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SOLVABILITY OF A FIRST ORDER SYSTEM IN THREE-DIMENSIONAL NON-SMOOTH DOMAINS

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1. INTRODUCTION

In this article we first deal with the validity of the inequality

$$\|v\|_0 \leqq C(\|\operatorname{div} v\|_0 + \|\operatorname{rot} v\|_0)\,, \tag{1.1}$$

where v is a vector function defined on a bounded and generally non-smooth domain $\Omega \subset \mathbb{R}^3$, and the vanishing normal component $n \,.\, v$ on the boundary $\partial\Omega$ is assumed. Following some preliminary lemmas in the next section, we show that (1.1) holds if and only if Ω is simply connected (Section 3). The inequality (1.1) was established earlier for a smooth domain which is homeomorphic to a ball even for the $\|\cdot\|_1$-norm on the left-hand side (see [3]). Other proofs are given in [8, 18–21]; they are mainly based on contradiction arguments. Estimates analogous to (1.1) for plane non-smooth domains are treated in [10] and in [11], where also mixed boundary conditions are prescribed. We also recall [15] that in the case of vanishing tangential components of v on $\partial\Omega$, the inequality (1.1) is valid iff $\partial\Omega$ is connected (in $\mathbb{R}^2$ and $\mathbb{R}^3$).

In Section 4 we apply (1.1) to the problem of solvability of the first order system of four partial differential equations

$$\begin{aligned} \operatorname{div} u &= f \quad \text{in} \quad \Omega\,, \\ \operatorname{rot} u &= g \quad \text{in} \quad \Omega\,, \\ n \,.\, u &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{1.2}$$

which play an important role in fluid flow and magnetostatic problems [4, 5, 16–22].

2. SOME FUNCTION SPACES

Throughout the paper, $\Omega \subset \mathbb{R}^3$ will always be a bounded domain with a Lipschitz boundary $\partial\Omega$ (see [14], p. 17) and with the outward unit normal n. Notations $H^k(\Omega)$, $k = 0, 1, \ldots$, are used for the (real valued) Sobolev spaces. The usual norm in $H^k(\Omega)$

and also in $(H^k(\Omega))^3$ will be denoted by $\|\cdot\|_k$. The scalar product on $(L^2(\Omega))^m, m = 1,3$, will be written as $(\cdot, \cdot)_0$ and we set

$$L_0^2(\Omega) = \{\chi \in L^2(\Omega) \mid (\chi, 1)_0 = 0\} .$$

Further, $H^{1/2}(\partial\Omega)$ is the space of traces of functions from $H^1(\Omega)$, and $\mathscr{D}(\Omega)$ is the space of infinitely differentiable functions with a compact support in Ω.

We note (see [9], p. 16) that the functional $v \mapsto n \,.\, v|_{\partial\Omega}$ defined on $(C^\infty(\bar{\Omega}))^3$ can be extended by continuity to a linear continuous mapping from the space

$$H(\text{div}; \Omega) = \{v \in (L^2(\Omega))^3 \mid \exists F \in L^2(\Omega) : (v, \text{grad } z)_0 + (F, z)_0 = 0 \ \forall z \in \mathscr{D}(\Omega)\}$$

into $H^{-1/2}(\partial\Omega)$, the latter being the dual space to $H^{1/2}(\partial\Omega)$. The function F is called the divergence of v (in the sense of distributions) and the Green formula can be rewritten as

$$(2.1) \qquad (\text{div } v, z)_0 + (v, \text{grad } z)_0 = \langle n \,.\, v, z\rangle_{\partial\Omega} \ \forall v \in H(\text{div}; \Omega), \ \forall z \in H^1(\Omega) .$$

