Miroslav Vondrák

Slab analogy in theory and practice of conforming equilibrium stress models for finite element analysis of plane elastostatics

Aplikace matematiky, Vol. 30 (1985), No. 3, 187–217

Persistent URL: http://dml.cz/dmlcz/104141

SLAB ANALOGY IN THEORY AND PRACTICE OF CONFORMING EQUILIBRIUM STRESS MODELS FOR FINITE ELEMENT ANALYSIS OF PLANE ELASTOSTATICS

MIROSLAV VONDRÁK

(Received January 26, 1984)

INTRODUCTION

It is well-known that finite element analysis of boundary value problems in elastostatics can be based on i) the principle of minimum potential energy, ii) the principle of minimum complementary energy, iii) the mixed variational principle (of the Hellinger-Reissner type). The most important purpose of calculations in engineering usually is to determine the state of stress in the body. One of the natural ways obtaining approximations of stress is to use the procedure of class ii) which we study in the present paper. The fundamental problem in the application of the principle of complementary energy is the construction of suitable subsets that approximate the set of all the statically admissible fields satisfying both the conditions of equilibrium inside the body and the static boundary conditions. A number of articles has been written on the dual finite element analysis (see e.g. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19]).

The recognition of an analogy between the Airy stress function in plane problems and the lateral displacement of plates (the so called slab analogy) was evident early [21, 3, 1] through the identical biharmonic relations valid for homogeneous and isotropic situations. B. M. Fraeijs de Veubeke and O. C. Zienkiewicz [19] made use of the slab analogy to indicate how suitable two-dimensional stress equilibrium models may be generated from conforming plate bending displacement models. V. B. Watwood and B. J. Hartz [14] developed the equilibrating stress element conjugated in the sense of slab analogy with Clought-Tocher's compatible displacement element. For Watwood and Hartz's element and the general domain, the theoretical convergence results have been presented by I. Hlaváček [7] without using the Airy stress function. M. Křížek [16] extended Watwood and Hartz's element for the plane problem to three dimensions. The author is obliged to bring the article of M. Křížek [17] to the reader's attention.

In Section 1 the notion "slab analogy", stated in [19], is motivated (from the mathematical point of view it generally includes the proof of existence of the Airy stress function) and the interface condition for the Airy stress function are established at the contact of two domains. Some spaces of types of conforming equilibrium stress elements, which can be obtained by slab analogy, are investigated in Section 2. In Section 4 a weak version of the Castigliano principle is established and the approximate variational problem is defined by using equilibrium stress fields. In Section 5 some subspaces of equilibrium stress elements are introduced and a priori error estimates in the L^2-norm (provided the solutions are smooth enough) and convergence results are obtained from the well-know results for compatible finite elements.

1. AIRY STRESS FUNCTION

Let $\Omega \subset \mathbb{R}^2$ be a non-empty multiply connected bounded domain with a Lipschitz boundary Γ (for the definition of a Lipschitz boundary see [1]). Let Γ_0 be the exterior boundary of Ω and Γ_i, $1 \leqq i \leqq p$, $(1 \leqq p < \infty)$ be all the remaining parts of the boundary Γ, i.e. $\Gamma = \bigcup_{i=0}^{p} \Gamma_i$. Note that a normal to the boundary Γ exist almost everywhere; the outward unit normal is always denoted by $\nu = (\nu_1, \nu_2)^{\mathsf{T}}$. Henceforth, $P_k(\Omega)$ will be understood as the space of polynomials of orders at most k. Let us denote the space of real infinitely differentiable functions with a compact support on Ω by $C_0^\infty(\Omega)$, and let us define the space $C^\infty(\bar{\Omega})$ by $C^\infty(\bar{\Omega}) = \{\varphi|_\Omega \mid \varphi \in C_0^\infty(\mathbb{R}^2)\}$. The Sobolev space of functions the derivatives of which up to the order m exist (in the sense of distributions) and are square-integrable in Ω, is a Hilbert space, denoted by $H^m(\Omega)$, for the scalar product $(u, v)_{m,\Omega} = \sum_{|\alpha| \leqq m} \int_\Omega \mathrm{D}^\alpha u\, \mathrm{D}^\alpha v \,\mathrm{d}x$, equipped with the norm $\|\cdot\|_{m,\Omega} = (\cdot, \cdot)_{m,\Omega}^{1/2}$. ($|\cdot|_{m,\Omega}$ is the usual seminorm).

