{for all} \quad v \in U_{ad}\,. \tag{4.10}$$

Then there exists a subsequence $\{y_k\}$, $y_k \equiv y(u_k)$, of $\{y(u_m)\}$ such that

$$y_k \to y_0 \quad \text{(weakly) in} \quad V\,, \quad y_0 \in V \tag{4.11}$$

and (cf. (3.19))

$$y_k = -C_0(y_k) + Mu_k\,, \quad k = 1, 2, \ldots. \tag{4.12}$$

The operator C_0 is completely continuous (see [4], Ch. 5) and $M : L^2(\Omega) \to V$ is linear bounded. Passing to the weak limit, (4.9), (4.11), (4.12) imply

$$y_0 = -C_0(y_0) + Mu\,. \tag{4.13}$$

We have verified in the third part that there exists a unique solution $y(u)$ of the equation (4.7). Hence $y_0 = y(u)$ and $y_k \to y(u)$ (weakly in V). Since the functionals $\mathscr{J}$, j are weakly lower semicontinuous, we obtain

$$J(u) = \mathscr{J}(y(u)) + j(u) \leqq \liminf_{k\to\infty} \mathscr{J}(y(u_k)) + \liminf_{k\to\infty} j(u_k) \leqq$$
$$\leqq \liminf_{k\to\infty} J(u_k) = \inf_{v\in U_{ad}} J(v)$$

and hence u is a solution of Optimal Control Problem P.

Remark 4.1. Instead of the set U_{ad} defined in (4.1), its arbitrary convex non-empty closed subset can be chosen for U_{ad} in Theorem 4.1.

5. DIFFERENTIABILITY OF THE STATE FUNCTION

We shall use the differential form of (4.6) in order to secure the uniqueness of the optimal control $u \in U_{ad}$. First we show that the mapping $v \mapsto y(v) \in V$, $v \in U_{ad}$, defined by the state equation

$$y(v) - Ly(v) + C(y(v)) = Mv\,, \quad v \in U_{ad}\,, \tag{5.1}$$

is Fréchet-differentiable with respect to $v \in U_{ad}$ and the derivative $y'(v) : L^2(\Omega) \to V$ is determined by the solution of the problem

$$[I - L + C'(y(v))]\, y'(v)\, h = Mh\,, \quad h \in L^2(\Omega)\,, \tag{5.2}$$

where

$$C'(y)\,\eta = 2C_1(C_2(y, \eta), y) + C_1(C_2(y, y), \eta)\,, \quad y, \eta \in V\,, \tag{5.3}$$

is the Fréchet differential of the operator C at the point $y \in V$. The following lemma presents some properties of the operator on the left-hand side of (5.2).

Lemma 5.1. *The operator $A(y(v)) = I - L + C'(y(v))$ is a linear, symmetric and positive definite mapping of V into V for every $v \in U_{ad}$.*

Proof. The linearity follows from the expression (5.3) and from the linearity of I, L. The symmetry results from the symmetry of the operators I, L (see Lemma 3.1) and from the relations

$$((C'(y)\,w, z)) = 2((C_1(C_2(y, w), y), z) + ((C_1(C_2(y, y), w), z)) =$$
$$= 2B(C_2(y, w), y, z) + B(C_2(y, y), w, z) = 2B(y, z, C_2(y, w)) +$$
$$+ B(C_2(y, y), z, w) = 2((C_2(y, z), C_2(y, w)))_0 + ((C_1(C_2(y, y), z), w)) =$$
$$= 2((C_1(C_2(y, z), y), w)) + ((C_1(C_2(y, y), z), w)) = ((C'(y)\,z, w)),$$

which hold for all $y, w, z \in V$.

It remains to verify the positive definiteness. Let $v \in U_{ad}$, $w \in V$. Using the definitions of C_1, C_2 and the estimates (3.13), (3.15), (4.4), we obtain

$$(5.4)\quad ((A(y(v))\,w, w)) = \|w\|^2 - ((Lw, w)) + 2((C_1(C_2(y(v), w), y(v)), w)) +$$
$$+ ((C_1(C_2(y(v), y(v)), w), w)) = \|w\|^2 - ((Lw, w)) +$$
$$+ 2\|C_2(y(v), w)\|_0^2 + ((C_1(C_2(y(v), y(v)), w), w)) \geqq$$
$$\geqq (1 - \gamma - \|C_1\|\,\|C_2\|\,\|y(v)\|^2)\,\|w\|^2 \geqq \tfrac{2}{3}(1 - \gamma)\,\|w\|^2,$$

where $1 - \gamma > 0$ by assumption.