Here $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Let $\partial\Omega_1, \ldots, \partial\Omega_r$ be the components of $\partial\Omega$. For $v \in H(\text{div}; \Omega)$ we define the functional $n \,.\, v \in H^{-1/2}(\partial\Omega_i)$, $i \in \{1, \ldots, r\}$, by

$$(2.2) \qquad \langle n \,.\, v, z\rangle_{\partial\Omega_i} = (\text{div } v, z)_0 + (v, \text{grad } z)_0 , \quad z \in Z_i ,$$

where

$$Z_i = \{z \in H^1(\Omega) \mid z = 0 \text{ on } \partial\Omega_j \ \forall j \in \{1, \ldots, r\} - \{i\}\}$$

and $\langle \cdot, \cdot \rangle_{\partial\Omega_i}$ is the duality pairing between $H^{-1/2}(\partial\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$.

Let us further introduce the space

$$H(\text{rot}; \Omega) = \{v \in (L^2(\Omega))^3 \mid \exists G \in (L^2(\Omega))^3 : (v, \text{rot } z)_0 = (G, z)_0 \ \forall z \in (\mathscr{D}(\Omega))^3\}$$

endowed with the norm

$$\|\cdot\|_{H(\text{rot};\Omega)} = (\|\cdot\|_0^2 + \|\text{rot} \cdot\|_0^2)^{1/2} .$$

The function G introduced above is called the rotation of v (in the sense of distributions) and the following Green formula holds:

$$(2.3) \qquad (\text{rot } v, z)_0 - (v, \text{rot } z)_0 = \langle n \times v, z\rangle_{\partial\Omega} \quad \forall v \in H(\text{rot}; \Omega) \quad \forall z \in (H^1(\Omega))^3 .$$

Here the vector product $n \times v$ is from $(H^{-1/2}(\partial\Omega))^3$ (see [9], p. 21) and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $(H^{-1/2}(\partial\Omega))^3$ and $(H^{1/2}(\partial\Omega))^3$.

Now, we define several subspaces of $H(\text{div}; \Omega)$ and $H(\text{rot}; \Omega)$:

$$\begin{aligned}
H_0(\text{div}; \Omega) &= \{v \in H(\text{div}; \Omega) \mid n \,.\, v = 0 \text{ on } \partial\Omega\} , \\
H(\text{div}^0; \Omega) &= \{v \in H(\text{div}; \Omega) \mid \text{div } v = 0 \text{ in } \Omega\} , \\
H_0(\text{div}^0; \Omega) &= H_0(\text{div}; \Omega) \cap H(\text{div}^0; \Omega) , \\
H_0(\text{rot}; \Omega) &= \{v \in H(\text{rot}; \Omega) \mid n \times v = 0 \text{ on } \partial\Omega\} ,
\end{aligned}$$

$$
\begin{aligned}
H(\mathrm{rot}^0; \Omega) &= \{v \in H(\mathrm{rot}; \Omega) \mid \mathrm{rot}\, v = 0 \text{ in } \Omega\}\,, \\
H_0(\mathrm{rot}^0; \Omega) &= H_0(\mathrm{rot}; \Omega) \cap H(\mathrm{rot}^0; \Omega)\,, \\
\mathscr{H}_{\mathscr{D}} &= H_0(\mathrm{div}^0; \Omega) \cap H(\mathrm{rot}^0; \Omega)\,, \\
\mathscr{H}_{\mathscr{R}} &= H(\mathrm{div}^0; \Omega) \cap H_0(\mathrm{rot}^0; \Omega)\,, \\
V &= H_0(\mathrm{div}; \Omega) \cap H(\mathrm{rot}; \Omega)\,, \\
D &= \{v \in H(\mathrm{div}^0; \Omega) \mid \langle n \,.\, v, 1\rangle_{\partial\Omega_i} = 0,\ i = 1, \ldots, r\}\,.
\end{aligned}
$$

From (2.1) we can easily derive

$$\mathrm{grad}\, z \in H(\mathrm{rot}^0; \Omega) \quad \text{for} \quad z \in H^1(\Omega)\,. \tag{2.4}$$

Henceforth, we shall present some other properties of the above spaces.