As $C_0^\infty(\Omega) \subset H^m(\Omega)$, we can define the space $H_0^m(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.

All vectors will be column vectors. The norm, seminorm and scalar product of vector and matrix functions, the components of which are from $H^k(\Omega)$, will be denoted in the same way as in $H^k(\Omega)$ and we put

$$(u, v)_{k,\Omega} = \sum_{i=1}^{n} (u_i, v_i)_{k,\Omega} \quad \text{for} \quad u = (u_1, \ldots, u_n)^{\mathsf{T}}, \quad v = (v_1, \ldots, v_n)^{\mathsf{T}} \in (H^k(\Omega))^n$$

and

$$\|v\|_{k,\Omega} = (v, v)_{k,\Omega}^{1/2} \quad \text{for} \quad v = (v_1, \ldots, v_n)^{\mathsf{T}} \in (H^k(\Omega))^n .$$

Throughout the paper let the symbols ∂_i, ∂_{ij}, ∂_ν, ∂_τ, δ_{ij} have the same meaning as in [2].

In the usual way we define the operator grad : $H^1(\Omega) \to (L^2(\Omega))^2$ by

$$\text{(1.1)} \qquad \operatorname{grad} v = (\partial_1 v, \partial_2 v), \quad v \in H^1(\Omega),$$

and the operator curl : $H^1(\Omega) \to (L^2(\Omega))^2$ by

$$\text{(1.2)} \qquad \operatorname{curl} v = (\partial_2 v, -\partial_1 v), \quad v \in H^1(\Omega).$$

We say, that $f \in L^2(\Omega)$ is the divergence of a vector function $q \in (L^2(\Omega))^2$ in the sense of distributions in Ω if

$$\text{(1.3)} \qquad (f, v)_{0,\Omega} = -(q, \operatorname{grad} v)_{0,\Omega} \quad \forall v \in C_0^\infty(\Omega)$$

holds and we write

$$\operatorname{div} q = f \quad \text{in} \quad L^2(\Omega).$$

Note that the first generalized derivatives of q need not exist. However, if $q \in (H^1(\Omega))^2$, then, evidently, $\operatorname{div} q = \partial_1 q_1 + \partial_2 q_2$.

Let us introduce the following space:

$$H(\operatorname{div}; \Omega) = \{q \in (L^2(\Omega))^2 \mid \operatorname{div} q \in L^2(\Omega)\}.$$

The next theorem concerns the boundary values of functions from the space $H(\operatorname{div}; \Omega)$ (see [4] Theorem 1.2.2):

Theorem 1.1. *A functional $\gamma_\nu : q \to q^\mathrm{T}\nu|_\Gamma$ defined on $(C^\infty(\bar{\Omega}))^2$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_ν, from $H(\operatorname{div}; \Omega)$ into $H^{-1/2}(\Gamma)$.* □

Here $H^{-1/2}(\Gamma)$ denotes the dual space to the space $H^{1/2}(\Gamma)$ of the traces on Γ of all functions from $H^1(\Omega)$.

Now Green's formula will be of the form:

$$\text{(1.4)} \quad (q, \operatorname{grad} v)_{0,\Omega} + (\operatorname{div} q, v)_{0,\Omega} = \langle \gamma_\nu q, \gamma_0 v \rangle_\Gamma \quad \forall q \in H(\operatorname{div}; \Omega) \ \forall v \in H^1(\Omega).$$

Here $\gamma_0 v$ denotes the traces of v and $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Particularly, if $\gamma_\nu q \in L^2(\Gamma)$, then

$$\langle \gamma_\nu q, \gamma_0 v \rangle_\Gamma = \int_\Gamma q^\mathrm{T} \nu v \, \mathrm{d}s \quad \forall v \in H^1(\Omega).$$

Now, for any $q \in H(\operatorname{div}; \Omega)$ we can define the functional $\gamma_\nu q \in H^{-1/2}(\Gamma_i)$, $i \in \{0, \ldots, p\}$ as

$$\langle \gamma_\nu q, \gamma_0 v \rangle_{\Gamma_i} = (q, \operatorname{grad} v)_{0,\Omega} + (\operatorname{div} q, v)_{0,\Omega} \quad \forall v \in V_i,$$

where

$$V_i = \{v \in H^1(\Omega) \mid \gamma_0 v = 0 \quad \text{on} \quad \Gamma_j \quad \forall j \in \{\{0, \ldots, p\} - \{i\}\},$$

i.e. $\langle \cdot, \cdot \rangle_{\Gamma_i}$ represents the duality between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$.