By virtue of Lemma 5.1 there exists a unique solution $z(h) \in V$ of the equation

$$(5.5)\qquad A(y(v))\,z(h) = [I - L + C'(y(v))]\,z(h) = Mh, \quad \forall h \in L^2(\Omega).$$

Let

$$(5.6)\qquad w = w(h) = y(v + h) - y(v) - z(h); \quad v, v + h \in U_{ad}.$$

If we verify $\|w\| = o(h)$, then $z(h) = y'(v)\,h$ is the differential of y in the sense of Fréchet.

Using (3.16), (5.5), (5.6) we have

$$A(y(v))\,w = C'(y(v))\,(y(v + h) - y(v)) - [C(y(v + h) - C(y(v))] =$$
$$= \int_0^1 [C'(y(v)) - C'(y(v) + s\eta)]\,\eta\,ds,$$

where $\eta = y(v + h) - y(v)$.

As $\|y(v)\|$, $\|y(v + h)\|$ are bounded for all $v, v + h \in U_{ad}$ (see (4.10)), the positive definiteness of $A(y(v))$ and the form (5.3) of $C'(y)\,\eta$ yield the estimate

$$(5.7)\qquad \|w\| \leqq K_1\|\eta\|^2,$$

where K_1 is a constant.

The function $\eta = y(v + h) - y(v) \in V$ fulfils the equation

$$\eta - L\eta + C(y(v + \eta) - C(y(v)) = Mh\,. \tag{5.8}$$

We have

$$((C(y(v + h) - C(y(v)), \eta)) = \int_0^1 ((C'(y(v) + s\eta)\,\eta, \eta))\,\mathrm{d}s =$$

$$= 2\int_0^1 \|C_2(y(v) + s\eta, \eta)\|_0^2\,\mathrm{d}s +$$

$$+ \int_0^1 ((C_1(C_2(y(v) + s\eta, y(v) + s\eta), \eta), \eta))\,\mathrm{d}s \geqq$$

$$\geqq -\|C_1\|\,\|C_2\| \max_{s\in\langle 0,1\rangle} \{\|(1 - s)\,y(v) + s\,y(v + h)\|^2\}\,\|\eta\|^2 \geqq$$

$$\geqq -\tfrac{1}{3}\alpha^2(1 - \gamma)\,\|\eta\|^2\,,$$

after having used the estimate (4.10). Using the last estimate we arrive at

$$((\eta - L\eta + C(y(v + h)) - C(y(v)), \eta)) \geqq (1 - \gamma)\,(1 - \tfrac{1}{3}\alpha^2)\,\|\eta\|^2$$

and (5.8), (3.7) imply

$$\|\eta\| \leqq K_2|h|_0\,, \quad K_2 = C_0[(1 - \gamma)\,(1 - \tfrac{1}{3}\alpha^2)]^{-1} \tag{5.9}$$

and comparing with (5.7) we obtain

$$\|w\| = \|y(v + h) - y(v) - z(h)\| = o(h)\,.$$

Thus we have proved the following theorem.

Theorem 5.1. *The mapping $y(\cdot) : U_{ad} \to V$ determined by the equation $y(v) - Ly(v) + C(y(v)) = Mv$, $v \in U_{ad}$, is Fréchet differentiable for all functions $v \in U_{ad}$. The differential $y'(v)\,h$ satisfies the equation*

$$[I - L + C'(y(v))]\,y'(v)\,h = Mh \tag{5.10}$$

for all $h \in L^2(\Omega)$ such that $v + h \in U_{ad}$, where $C'(y(v))$ is defined in (5.3).

