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ON EXPONENTIAL APPROXIMATION

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1. INTRODUCTION

A large number of events (processes) especially in economy, medicine, biology, physics, chemistry etc. have an exponential character. Therefore, when constructing the formula expressing such a process, the natural starting point is the exponential function, which often reproduces the process more truly than other functions (e.g. polynomials). The aim of this article is to show one kind of exponential approximation.

Problem. One has to find a real function

$$y(x_1, x_2, \ldots, x_n) \quad \text{for} \quad (x_1, x_2, \ldots, x_n) \in E_n \tag{1}$$

so that it assumes given real values for prescribed fixed values of $(x_1, x_2, \ldots, x_n)$. In this article we will limit ourselves only to the case that for the whole definition domain the following inequality holds:

$$y(x_1, x_2, \ldots, x_n) > 0\,. \tag{2}$$

If the given values have an exponential distribution, so that a function which follows this distribution has an exponential character, then it is of advantage to choose as an approximation function a function of the form

$$\begin{aligned} y(x_1, x_2, \ldots, x_n; i_1, i_2, \ldots, i_n) = \\ = \prod_{j_1=0}^{p_1} \prod_{j_2=0}^{p_2} \cdots \prod_{j_n=0}^{p_n} a(i_1^{j_1}, i_2^{j_2}, \ldots, i_n^{j_n})^{\Pi(x_1, \ldots, x_n)}\,, \end{aligned} \tag{3}$$

where $\Pi(x_1, \ldots, x_n) = \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda}$ and the order of the approximation is $p = \sum_{\lambda=1}^{n} p_\lambda$.

Note 1. The n-tuple $(i_1, i_2, \ldots, i_n)$ is an arbitrary permutation of the numbers $1, 2, \ldots, n$ which denotes the order of the products. In the simplest case we have the 1st permutation, i.e. $i_\lambda = \lambda$ for $\lambda = 1, 2, \ldots, n$. *The symbols* $a(i_1^{j_1}, i_2^{j_2}, \ldots, i_n^{j_n})$ *denote* *only the constants belonging to the exponents* $\prod_{\lambda=1}^{n} x_\lambda^{j_\lambda} = x_1^{j_1} \cdot x_2^{j_2} \cdot \ldots \cdot x_n^{j_n}$ *in the relation* (3). At the same time, j_λ in the expression $i_\lambda^{j_\lambda}$ *is no exponent*, it only indicates how many times i_λ occurs.

Now let us consider an n-dimensional "table" with equidistant arguments for the function y. If the values in the table are exact, then one can determine the coefficients a by means of *exponential interpolation*. However, if the values are charged with errors, then it is necessary to use the *exponential approximation*.

We will show two methods: The former is based on the *method of the least squares* while the latter is connected with the method occuring when deriving the *King formula* used for determining the parameters of the Gompertz-Makeham law.

2. NOTATION AND SYMBOLS

For the sake of brevity it is advantageous to introduce some symbols (vectors):

$$\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$$

where i_λ is the index of the λth sum (λth product);

$$\boldsymbol{l} = (l_1, l_2, \ldots, l_p)$$

where l_λ is the λth cumulated group;

$$\boldsymbol{Y_l} = (Y_{1,l_1}, Y_{2,l_2}, \ldots, Y_{p,l_p})$$

where Y_{λ,l_λ} is the value of the λth cumulated group;

$$\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$$

where s_λ is the number of terms (factors) in the λth sum (product);

$$\boldsymbol{m_l} = (m_{l,1}, m_{l,2}, \ldots, m_{l,n})$$

where $m_{l,\lambda}$ is the number of terms in the λth group;

$$y(\boldsymbol{x}; \boldsymbol{i}) = y(x_1, x_2, \ldots, x_n;\ i_1, i_2, \ldots, i_n)\ ;$$

$$\eta(\boldsymbol{x}; \boldsymbol{i}) = \eta(x_1, x_2, \ldots, x_n;\ i_1, i_2, \ldots, i_n) = \log y(\boldsymbol{x}; \boldsymbol{i})\ ;$$