The next theorem (see [4], Theorem 1.3.1) yields necessary and sufficient conditions for the existence of a stream function of a divergence-free vector.

Theorem 1.2. *A function $q \in (L^2(\Omega))^2$ satisfies*

$$(1.5) \qquad \operatorname{div} q = 0\,, \quad \langle \gamma_\nu q, 1 \rangle_{\Gamma_i} = 0 \quad \textit{for} \quad i = 0, \ldots, p\,,$$

if and only if there exists stream function φ in $H^1(\Omega)$ such that

$$(1.6) \qquad q = \operatorname{curl} \varphi\,,$$

and this function φ is unique apart from an additive constant. □

Let us introduce some further notations. Henceforth,

$$\boldsymbol{H} \equiv H(\Omega) = \{\tau \in (L^2(\Omega))^4 \mid \tau = \tau^\mathrm{T}\}$$

denotes the space of symmetric tensors. It is a Hilbert space for the scalar product

$$(\tau', \tau'')_{0,\Omega} = \sum_{i,j=1}^{2} (\tau'_{ij}, \tau''_{ij})_{0,\Omega}\,, \quad \tau', \tau'' \in (L^2(\Omega))^4\,.$$

Define the operator ε: $(H^1(\Omega))^2 \to H(\Omega)$ by

$$(1.7) \qquad \varepsilon(v) = \begin{pmatrix} \partial_1 v_1, & \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) \\ \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1), & \partial_2 v_2 \end{pmatrix}, \quad v = (v_1, v_2)^\mathrm{T} \in (H^1(\Omega))^2\,.$$

Further, we say, that $f \in (L^2(\Omega))^2$ is the divergence of a tensor function $\tau \in H(\Omega)$ in the sense of distributions in Ω if

$$(1.8) \qquad (f, v)_{0,\Omega} = -(\tau, \varepsilon(v))_{0,\Omega} \quad \forall v \in (C_0^\infty(\Omega))^2$$

holds and we write

$$\operatorname{Div} \tau = f \quad \text{in} \quad (L^2(\Omega))^2\,.$$

Evidently, for $\tau \in H(\Omega) \cap (H^1(\Omega))^4$ we have $\operatorname{Div} \tau = (\partial_1 \tau_{11} + \partial_2 \tau_{12}, \partial_1 \tau_{12} + \partial_2 \tau_{22})^\mathrm{T}$. Now, we introduce the following space:

$$H(\mathrm{Div}; \Omega) = \{\tau \in H(\Omega) \mid \operatorname{Div} \tau \in (L^2(\Omega))^2\}\,.$$

For $\tau \in H(\mathrm{Div}; \Omega)$ we can define the linear functional $\gamma_\nu \tau \in (H^{-1/2}(\Delta))^2$ by

$$\langle \gamma_\nu \tau, w \rangle_\Delta = \langle \gamma_\nu q^1, w_1 \rangle_\Delta + \langle \gamma_\nu q^2, w_2 \rangle_\Delta\,, \quad w = (w_1, w_2)^\mathrm{T} \in (H^{1/2}(\Delta))^2\,,$$

where q^1 and q^2 are the columns of the tensor τ and Δ is either Γ or Γ_i for $i \in \{0, \ldots, p\}$. The functional $\gamma_\nu \tau \in (H^{1/2}(\Delta))^2$ is called *the stress vector* and its components are $\gamma_\nu q^j$, $j = 1, 2$ (for $\tau \in (C(\bar{\Omega}))^4$ the components of the stress vector will be denoted in the usual way by $t_j = (q^j)^\mathrm{T} \nu$). Thus Green's formula will clearly have the form:

$$(1.9) \quad (\tau, \varepsilon(v))_{0,\Omega} + (\operatorname{Div} \tau, v)_{0,\Omega} = \langle \gamma_\nu \tau, \gamma_0 v \rangle_\Gamma \quad \forall \tau \in H(\mathrm{Div}; \Omega) \quad \forall v \in (H^1(\Omega))^2\,.$$

Moreover, we define the operator ϱ: $H^2(\Omega) \to H(\Omega)$ by

$$(1.10) \qquad \varrho U = \begin{pmatrix} \partial_{22} U, & -\partial_{12} U \\ -\partial_{12} U, & \partial_{11} U \end{pmatrix}, \quad U \in H^2(\Omega)\,.$$