6. UNIQUENESS OF THE OPTIMAL CONTROL

Let us assume, moreover, that the functionals $\mathscr{J}$, j are Fréchet differentiable, satisfying the conditions

$$\langle \mathscr{J}'(y_1) - \mathscr{J}'(y_2), y_1 - y_2\rangle \geqq m\|y_1 - y_2\|^2\,, \quad m > 0 \tag{6.1}$$

$$\text{for all} \quad y_1, y_2 \in V\,,$$

(6.2) $$(j'(v_1) - j'(v_2), v_1 - v_2)_0 \geqq N|v_1 - v_2|_0^2, \quad N > 0 \quad \text{for all} \quad v_1, v_2 \in L^2(\Omega)$$

and $\mathscr{J}'$ satisfies the growth condition

(6.3) $$\|\mathscr{J}'(y)\|_* \leqq d_0\|y\| + d_1 \quad \text{for all} \quad y \in V,$$

where d_0 and d_1 are some constants.

If $u \in U_{ad}$ is the optimal control, i.e. a solution of Problem P, then $\langle J'(u), v - u\rangle_0 \geqq 0$ for all $v \in U_{ad}$. Let u_1, u_2 be two optimal controls. Then

(6.4) $$\langle J'(u_1), v - u_1\rangle_0 = \langle \mathscr{J}'(y(u_1)), y'(u_1)(v - u_1)\rangle + \\ + (j'(u_1), v - u_1)_0 \geqq 0,$$

$$\langle J'(u_2), v - u_2\rangle_0 = \langle \mathscr{J}'(y(u_2)), y'(u_2)(v - u_2)\rangle + \\ + (j'(u_2), v - u_2)_0 \geqq 0$$

for all $v \in U_{ad}$.

Inserting u_2, u_1 into (6.4) and adding we obtain

(6.5) $$0 \leqq \langle \mathscr{J}'(y(u_1)) - \mathscr{J}'(y(u_2)), y(u_2) - y(u_1)\rangle + \\ + (j'(u_1) - j'(u_2), u_2 - u_1)_0 - \\ - \langle \mathscr{J}'(y(u_1)), y(u_2) - y(u_1) - y'(u_1)(u_2 - u_1)\rangle - \\ - \langle \mathscr{J}'(y(u_2)), y(u_1) - y(u_2) - y'(u_2)(u_1 - u_2)\rangle.$$

Let us denote

(6.6) $$w_1 = y(u_2) - y(u_1) - y'(u_1)(u_2 - u_1), \\ w_2 = y(u_1) - y(u_2) - y'(u_2)(u_1 - u_2), \\ \eta = y(u_2) - y(u_1).$$

We derive an estimate for w_1. Using (4.7) and (5.10) we have

$$[I - L + C'(y(u_1))]\, w_1 = C(y(u_1)) - C(y(u_2)) + C'(y(u_1))(y(u_2) - y(u_1)) = \\ = \int_0^1 [C'(y(u_1)) - C'(y(u_1 + s\eta)]\, \eta \, ds = \psi.$$

The mean value theorem implies

(6.8) $$\|\psi\| = \sup_{\|h\|=1} \left| \left(\left(\int_0^1 C''(y(u_1) + \tau(s)\,\eta)(\eta, \eta)\, s \, ds, h \right) \right) \right|, \quad \tau(s) \in (0, s),$$

where the second derivative has the form

(6.9) $$C''(y)(\eta, \eta) = 2C_1(C_2(\eta, \eta), y) + 4C_1(C_2(y, \eta), \eta) \quad \text{for all} \quad y, \eta \in V.$$

Using the estimate (4.10) we have

$$\|\psi\| \leqq \alpha[3(1-\gamma)\,\|C_1\|\,\|C_2\|]^{1/2}\,\|\eta\|^2\,. \tag{6.10}$$

Taking into account the positive definiteness (5.4) of the operator $A(y(u_i)) = I - L + C'(y(u_i))$, $i = 1, 2$, and (6.7), (6.10) we obtain the estimate

$$\|w_i\| \leqq \tfrac{3}{2}\alpha[3(1-\gamma)^{-1}\,\|C_1\|\,\|C_2\|]^{1/2}\,\|\eta\|^2\,. \tag{6.11}$$