$$a(\boldsymbol{i^j}) = a(i_1^{j_1}, i_2^{j_2}, \ldots, i_n^{j_n})$$

is the coefficient determined from the exact values;

$$c(\boldsymbol{i^j}) = c(i_1^{j_1}, i_2^{j_2}, \ldots, i_n^{j_n})$$

is the coefficient determined from the values charged with errors;

$$\alpha(\boldsymbol{i^j}) = \log a(\boldsymbol{i^j})\ ;$$

$$\gamma(\boldsymbol{i^j}) = \log c(\boldsymbol{i^j})\ ;$$

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$$

where p_λ is the order of the function for the λth variable;

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathsf{T}}$$

is the (column) vector of the variable x;

$$\boldsymbol{h} = (h_1, h_2, \ldots, h_n)$$

where h_λ is the step of x_λ;

$$\varkappa(\boldsymbol{j}) = \frac{(\sum_{\lambda=1}^{n} j_\lambda)!}{\prod_{\lambda=1}^{n} j_\lambda!}$$

is the polynomial coefficient;

$$\boldsymbol{x}^k = [(\sum_{\mu=1}^{n} x_\mu)]^k$$

is the formal power of the sum (without the sign $+$);

e.g. for $k = 2$ we have $\boldsymbol{x}^2 = (x_1^2, 2x_1x_2, x_2^2, 2x_1x_3, 2x_2x_3, x_3^2, \ldots, x_n^2)^{\mathsf{T}}$;

$$\gamma = [\gamma(1^1, 2^0, 3^0, \ldots, n^0), \gamma(1^0, 2^1, 3^0, \ldots, n^0), \ldots, \gamma(1^0, 2^0, 3^0, \ldots, (n-1)^0, n^1)];$$

$$\gamma^2 = [\gamma(1^2, 2^0, 3^0, \ldots, n^0), \gamma(1^1, 2^1, 3^0, \ldots, n^0), \gamma(1^0, 2^2, 3^0, \ldots, n^0),$$
$$\gamma(1^1, 2^0, 3^1, \ldots, n^0), \gamma(1^0, 2^1, 3^1, \ldots, n^0), \ldots, \gamma(1^0, 2^0, 3^0, \ldots, n^2)].$$

The scalar product is

$$(\gamma^2, \boldsymbol{x}^2) = \gamma(1^2, 2^0, 3^0, \ldots, n^0)\, x_1^2 + 2\gamma(1^1, 2^1, 3^0, \ldots, n^0)\, x_1 x_2 +$$
$$+ \gamma(1^0, 2^2, 3^0, \ldots, n^0)\, x_2^2 + \ldots + \gamma(1^0, 2^0, 3^0, \ldots, n^2)\, x_n^2$$

or generally

$$(\gamma^k, \boldsymbol{x}^k) = \sum_{\boldsymbol{j}=0}^{k} \varkappa(\boldsymbol{j})\, \gamma(\boldsymbol{i}^{\boldsymbol{j}}) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda}, \quad \text{where} \quad \sum_{\lambda=1}^{n} j_\lambda = k, \quad k = 0, 1, 2, \ldots.$$

Using this notation we can transcribe the formula (3) into the form

$$y(\boldsymbol{x}; \boldsymbol{i}) = \prod_{\boldsymbol{j}=0}^{\boldsymbol{p}} a(\boldsymbol{i}^{\boldsymbol{j}})^{\Pi(x_1, \ldots, x_n)} \tag{4}$$

and further

$$\eta(\boldsymbol{x}; \boldsymbol{i}) = \sum_{\boldsymbol{j}=0}^{\boldsymbol{p}} \log a(\boldsymbol{i}^{\boldsymbol{j}}) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda} = \sum_{\boldsymbol{j}=0}^{\boldsymbol{p}} \alpha(\boldsymbol{i}^{\boldsymbol{j}}) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda}. \tag{5}$$