Lemma 2.1. *The following inclusions hold*:

$$\mathrm{rot}\, v \in D \quad \text{for} \quad v \in H(\mathrm{rot}; \Omega)\,, \tag{2.5}$$

and

$$\mathrm{rot}\, v \in H_0(\mathrm{div}^0; \Omega) \quad \text{for} \quad v \in H_0(\mathrm{rot}; \Omega)\,. \tag{2.6}$$

Proof. Let $v \in H(\mathrm{rot}; \Omega)$ and $z \in \mathscr{D}(\Omega)$ be given. Then by (2.3) we obtain

$$(\mathrm{rot}\, v, \mathrm{grad}\, z)_0 = (v, \mathrm{rot}\,\mathrm{grad}\, z)_0 + \langle n \times v, \mathrm{grad}\, z\rangle_{\partial\Omega} = 0\,. \tag{2.7}$$

Hence, (2.1) yields

$$\mathrm{rot}\, v \in H(\mathrm{div}^0; \Omega)\,. \tag{2.8}$$

Let us choose $i \in \{1, \ldots, r\}$ arbitrarily and let $\eta \in C^\infty(\bar{\Omega})$ be such that $\eta = 1$ in a neighbourhood of $\partial\Omega_i$ and $\eta = 0$ in some neigbourhoods of the other components $\partial\Omega_j$, $j \neq i$, that is $\eta \in Z_i$. Thus (2.2), (2.8) and (2.3) imply

$$\langle n \,.\, \mathrm{rot}\, v, 1\rangle_{\partial\Omega_i} = (\mathrm{rot}\, v, \mathrm{grad}\, \eta)_0 = \langle n \times \mathrm{grad}\, \eta, v\rangle_{\partial\Omega} = 0\,.$$

Consequently, (2.5) is valid. The relation (2.7) holds for any $v \in H_0(\mathrm{rot}; \Omega)$ and $z \in C^\infty(\bar{\Omega})$ as well. Therefore, $\mathrm{rot}\, v \in H_0(\mathrm{div}^0; \Omega)$. □

Lemma 2.2. *The identity*

$$(\mathrm{rot}\, \varphi, \mathrm{rot}\, \varphi)_0 = (\varphi, \mathrm{rot}\,\mathrm{rot}\, \varphi)_0$$

holds for all $\varphi \in H_0(\mathrm{rot}; \Omega)$ *such that* $\mathrm{rot}\, \varphi \in H(\mathrm{rot}; \Omega)$.

Proof. Let $\varphi \in H_0(\mathrm{rot}; \Omega)$ with $\mathrm{rot}\, \varphi \in H(\mathrm{rot}; \Omega)$ be given. As $(C^\infty(\bar{\Omega}))^3$ is dense in $H(\mathrm{rot}; \Omega)$ (see [6, 9]), there exists a sequence $\psi_j \in (C^\infty(\bar{\Omega}))^3$ such that

$$\|\mathrm{rot}\, \varphi - \psi_j\|_{H(\mathrm{rot};\Omega)} \to 0 \quad \text{as} \quad j \to \infty\,. \tag{2.9}$$

Applying the Green formula (2.3), we get

$$(\mathrm{rot}\, \varphi, \psi_j)_0 - (\varphi, \mathrm{rot}\, \psi_j)_0 = \langle n \times \varphi, \psi_j\rangle_{\partial\Omega} = 0\,,$$

since $\varphi \in H_0(\mathrm{rot}; \Omega)$. From (2.9) we conclude that

$$(\mathrm{rot}\, \varphi, \psi_j)_0 \to (\mathrm{rot}\, \varphi, \mathrm{rot}\, \varphi)_0$$

and

$$(\varphi, \operatorname{rot} \psi_j)_0 \to (\varphi, \operatorname{rot} \operatorname{rot} \varphi)_0$$

for $j \to \infty$, which yields the result as required. $\square$

3. STUDY OF THE INEQUALITY (1.1)

First, let us recall the definition of a simply connected domain (see e.g. [2, 7, 12, 14]).