From (6.5), (6.6), (4.10), (6.1), (6.2), (6.3), (6.11) we derive the inequality

$$\begin{aligned} 0 \leqq \{&-m + [d_0\,\alpha(1-\gamma)^{1/2}\,(3\|C_1\|\,\|C_2\|)^{-1/2} + d_1]\,. \\ &.\,3\alpha[3(1-\gamma)^{-1}\,\|C_1\|\,\|C_2\|]^{1/2}\}\,\|\eta\|^2 - N|u_1 - u_2|_0^2\,. \end{aligned} \tag{6.12}$$

Setting $h = u_1 - u_2$ in (5.9) we obtain

$$\|\eta\| \leqq c_0[(1-\gamma)(1-\tfrac{1}{3}\alpha^2)]^{-1}\,|u_1 - u_2|_0\,. \tag{6.13}$$

It is now easy to deduce sufficient conditions for the uniqueness of the optimal control combining (6.13) with (6.12):

Theorem 6.1. *Let the functionals $\mathscr{J}$, j be weakly lower semicontinuous with Fréchet derivatives satisfying the conditions* (6.1), (6.2), (6.3). *If*

$$m \geqq [d_0\,\alpha(1-\gamma)^{1/2}\,(3\|C_1\|\,\|C_2\|)^{-1/2} + d_1]\,3\alpha[3(1-\gamma)^{-1}\,\|C_1\|\,\|C_2\|]^{1/2}$$

or

$$\begin{aligned} N > \{&-m + [d_0\,\alpha(1-\gamma)^{1/2}\,(3\|C_1\|\,\|C_2\|)^{-1/2} + d_1]\,. \\ &.\,3\alpha[3(1-\gamma)^{-1}\,\|C_1\|\,\|C_2\|]^{1/2}\}\,c_0^2[(1-\gamma)(1-\tfrac{1}{3}\alpha^2)]^{-2} > 0\,. \end{aligned}$$

where $0 < \alpha < 1$, γ is defined in (3.13), (3.14) *and c_0 in* (4.2), *then there exists a unique solution $u \in U_{ad}$ of Optimal Control Problem P.*

7. NECESSARY CONDITIONS OF OPTIMALITY

We assume that the functionals $\mathscr{J}$, j are Fréchet differentiable. As we have mentioned above (cf. (6.4)), if $u \in U_{ad}$ is the optimal control, then the following relations hold:

$$\langle \mathscr{J}'(y(u)), y'(u)(v-u)\rangle + (j'(u), v-u)_0 \geqq 0 \quad \forall v \in U_{ad}\,, \tag{7.1}$$

$$[I - L + C'(y(u))]\,y'(u)\,h = Mh \quad \forall h \in L^2(\Omega)\,. \tag{7.2}$$

We recall that the operator $A(y(u)) = I - L + C'(y(u))$ is symmetric. Then the system (7.1), (7.2) can be rewritten in the form

$$(p + j'(u), v-u)_0 \geqq 0 \quad \forall v \in U_{ad}\,, \tag{7.3}$$

$$[I - L + C'(y(u))]\,p = \mathscr{R}\mathscr{J}'(y(u))\,, \tag{7.4}$$

where $\mathscr{R} : V^* \to V$ is the Riesz representative operator and we have used the relations

$$\langle \mathscr{J}'(y(u)), y'(u)(v-u)\rangle = ((\mathscr{R}\mathscr{J}'(y(u)), y'(u)(v-u))) =$$
$$= ((A(y(u))\, p, y'(u)(v-u))) = ((p, A(y(u))\, y'(u)(v-u))) =$$
$$= ((p, M(v-u))) = (p, v-u)_0 .$$

If we add the state equation

$$y(u) - L\,y(u) + C(y(u)) = Mu \tag{7.5}$$

we obtain the optimality system (7.3), (7.4), (7.5) for Optimal Control Problem P. The equation (7.4) is the adjoint equation to (7.5), $p \in V$ is the adjoint state and $(p + j'(u))$ represents the gradient $J'(u)$.