If the values are charged with errors, we have

$$\bar{y}(\boldsymbol{x}; \boldsymbol{i}) = \prod_{\boldsymbol{j}=0}^{\boldsymbol{p}} c(\boldsymbol{i}^{\boldsymbol{j}})^{\Pi(x_1, \ldots, x_n)} \tag{6}$$

and

$$\bar{\eta}(\boldsymbol{x}; \boldsymbol{i}) = \sum_{\boldsymbol{j}=0}^{\boldsymbol{p}} \log c(\boldsymbol{i}^{\boldsymbol{j}}) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda} = \sum_{\boldsymbol{j}=0}^{\boldsymbol{p}} \gamma(\boldsymbol{i}^{\boldsymbol{j}}) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda} \tag{7}$$

where $c(\boldsymbol{i}^j)$ are the coefficients determined from the values charged with errors.

If we introduce, for the sake of brevity, the notation

$$(8) \qquad (\gamma^k, x^k) = \Gamma_n^k = \sum_{j=0}^{k} \varkappa(j)\, \gamma(\boldsymbol{i}^j) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda}$$

then for the pth order formula we can write

$$(9) \qquad \eta(x; j) = \sum_{k=0}^{p} \Gamma_n^k \quad \text{where} \quad \Gamma_n^0 = \gamma(j^0)\,.$$

One determines the partial derivatives of Γ_n^k from the formula

$$(10) \qquad \frac{\partial \Gamma_n^k}{\partial[\varkappa(j)\, \gamma(i_1^{\mu_1}, i_2^{\mu_2}, \ldots, i_n^{\mu_n})]} = \sum_{j=0}^{k} \varkappa(j)\, \gamma(\boldsymbol{i}^j) \prod_{\lambda=1}^{n} x_\lambda^{j_\lambda + \mu_\lambda}\,, \quad \sum_{\lambda=1}^{n} \mu_\lambda = k\,.$$

E.g. for $n = 2$, $k = 3$ we have

$$\frac{\partial \Gamma_2^3}{\partial[\varkappa(2; 1)\, \gamma(i_1^2, i_2^1)]} = \sum_{j=0}^{3} \varkappa(j)\, \gamma(i_1^{j_1}, i_2^{j_2})\, x_1^{j_1+2} x_2^{j_2+1}$$

3. THE LEAST SQUARES METHOD

Theorem 1. *In E_n let the arguments $\boldsymbol{x}_q = (x_{q,1}, x_{q,2}, \ldots, x_{q,n})$ for $q = 1, 2, \ldots, s$ correspond to the error charged values $\boldsymbol{y}_q$ (e.g., the results of a measurement). If the values $\boldsymbol{y}_q$ show an exponential behaviour, then the approximation function $\bar{y}(\boldsymbol{x})$ of the* p*th order according to (6) can be written in the form*

$$(11) \qquad \bar{y}(x; \boldsymbol{i}) = \prod_{j=0}^{p} c(\boldsymbol{i}^j)^{\Pi(x_1, \ldots, x_2)}\,.$$

Proof. Let us construct the function of the sum of squares of the derivatives

$$(12) \qquad \Phi[\gamma] = \sum_{i} \left[\sum_{k=1}^{p} \Gamma_n^k - \eta(x; i)\right]^2 .$$

A necessary condition for the minimum of the function $\Phi[\gamma]$ is the validity of the relation

$$(13) \qquad \frac{\partial \Phi[\gamma]}{\partial \gamma_\tau} = 0 \quad \text{for} \quad \tau = 1, 2, \ldots, T$$

where T denotes the number of all variables γ of $\Phi[\gamma]$. After executing the calculation the relation (13) assumes the form

$$(14) \qquad 2 \sum_{\boldsymbol{i}} \left[\sum_{k=1}^{p} \Gamma_n^k - \eta(x; \boldsymbol{i})\right] \prod_{\lambda=1}^{n} x_\lambda^{\mu_{\tau,\lambda}} = 0 \quad \text{for} \quad \tau = 1, 2, \ldots, T$$

where $\mu_{\tau,\lambda}$ is an element of

$$\gamma_\tau\left(i_1^{\mu_{\tau,1}}, i_2^{\mu_{\tau,2}}, \ldots, i_\lambda^{\mu_{\tau,\lambda}}, \ldots, i_n^{\mu_{\tau,n}}\right).$$

The formula (14) is a system of T linear equation with T unknowns γ_τ for $\tau = 1, 2, \ldots$ $\ldots, T$. Solving this system we obtain the values γ_τ, and taking antilogarithms we get the parameters c which, put into (3), determine the approximation function (11).