Definition 3.1. *A domain Ω in $\mathbb{R}^d$ is said to be simply connected if it has the following property: Given any simple closed curve γ: $x = h(t)$, $t \in [a, b]$, with range in Ω, there is a continuous function $x = F(s, t)$ defined for $s \in [0, 1]$, $t \in [a, b]$ such that:*

(i) $F(0, t) = h(t)$, $t \in [a, b]$;
(ii) $F(1, t) = P$, $t \in [a, b]$, *where P is some point in Ω;*
(iii) $F(s, t)$ *lies in Ω for all* $s \in [0, 1]$, $t \in [a, b]$.
(iv) $F(s, a) = F(s, b)$ *for all* $s \in [0, 1]$.

Defining (closed) curves γ_s by $x = F(s, t)$, $t \in [a, b]$, we say that the family $\{\gamma_s\}$ represents a continuous deformation of γ into a point P.

Domains which are not simply connected are called multiply connected.

The main task of this section will be to prove the following theorem.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Then*

$$\|v\|_0 \leqq C(\|\operatorname{div} v\|_0 + \|\operatorname{rot} v\|_0) \quad \forall v \in V = H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega) \tag{3.1}$$

if and only if Ω is simply connected.

The proof is based on an auxiliary lemma:

Lemma 3.3. *Let Ω be a simply connected domain with a Lipschitz boundary and let $\psi \in H_0(\operatorname{div}^0; \Omega)$. Then there exists exactly one stream function $\varphi \in D \cap$ $\cap H_0(\operatorname{rot}; \Omega)$ such that*

$$\psi = \operatorname{rot} \varphi\,.$$

Moreover,

$$\|\varphi\|_0 \leqq C\|\operatorname{rot} \varphi\|_0\,, \tag{3.2}$$

where $C > 0$ does not depend on φ (and ψ).

Proof. For the existence of precisely one divergence-free stream function $\varphi \in$ $\in D \cap H_0(\operatorname{rot}; \Omega)$ corresponding to $\psi \in H_0(\operatorname{div}^0; \Omega)$ see e.g. [1, 24]. We only prove the inequality (3.2).

From the unicity of φ and (2.6), the linear operator

$$\operatorname{rot}\colon D \cap H_0(\operatorname{rot}; \Omega) \to H_0(\operatorname{div}^0; \Omega) \tag{3.3}$$

is bijective. The space $H_0(\mathrm{div}^0; \Omega)$ equipped with the $\|\cdot\|_0$-norm is a Banach space. One can easily find that the space $D \cap H_0(\mathrm{rot}; \Omega)$ with the norm $\|\cdot\|_{H(\mathrm{rot};\Omega)}$ is a Banach space as well. As the operator (3.3) is continuous, i.e.

$$\|\mathrm{rot}\ \varphi\|_0 \leqq C' \|\varphi\|_{H(\mathrm{rot};\Omega)},$$

by the closed graph theorem the inverse (closed) operator is continuous as well. Thus (3.2) holds. $\square$

Proof of Theorem 3.2. $\Rightarrow$: It is known (see e.g. [1], p. 153) that $\Omega \subset \mathbb{R}^3$ is simply connected if and only if the components of $\mathbb{R}^3 - \bar{\Omega}$ are simply connected. Suppose that Ω is multiply connected. Then there exists a component ω of $\mathbb{R}^3 - \bar{\Omega}$ which is also multiply connected, and we show that (3.1) does not hold.