8. OPTIMAL CONTROL WITH RESPECT TO THE STRESS FUNCTION

Let us rewrite the equation (3.16) in the form

$$y - L(F)\, y + C(y) = Mv\,, \tag{8.1}$$

where $L(F)$: $V \to V$ is the operator defined by (cf. (3.10))

$$((L(F)\, y, \varphi)) = B(F, y, \varphi) \quad \text{for all} \quad y, \varphi \in V \tag{8.2}$$

with a function $F \in H^2(\Omega)$ and the trilinear form B defined by (cf. (2.5))

$$B(F, y, \varphi) = \int_\Omega [F_{12}(y_2\varphi_1 + y_1\varphi_2) - F_{22}y_1\varphi_1 - F_{11}y_2\varphi_2]\, dx\,. \tag{8.3}$$

In Lemma 3.1 it was shown that $L(F) : V \to V$ is for every $F \in H^2(\Omega)$ linear, self-adjoint and compact, its norm being estimated by

$$\|L(F)\|_{\mathscr{L}(V,V)} \leqq c_4\|F\|_2\,, \tag{8.4}$$

where c_4 depends only on the domain Ω.

Setting $\gamma = c_4\|F\|_2$ in Theorem 3.1 and using (3.7), we obtain

Theorem 8.1. *If* $v \in L^2(\Omega)$, $|v|_0 < c_0^{-1}(3\|C_1\|\,\|C_2\|)^{-1/2}$ *and*

$$\|F\|_2 < c_4^{-1}[1 - (3c_0^2\|C_1\|\,\|C_2\|\,|v|_0^2)^{1/3}\,, \tag{8.5}$$

then there exists a unique solution $y \equiv y(F)$ of the equation (8.1). Moreover, the estimate

$$\|y(F)\| \leqq (1 - c_4\|F\|_2)^{-1}\, c_0|v|_0 \tag{8.6}$$

holds.

Henceforth we shall assume that $|v|_0 < c_0^{-1}(3\|C_1\| \|C_2\|)^{-1/2}$ and $v \in L^2(\Omega)$ is fixed.

Let us consider the set of admissible stress functions

$$\tilde{U}_{ad} = \{F \mid F \in H^2(\Omega), \|F\|_2 \leqq \alpha c_4^{-1}[1 - (3c_0^2\|C_1\| \|C_2\| |v|_0^2)^{1/3}]\}, \tag{8.7}$$

where $0 < \alpha < 1$ and c_0, c_4 are defined in (4.2) and (8.4).

We introduce the cost functional

$$\tilde{J}(F) = \mathscr{J}(y(F)) + \tilde{j}(F), \quad F \in \tilde{U}_{ad} \tag{8.8}$$

with functionals $\mathscr{J} : V \to R$, $\tilde{j} : H^2(\Omega) \to R$. Now we can define:

Optimal Control Problem $\tilde{P}$. To find a function $F_0 \in \tilde{U}_{ad}$ such that

$$\tilde{J}(F_0) = \min_{F \in \tilde{U}_{ad}} \tilde{J}(F). \tag{8.9}$$

$$y(F_0) - L(F_0)\, y(F_0) + C(y(F_0)) = Mv. \tag{8.10}$$

Theorem 8.2. *If the functionals $\mathscr{J}$, $\tilde{j}$ are weakly lower semicontinuous on V and $H^2(\Omega)$, respectively, then there exists a solution $F_0 \in \tilde{U}_{ad}$ of Optimal Control Problem $\tilde{P}$.*

Proof. We proceed in a similar way as in the proof of Theorem 4.1.

Let $\{F_n\} \subset \tilde{U}_{ad}$ be a minimizing sequence for $\tilde{J}$:

$$\lim_{n \to \infty} \tilde{J}(F_n) = \inf_{F \in \tilde{U}_{ad}} \tilde{J}(F). \tag{8.11}$$

Since the set $\tilde{U}_{ad}$ is a closed bounded ball in $H^2(\Omega)$, there exists a subsequence $\{F_m\}$ such that

$$F_m \rightharpoonup F_0 \quad \text{(weakly) in} \quad H^2(\Omega), \quad F_0 \in \tilde{U}_{ad}. \tag{8.12}$$

The set $\{y(F)\}$ is bounded by

$$\|y(F)\| \leqq \{1 - \alpha[1 - (3c_0^2\|C_1\| \|C_2\| |v|_0^2)^{1/3}]\}^{-1} c_0|v|_0 \quad \forall F \in \tilde{U}_{ad}, \tag{8.13}$$

as follows from (8.6), (8.7).