Note 2. It is necessary to remark that one may use the above method only if the "multidimensional table" contains the knots of the values whose coordinates occur in the formulas of the method.

4. METHOD OF CUMULATED VALUES

This method, which is closely related to the King formula method, consists in cumulating (by addition or multiplication) some successive values. From the table obtained in this way it is possible to determine the parameters of the approximation function. Here also the condition of knots existence must be satisfied. Now, if we want to construct a $\boldsymbol{p}$-order formula, we must divide the number of all arguments $\boldsymbol{s}$ by $\boldsymbol{p} + 1$.

In this manner we obtain the number of the cumulated values $\boldsymbol{m}$, namely

$$(15) \qquad \boldsymbol{m} = \frac{\boldsymbol{s}}{\boldsymbol{p} + 1} \quad \text{where} \quad \boldsymbol{s} = (s_1, s_2, \ldots, s_n) \quad \text{and} \quad \boldsymbol{p} = (p_1, p_2, \ldots, p_n).$$

So instead of all values of the table

$$(16) \qquad y_i = \prod_{j=0}^{p-1} c_j^{x_i^j} \quad \text{for} \quad \boldsymbol{i} = 1, 2, \ldots, s_i$$

we have only the cumulated values

$$(17) \qquad \boldsymbol{Y}_l = \prod_{i=(l-1)m+1}^{lm} y_i = \prod_{k=0}^{p} c_k^{\sum_{i=(l-1)m+1}^{lm} x_i^k} \quad \text{for} \quad l = 1, 2, \ldots.$$

The number of $\boldsymbol{Y}_l$ is essentially less than the number of $\boldsymbol{y}_i$.

Theorem 2. *In E_n let the arguments $\boldsymbol{x}_q = (x_{q,1}, x_{q,2}, \ldots, x_{q,n})$ for $q = 1, 2, \ldots, s$ correspond to the error charged values $\boldsymbol{y}_q$ (e.g., the results of a measurement). Further, let the values $\boldsymbol{y}_q$ show an exponential behaviour. Then the approximation function $K(x)$ of the* p*th order has the form*

$$(18) \qquad K(\boldsymbol{x}; \boldsymbol{i}) = \prod_{j=0}^{p} c(\boldsymbol{i}^j)^{\Pi(x_1, \ldots, x_n)}.$$

Proof. The function $\bar{y}(\boldsymbol{x}; \boldsymbol{i})$ of the $\boldsymbol{p}$th order has $\boldsymbol{p} + 1$ constants. We assume that the quotient of this function is a function of the $(\boldsymbol{p} - 1)$st order and so it will have $\boldsymbol{p}$

constants. One has to prove that the order of the quotient of the function is indeed lower than that of the function.

We transform the primary table with s values into a table with $p + 1$ values by introducing cumulated values the number of which is $m = s/(p + 1)$. Moreover, introducing the notation

$$\sigma_t = \sum_{u=1}^{m} u^t \quad \text{for} \quad \sigma_t = (\sigma_{1,t_1}, \sigma_{2,t_2}, \ldots, \sigma_{n,t_n}) \tag{19}$$

we can write the exponent in the formula (17) in the form

$$\sum_{i=(l-1)m+1}^{lm} x_i^k = \sum_{t=0}^{k} \binom{k}{t} x_{(l-1)m}^{k-t} h^t \sigma_t . \tag{20}$$