In accordance with Definition 3.1 there exists a simple closed curve $\gamma \subset \omega$ which cannot be continuously deformed into a point without leaving the domain ω. Clearly, γ can be chosen in such a way that it is smooth enough. Let $\tilde{\Gamma}$ be a sufficiently smooth orientable surface bounded by γ (see Fig. 1) and let

$$\Gamma = \tilde{\Gamma} \cap \Omega .$$

By a regularization technique (see e.g. [13], p. 58), it is easy to construct a function $q \in C^\infty(\Omega - \Gamma)$ with bounded derivatives such that $q = 1$ in an exterior neighbourhood of Γ (with respect to a given orientation of $\tilde{\Gamma}$), and $q = 0$ in an interior neighbourhood of Γ. Setting

$$w = \begin{cases} \mathrm{grad}\ q & \text{in} \quad \Omega - \Gamma , \\ 0 & \text{on} \quad \Gamma , \end{cases}$$

we see that $w \in (C^\infty(\bar{\Omega}))^3$ and that w is not a potential field globally on Ω.

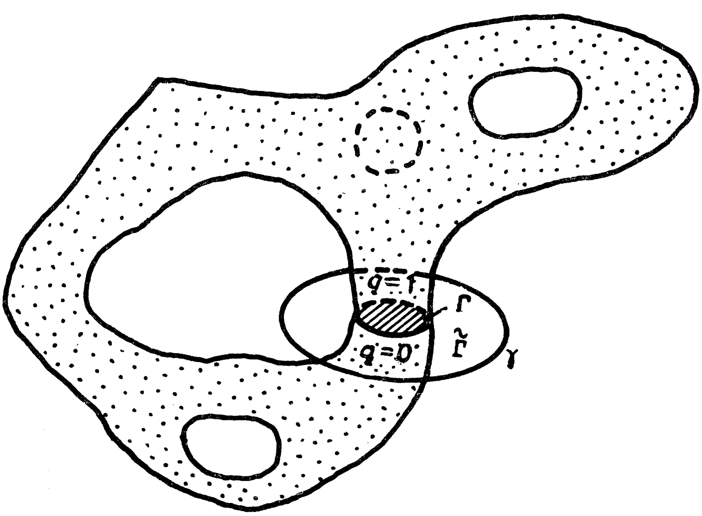

Fig. 1.

Consider the Neumann problem: Find $p \in H^1(\Omega)$ such that

$$\begin{aligned} \Delta p &= \operatorname{div} w \quad \text{in} \quad \Omega\,, \\ \partial_n p &= n \,.\, w \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{3.4}$$

(∂_n being the normal derivative), which is solvable because by (2.1)

$$(\operatorname{div} w, 1)_0 = \langle n \,.\, w, 1 \rangle_{\partial\Omega}\,.$$

Now, let us define

$$v = \operatorname{grad} p - w\,. \tag{3.5}$$

Making use of (2.4) and (3.4), we arrive at

$$(v, \operatorname{grad} z)_0 = (\operatorname{grad} p - w, \operatorname{grad} z)_0 = \langle \partial_n p - n \,.\, w, z \rangle_{\partial\Omega} = 0 \quad \forall z \in H^1(\Omega)\,,$$

that is $v \in H_0(\operatorname{div}^0; \Omega)$.

Furthermore, $v \in H(\operatorname{rot}^0; \Omega)$ which follows from (3.5), (2.4) and the fact that $w \in (C^\infty(\bar{\Omega}))^3$ vanishes in some neighbourhood of Γ. Consequently, v satisfies (1.2) with zero right-hand sides. On the other hand $v \neq 0$, since it is not a potential field by (3.5). So the inequality (3.1) is not valid for multiply connected domains.