Denoting

$$y_m \equiv y(F_m), \quad m = 1, 2, \dots, \tag{8.14}$$

we can find a subsequence $\{y_k\}$ such that

$$y_k \rightharpoonup y_0 \quad \text{(weakly) in} \quad V, \quad y_0 \in V, \tag{8.15}$$

$$y_k - L(F_k)\, y_k + C(y_k) = Mv. \tag{8.16}$$

As the imbedding $V \subset W^{1,4}(\Omega)$ is compact ([5]), we have

$$y_k \to y_0 \quad \text{(strongly) in} \quad W^{1,4}(\Omega)\,. \tag{8.17}$$

Combining (8.17) with (8.12) and using (3.11) we obtain $B(F_k, y_k, \varphi) \to B(F_0, y_0, \varphi)$. Consequently,

$$L(F_k)\, y_k \rightharpoonup L(F_0)\, y_0 \quad \text{(weakly) in} \quad V \tag{8.18}$$

holds by virtue of the relation (8.2).

The operator $C : V \to V$ is completely continuous (see e.g. [4] – (5.13)). Then $C(y_k) \to C(y_0)$ in V follows from (8.15). Passing to the weak limit with $k \to \infty$ in (8.16), we arrive at the equation

$$y_0 - L(F_0)\, y_0 + C(y_0) = Mv\,. \tag{8.19}$$

From the uniqueness of the solution of (8.1) for $F_0 \in \tilde{U}_{ad}$ we conclude that $y_0 \equiv y(F_0)$ and $y_k \rightharpoonup y(F_0)$ (weakly) in V. The rest of the proof is the same as that of Theorem 4.1.

It is possible to obtain similar results as in Chapters 5–7. The mapping $y(\cdot) : : \tilde{U}_{ad} \to V$, determined by the equation (8.1), is Fréchet differentiable and

$$[I - L(F) + C'(y(F))]\, y'(F)\, h = L(h)\, y(F) \quad \forall h \in H^2(\Omega) \tag{8.20}$$

holds, where C' is defined by (5.3).

If the functional $\mathscr{J} : V \to R$ satisfies the assumptions (6.1), (6.3) and the functional $\tilde{\jmath} : H^2(\Omega) \to R$ is Fréchet differentiable with a strongly monotone derivative, then a uniqueness theorem parallel to Theorem 6.1 holds for Optimal Control Problem $\tilde{P}$.,

In the end we introduce necessary conditions of optimality for Problem $\tilde{P}$. They have the form of the optimality system

$$B(F - F_0, y(F_0), p) + \langle \tilde{\jmath}'(F_0), F - F_0 \rangle_2 \geqq 0 \quad \text{for all} \quad F \in \tilde{U}_{ad}\,, \tag{8.21}$$

$$[I - L(F_0) + C'(y(F_0))]\, p = \mathscr{R}\mathscr{J}'(y(F_0))\,, \tag{8.22}$$

$$y(F_0) - L(F_0)\, y(F_0) + C(y(F_0)) = Mv\,, \tag{8.23}$$

where $\mathscr{R} : V^* \to V$ is the Riesz representative operator and $\langle \cdot, \cdot \rangle_2$ denotes the duality between $(H^2(\Omega))^*$ and $H^2(\Omega)$.

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Souhrn

O PROBLÉMU OPTIMÁLNÍHO ŘÍZENÍ PRO KÁRMÁNOVY ROVNICE. II. KOMBINOVANÉ OKRAJOVÉ PODMÍNKY

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Je studována úloha řízení systému Kármánových rovnic pro rovnováhu tenké pružné desky, uložené různým způsobem na okrajích.

Dokazuje se existence optimálního příčného, resp. bočního zatížení. Množina přípustných funkcí je zvolena tak, že stavová úloha má jediné řešení. Je podán důkaz diferencovatelnosti řešení stavové úlohy vzhledem k řídící proměnné, důkaz jednoznačnosti za určitých podmínek a odvozují se nutné podmínky optimality.

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