Hence the exponent of the quotient of two successive terms is given by the formula

$$\sum_{i=lm+1}^{(l+1)m} x_i^k - \sum_{i=(l-1)m+1}^{lm} x_i^k = \sum_{t=0}^{k-1} \binom{k}{t} (x_{lm}^{k-t} - x_{(l-1)m}^{k-t}) h^t \sigma_t . \tag{21}$$

As the right-hand side of the formula (21) shows, the degree of the polynomial in the exponent is $k - 1$ and therefore the quotient of the primary function also diminishes by 1. Consequently, if the order of the function $\bar{y}(x; i)$ is p, the quotient $Q\bar{y}(x; i)$ will have the order $p - 1$ and $Q^\varrho \bar{y}(x; i)$ the order $p - \varrho$. Let the φth component of the function $\bar{y}(x; i)$, i.e. $\bar{y}_\varphi$, be of the p_φth order, then $Q^{p_\varphi} \bar{y}_\varphi$ will be constant and $Q^{p_\varphi + 1} = 1$.

From $\bar{y}(x; i)$ by successive calculation we get $Q^g \bar{y}(x; i)$ for $g = 1, 2, \ldots, p$, which together with the primary function forms a system of $p + 1$ equations with $p + 1$ unknowns $c(i^j)$. Substituting the result into the formula (11) we determine the approximation function of the $(p + 1)$st order, q.e.d.

5. THE POLYNOMIAL METHOD

The polynomial method is a classical one, which follows the least square method for the primary values and yields a polynomial as an approximation function. This method gives, even for a large number of values, results with a considerably smaller accuracy and therefore it is less suitable for determining an approximation function for values showing exponential behaviour.

6. APPLICATIONS

We give an example of constructing an approximation function for a function of 1 variable (in E_1). If the values y_i are not charged with errors, we actually have an exponential interpolation, which has the form

$$y_i = \prod_{j=0}^{s-1} a_j^{x_i^j}, \quad i = 1, 2, \ldots, s . \tag{22}$$

The parameters of the function are defined by means of rooted quotients (because this will be the case of nonequal intervals) [1].

Note 3. In contradistinction to the main text, in this case the symbols x_1, x_2, x_3 do not mean different variables but only concrete numerical values of the variable x.

A. The values of y_i not charged with errors

Example 1. One has to find a function $y(x)$ satisfying $y_i = f(x_i)$ for $i = 1, 2, 3, 4, 5$, if the values are not charged with errors. The values x_i and y_i are given in Table I.

Table I

i	$y(x_i) = y_i$	Q	Q^2
1	$y(1{\cdot}12) = 18{\cdot}860280$	$Q[1{\cdot}12;\ 1{\cdot}23] = 19{\cdot}225719$	$Q^2[1{\cdot}12;\ 1{\cdot}23;\ 1{\cdot}35] = 3{\cdot}833175$
2	$y(1{\cdot}23) = 26{\cdot}108098$	$Q[1{\cdot}23;\ 1{\cdot}35] = 26{\cdot}187925$	$Q^2[1{\cdot}23;\ 1{\cdot}35;\ 1{\cdot}48] = 4{\cdot}165739$
3	$y(1{\cdot}35) = 38{\cdot}632015$	$Q[1{\cdot}35;\ 1{\cdot}48] = 37{\cdot}413134$	$Q^2[1{\cdot}35;\ 1{\cdot}48;\ 1{\cdot}56] = 4{\cdot}495876$
4	$y(1{\cdot}48) = 61{\cdot}864355$	$Q[1{\cdot}48;\ 1{\cdot}56] = 51{\cdot}299731$	
5	$y(1{\cdot}56) = 84{\cdot}771466$		

i	Q^3	Q^4
1	$Q^3[1{\cdot}12;\ 1{\cdot}23;\ 1{\cdot}35;\ 1{\cdot}48] = 1{\cdot}260000$	$Q^4[1{\cdot}12;\ 1{\cdot}23;\ 1{\cdot}35;\ 1{\cdot}48;\ 1{\cdot}56] = 1{\cdot}000000$
2	$Q^3[1{\cdot}23;\ 1{\cdot}35;\ 1{\cdot}48;\ 1{\cdot}56] = 1{\cdot}260000$	