$\Leftarrow$: Let Ω be simply connected and let $v \in V$ be given. Consider the problem

$$\begin{aligned} \Delta z &= \operatorname{div} v \quad \text{in} \quad \Omega\,, \\ \partial_n z &= 0 \qquad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{3.6}$$

which has exactly one weak solution z in $L_0^2(\Omega) \cap H^1(\Omega)$, because $\operatorname{div} v \in L_0^2(\Omega)$ by (2.1), and it holds that

$$\|z\|_1 \leqq C_1 \|\operatorname{div} v\|_0\,. \tag{3.7}$$

The relations (2.1), (2.4) and (3.6) give $\operatorname{grad} z \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}^0; \Omega)$, i.e. again by (3.6)

$$\psi = v - \operatorname{grad} z \in H_0(\operatorname{div}^0; \Omega) \cap H(\operatorname{rot}; \Omega)\,. \tag{3.8}$$

In accordance with Lemma 3.3 there exists exactly one stream function $\varphi \in D \cap$ $\cap\, H_0(\operatorname{rot}; \Omega)$ such that

$$\psi = \operatorname{rot} \varphi\,. \tag{3.9}$$

Applying now Lemma 2.2 and (3.2), we come to

$$\begin{aligned} \|\operatorname{rot} \varphi\|_0^2 &= (\operatorname{rot} \varphi, \operatorname{rot} \varphi)_0 = (\varphi, \operatorname{rot} \operatorname{rot} \varphi)_0 \leqq \\ &\leqq \|\varphi\|_0 \|\operatorname{rot} \operatorname{rot} \varphi\|_0 \leqq C_2 \|\operatorname{rot} \varphi\|_0 \|\operatorname{rot} \operatorname{rot} \varphi\|_0\,. \end{aligned} \tag{3.10}$$

So by (3.8), (3.9), (3.10), (3.7) and (2.4) we obtain

$$\begin{aligned} \|v\|_0 &\leqq \|\operatorname{grad} z\|_0 + \|\operatorname{rot} \varphi\|_0 \leqq \|z\|_1 + C_2 \|\operatorname{rot} \operatorname{rot} \varphi\|_0 \leqq \\ &\leqq C_1 \|\operatorname{div} v\|_0 + C_2 \|\operatorname{rot} \psi\|_0 \leqq C(\|\operatorname{div} v\|_0 + \|\operatorname{rot} v\|_0)\,. \quad \square \end{aligned}$$

Remark 3.4. The spaces $\mathscr{H}_{\mathscr{D}}$ and $\mathscr{H}_{\mathscr{R}}$ are finite-dimensional (cf. [18, 19, 22, 23]). From Theorem 3.2 we see that $\mathscr{H}_{\mathscr{D}}$ is trivial iff Ω is simply connected; (note that $\mathscr{H}_{\mathscr{R}}$ is trivial iff $\partial\Omega$ is connected [15]). The proof of the inequality (3.1) can be modified for $v \in V \cap (\mathscr{H}_{\mathscr{D}})^{\perp}$ without any assumptions on the connectivity of Ω (the symbol $\perp$ denotes the orthocomplement in $(L^2(\Omega))^3$). This was proved e.g. in [21] for smooth domains.

4. APPLICATION TO A VARIATIONAL PROBLEM

In this section we shall deal with a variational formulation of the problem (1.2).

For $f \in L^2(\Omega)$ and $g \in (L^2(\Omega))^3$ we define the linear form

$$b(v) = (f, \operatorname{div} v)_0 + (g, \operatorname{rot} v)_0\,, \quad v \in V\,, \tag{4.1}$$

and the bilinear form

$$a(w, v) = (\operatorname{div} w, \operatorname{div} v)_0 + (\operatorname{rot} w, \operatorname{rot} v)_0\,, \quad w, v \in V\,. \tag{4.2}$$

Assume that a sufficiently smooth u satisfies (1.2) in the classical sense. Then we immediately see that $u \in V$ and

$$\begin{aligned}(\operatorname{div} u,\ \operatorname{div} v)_0 &= (f, \operatorname{div} v)_0\,,\\ (\operatorname{rot} u,\ \operatorname{rot} v)_0 &= (g, \operatorname{rot} v)_0\end{aligned}$$

for all $v \in V$. Consequently,

$$a(u, v) = b(v) \quad \forall v \in V\,, \tag{4.3}$$

and moreover, by (2.1) and (2.5) we have

$$f \in L^2_0(\Omega)\,, \quad g \in D\,. \tag{4.4}$$

Conversely, let (4.4) hold and let (4.3) be satisfied for a sufficiently smooth $u \in V$. Assuming that Ω is simply connected, we show that u fulfils (1.2).