In view of the fact that we have 5 values, we assume a formula with 5 parameters a_j for $j = 0, 1, 2, 3, 4$. Therefore we write the given function in the form

$$y = a_0 \,.\, a_1^x \,.\, a_2^{x^2} \,.\, a_3^{x^3} \,.\, a_4^{x^4} \,.$$

By calculating all rooted quotients in Table I one can see that the fourth rooted quotient has the value 1 and then the above mentioned function reduces to

$$y = a_0 \,.\, a_1^x \,.\, a_2^{x^2} \,.\, a_3^{x^3} \,.$$

The rooted quotients of this function (defined in [1]) for $i = 1$ are

$$Q[x_1, x_2] = a_1 \,.\, a_2^{x_1 + x_2} \,.\, a_3^{x_1^2 + x_1 x_2 + x_2^2}\,, \quad Q^2[x_1, x_2, x_3] = a_2 \,.\, a_3^{x_1 + x_2 + x_3}\,,$$

$$Q^3[x_1, x_2, x_3, x_4] = a_3 \,.$$

These expressions together with

$$y_1 = a_0 \,.\, a_1^{x_1} \,.\, a_2^{x_1^2} \,.\, a_3^{x_1^3}$$

give the left-hand sides of the system of equations for the calculation of the values a_j for $j = 0, 1, 2, 3$. By solving this system we obtain with an accuracy of 10^{-3} the values

$$a_3 = 1{\cdot}26000\,, \quad a_2 = 1{\cdot}63000\,, \quad a_1 = 2{\cdot}34000 \quad \text{and} \quad a_0 = 2{\cdot}85000\,.$$

The desired expression has the form

$$y = 2{\cdot}85 \,.\, 2{\cdot}34^x \,.\, 1{\cdot}63^{x^2} \,.\, 1{\cdot}26^{x^3}\,.$$

The polynomial approximation function calculated from the values y_i for $i = 1, 2, 3, 4, 5$ has the form

$$\bar{y}(x) = 660{\cdot}420895x^4 - 3059{\cdot}626853x^3 + 5451{\cdot}728535x^2 - 4358{\cdot}644910x + 1321{\cdot}264676\,.$$

For the value x not included in Table I the values $\bar{y}(x)$ differ from those of $y(x)$. In order to compare both formulas we introduce the values of the both functions for some arguments not included in Table I:

$$\begin{aligned} y(1{\cdot}18) &= 22{\cdot}43270 & \bar{y}(1{\cdot}18) &= 22{\cdot}39501 \\ y(1{\cdot}29) &= 31{\cdot}60087 & \bar{y}(1{\cdot}29) &= 31{\cdot}62346 \\ y(1{\cdot}41) &= 47{\cdot}71448 & \bar{y}(1{\cdot}41) &= 47{\cdot}68768 \\ y(1{\cdot}52) &= 72{\cdot}23926 & \bar{y}(1{\cdot}52) &= 72{\cdot}26846 \end{aligned}$$

B. The values of y_i charged with errors

Let us assume that the approximation function of the pth order for the given function values has the form

$$y = \prod_{j=0}^{p} c_j^{x^j}\,. \tag{23}$$

The system of equations for the parameters is

$$\prod_{i=1}^{s} \prod_{j=0}^{p} c_j^{x_i^{j+r}} = \prod_{i=1}^{n} z_i^{x_i^r} \quad \text{for} \quad r = 0, 1, 2, \ldots, p\,, \tag{24}$$

where the values z_i are charged with errors.

Example 2. In the second column of Table II the values of $z(x)$ corresponding to the arguments x are given. The values of $z(x)$ are charged with errors (we can assume them e.g. to be results of a measurement). If we either know or can justifiably suppose that the values of $z(x)$ show an exponential behaviour, one has to find a function which agrees with the given values as well as possible. We show the solution by each of the three methods.

Table II

1	2	3	4	5	6	7	8
x	$z(x)$	$y(x)$	$K(x)$	$P(x)$	$y(x) - z(x)$	$K(x) - z(x)$	$P(x) - z(x)$
1·4	3·316	3·306	3·298	4·455	−0·010	−0·018	+1·139
1·5	3·531	3·539	3·533	4·215	+0·008	+0·002	+0·684
1·6	3·802	3·798	3·793	4·085	−0·004	−0·009	+0·283
1·7	4·058	4·086	4·084	4·064	+0·028	+0·026	+0·006
1·8	4·413	4·408	4·408	4·154	−0·005	−0·005	−0·259
1·9	4·759	4·768	4·769	4·353	+0·009	+0·010	−0·406
2·0	5·178	5·171	5·173	4·662	−0·007	−0·005	−0·516
2·1	5·615	5·623	5·626	5·081	+0·008	+0·011	−0·534
2·2	6·142	6·130	6·134	5·610	−0·012	−0·008	−0·532
2·3	6·686	6·700	6·704	6·249	+0·014	+0·018	−0·437
2·4	7·356	7·343	7·347	6·998	−0·013	−0·009	−0·358
2·5	8·081	8·069	8·071	7·856	−0·012	−0·010	−0·225
2·6	8·872	8·889	8·888	8·825	+0·017	+0·016	−0·047
2·7	9·791	9·819	9·814	9·903	+0·028	+0·023	+0·112
2·8	10·903	10·874	10·863	11·092	−0·029	−0·040	+0·189
2·9	12·001	12·074	12·055	12·390	+0·073	+0·054	+0·389
3·0	13·480	13·442	13·411	13·798	−0·038	−0·069	+0·318
3·1	14·957	15·004	14·958	15·316	+0·047	+0·001	+0·359
					+0·102	−0·012	+0·165

a. The least squares method

Here we use the formula (24). If we denote $x_i = 1{\cdot}3 + 0{\cdot}1 \,.\, i$ for $i = 1, 2, \ldots, 18$, we have

$$\prod_{i=1}^{18} z(x_i) = 6{\cdot}170598 \,.\, 10^{14} \quad \text{for} \quad r = 0\,,$$

$$\prod_{i=1}^{18} z(x_i)^{x_i} = 1{\cdot}410650 \,.\, 10^{35} \quad \text{for} \quad r = 1\,,$$

$$\prod_{i=1}^{18} z(x_i)^{x_i^2} = 2{\cdot}167399 \,.\, 10^{87} \quad \text{for} \quad r = 2\,.$$

By solving this system we obtain

$$y(x) = 1{\cdot}67534 \,.\, 1{\cdot}35341^x \,.\, 1{\cdot}13925^{x^2}\,.$$

b. Method of cumulated values

In the case of the modified King's formula we start from the formula (17). Here $p = 2$ and $m = 6$. Therefore

$$Y_1 = \prod_{i=1}^{6} c_0 \,.\, c_1^{x_i} \,.\, c_2^{x_i^2}\,, \quad Y_2 = \prod_{i=7}^{12} c_0 \,.\, c_1^{x_i} \,.\, c_2^{x_i^2}\,, \quad Y_3 = \prod_{i=13}^{18} c_0 \,.\, c_1^{x_i} \,.\, c_2^{x_i^2}\,.$$

The quotients are given by the expressions

$$Q[1, 2] = \frac{Y_2}{Y_1}, \quad Q[2, 3] = \frac{Y_3}{Y_2}, \quad Q^2[1, 2, 3] = \frac{Q[2, 3]}{Q[1, 2]}\,.$$