So let $\chi \in L^2_0(\Omega)$ be arbitrary and let $z \in L^2_0(\Omega) \cap H^1(\Omega)$ be the weak solution of the problem

$$\begin{aligned}\Delta z &= \chi \quad \text{in} \quad \Omega\,,\\ \partial_n z &= 0 \quad \text{on} \quad \partial\Omega\,.\end{aligned} \tag{4.5}$$

Then $v = \operatorname{grad} z \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}^0; \Omega) \subset V$ and from (4.5), (4.2),(4.3) and (4.1) we get

$$(\operatorname{div} u, \chi)_0 = (\operatorname{div} u, \operatorname{div} v)_0 = a(u, v) = b(v) = (f, \operatorname{div} v)_0 = (f, \chi)_0\,.$$

Hence, $\operatorname{div} u = f$ in $L^2_0(\Omega)$.

Furthermore, let $\psi \in D$ be arbitrary. Then by [9], p. 28, there exists a divergence-free stream function $v' \in H(\operatorname{div}^0; \Omega) \cap (H^1(\Omega))^3$ (not uniquely determined) such that $\psi = \operatorname{rot} v'$. As $\langle n \,.\, v', 1 \rangle_{\partial\Omega} = 0$ due to (2.1), the following problem is solvable:

$$\begin{aligned} \Delta\eta &= 0 \quad &&\text{in} \quad \Omega\,, \\ \partial_n \eta &= n \,.\, v' \quad &&\text{on} \quad \partial\Omega\,. \end{aligned}$$

Then clearly the function $v = v'$-grad η is from $H_0(\operatorname{div}^0; \Omega) \cap H(\operatorname{rot}; \Omega) \subset V$ and v is also a divergence-free stream function to ψ, that is

$$\psi = \operatorname{rot} v$$

(cf. [1, 15, 24]). Using (4.2), (4.3) and (4.1), we arrive at

$$(\operatorname{rot} u, \psi)_0 = (\operatorname{rot} u, \operatorname{rot} v)_0 = a(u, v) = b(v) = (g, \operatorname{rot} v)_0 = (g, \psi)_0\,,$$

i.e. $\operatorname{rot} u = g$ in D.

Thus we have justified the following definition.

Definition 4.1. *Let Ω be simply connected. The problem of finding $u \in V$ which satisfies* (4.3) *is called the variational formulation of the problem* (1.2).

Theorem 4.2. *Let Ω be simply connected. Then the variational formulation of the problem* (1.2) *has precisely one solution.*

Proof. By Theorem 3.2 the bilinear form (4.2) is a scalar product on V. It is easy to show that V is a Hilbert space and that the linear form (4.1) is continuous on V. Now the assertion follows from the Riesz theorem. □

Remark 4.3. When Ω is multiply connected the bilinear form (4.2) is a scalar product on $V \cap (\mathscr{H}_{\mathscr{D}})^{\perp}$ (cf. Remark 3.4), i.e. the solution of (1.2) exists and is unique apart from a function of $\mathscr{H}_{\mathscr{D}}$.

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Souhrn

ŘEŠITELNOST JISTÉHO SYSTÉMU PRVNÍHO ŘÁDU NA TROJROZMĚRNÝCH NEHLADKÝCH OBLASTECH

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Je studován systém parciálních diferenciálních rovnic prvního řádu, který je definován pomocí operátorů divergence a rotace na ohraničené oblasti $\Omega \subset \mathbb{R}^3$ s nehladkou hranicí. Na hranici $\partial\Omega$ je předepsána nulová normálová složka řešení. Je podána variační formulace a vyšetřována její řešitelnost.

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