The calculation can proceed according to the following scheme:

$$Y_1 = \prod_{i=1}^{6} z(x_i) = 3793{\cdot}9013$$

$$Q[1, 2] = 18{\cdot}707200$$

$$Y_2 = \prod_{i=7}^{12} z(x_i) = 70973{\cdot}271 \qquad\qquad Q^2[1, 2, 3] = 1{\cdot}7260080\,.$$

$$Q[2, 3] = 32{\cdot}288777$$

$$Y_3 = \prod_{i=13}^{18} z(x_i) = 2291640{\cdot}1$$

From the formulas for $Q^2[1, 2, 3]$, $Q[1, 2]$ and Y_1 we successively obtain $a_2 = 1{\cdot}13467$, $a_1 = 1{\cdot}37828$, $a_0 = 1{\cdot}64293$ and so the approximation function has the form $K(x) = 1{\cdot}64293 \,.\, 1{\cdot}37828^x \,.\, 1{\cdot}13467^{x^2}$.

c. The polynomial method

Let the 2nd order polynomial approximation function have the form $P(x) = c_0 + c_1 x + c_2 x^2$. The classical least squares method used for the primary values leads to a system of equations for c_0, c_1 and c_2. The right-hand sides of these equations are the sums

$$\sum_{i=1}^{18} z(x_i) = 132{\cdot}941\,, \quad \sum_{i=1}^{18} x_i z(x_i) = 330{\cdot}4417\,, \quad \sum_{i=1}^{18} x_i^2\, z(x_i) = 854{\cdot}64595\,.$$

By solving this system we get the parameters $c_0 = 19{\cdot}35604$, $c_1 = -18{\cdot}33557$, $c_2 = 5{\cdot}49430$ and the approximation function will have the form $P(x) = 19{\cdot}35604 - 18{\cdot}33557x + 5{\cdot}49430x^2$.

The columns 3, 4 and 5 of Table II contain the values of the approximation functions $y(x)$, $K(x)$ and $P(x)$ while the columns 6, 7 and 8 give the deviations of the values of these functions from the corresponding values of the function $z(x)$. In the last rows of the 6th, 7th and 8th columns, the sums of the deviations are given. To the purpose of comparing the accuracy of the particular approximation functions we give the sums of squares of the deviations:

$$\sum_{i=1}^{18} [y(x_i) - z(x_i)]^2 = 0{\cdot}013\,, \quad \sum_{i=1}^{18} [K(x_i) - z(x_i)]^2 = 0{\cdot}012\,,$$

$$\sum_{i=1}^{18} [P(x_i) - z(x_i)]^2 = 3{\cdot}713\,.$$

As follows from these values, the polynomial function used for the approximation of functions with an exponential character is significantly worse than the functions based on the exponential function. Finally, it is necessary to remark that the technique of calculation of the function $K(x)$ is much simpler than that of the function $y(x)$ with roughly the same accuracy, as the sum of squares of the deviations shows.

Reference

[1] *A. Huťa:* On exponential interpolation. Acta Facultatis Rerum Naturalium Universitatis Comenianae, Mathematica XXXV (1979), 157—183.

Súhrn

O EXPONENCIÁLNEJ APROXIMÁCII

ANTON HUŤA

Problém. Treba nájsť reálnu funkciu (1) premenných x_i pre $i = 1, 2, \ldots, n$ v E_n, aby pre vopred dané pevné hodnoty $(x_1, x_2, \ldots, x_n)$ nadobudla určité reálne hodnoty, pre ktoré platí (2).

Ak funkcia sledujúca rozdelenie týchto hodnôt má exponenciálny charakter, potom za aproximujúcu funkciu je vhodné zvoliť funkciu tvaru (3), ktorá uvedené rozdelenie lepšie vystihuje ako iné aproximujúce funkcie (napr. založené na polynómoch). V práci sú uvedené tri metódy: 1. Metóda najmenších štvorcov prispôsobená pre exponenciálny priebeh funkcie. 2 .Metóda kumulatívnych hodnôt tzv. Kingova formula. 3. Metóda polynomická spomenutá len okrajove za účelom porovnania s predchádzajúcimi.

V aplikácii je poukázané na numerický výpočet aproximujúcej funkcie podľa uvedených metód ako aj ich nepresnosti.

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