We introduce *the conditions of total equilibrium* for $\tau \in H(\mathrm{Div}; \Omega)$, $\mathrm{Div}\,\tau = 0$ (see [1])

$$\langle \gamma_\nu q^j, 1 \rangle_\Gamma = 0 \,, \quad j = 1, 2 \,, \tag{1.11}$$

$$\langle x_1 \gamma_\nu q^2 - x_2 \gamma_\nu q^1, 1 \rangle_\Gamma = 0 \,, \tag{1.12}$$

which immediately follow from (1,9) (q^j are the columns of τ).

Now, let us denote by $\mathfrak{M}$ the set of all non-empty simply connected bounded domains in $\mathbb{R}^2$ with a Lipschits boundary Γ. Necessary and sufficient conditions for the existence of the Airy stress function are given in

Theorem 1.3. *Let $\Omega \in \mathfrak{M}$. A function $\tau \in H(\mathrm{Div}; \Omega)$ satisfies*

$$\mathrm{Div}\,\tau = 0 \quad \text{in} \quad \Omega \,, \tag{1.13}$$

if and only if there exists an Airy stress function $U \in H^2(\Omega)$ such that

$$\tau = \varrho U \tag{1.14}$$

and this function U is unique apart from a linear function.

Proof. Let $U \in H^2(\Omega)$. Write $\tau = \varrho U$, $q^1 = \mathrm{curl}\,(\partial_2 U)$, $q^2 = \mathrm{curl}\,(-\partial_1 U)$. By Theorem 1.2 we see that $\mathrm{div}\, q^1 = \mathrm{div}\, q^2 = 0$. Hence $\mathrm{Div}\,\tau = 0$ in Ω.

Conversely, let $\tau \in H(\mathrm{Div}; \Omega)$, $\mathrm{Div}\,\tau = 0$. Then $\mathrm{div}\, q^j = 0, j = 1, 2$. By Theorem 1.2 there exist $\varphi^1, \varphi^2 \in H^1(\Omega)$ such, that $q^j = \mathrm{curl}\, \varphi^j$, $j = 1, 2$ and (1.11) holds. Since $\tau_{12} = \tau_{21}$ we have $-\partial_1 \varphi^1 = \partial_2 \varphi^2$. Hence putting $\varphi = (\varphi^1, \varphi^2)$, we obtain $\mathrm{div}\, \varphi = 0$ in Ω and using (1.4) we obtain $\langle \gamma_\nu \varphi, 1 \rangle_\Gamma = 0$. Applying Theorem 1.2 once again we see that there exists $U \in H^1(\Omega)$ such that $\mathrm{curl}\, U = \varphi$ in Ω. But $U \in H^2(\Omega)$, as $\partial_1 U = -\varphi^2$ and $\partial_2 U = \varphi^1$ are from the space $H^1(\Omega)$.

The Airy stress function of a divergence-free tensor τ is unique apart from a linear function, since if $U^1, U^2 \in H^2(\Omega)$ and $\varrho U^1 = \varrho U^2 = \tau$, then $\varrho(U^1 - U^2) = 0$, which yields $U^1 - U^2 \in P_1(\Omega)$. □

Let $\Omega \in \mathfrak{M}$ and let us denote by γ the measurable part of the boundary Γ for which $\mathrm{mes}\,\gamma > 0$.

Let us introduce the space

$$V^0 \equiv V^0(\Omega) = \{v \in H^1(\Omega) \mid v \equiv 0 \text{ on } \Gamma \dot{-} \gamma\}$$

and on γ let us define the functional $\gamma_\nu \tau$ by

$$\langle \gamma_\nu \tau, \gamma_0 v \rangle_\gamma = (\tau, \varepsilon(v))_{0,\Omega} + (\mathrm{Div}\,\tau, v)_{0,\Omega} \,, \quad v \in (V^0)^2 \,. \tag{1.15}$$

Let Ω_γ denote the $\Omega \in \mathfrak{M}$ which is divided into two simply connected domains Ω^1, Ω^2 with Lipschitz boundaries Γ^1, Γ^2. The common part of this boundaries will be denoted by γ, i.e. $\gamma = \Gamma^1 \cap \Gamma^2$. Let a normal n to γ be oriented so that it is an outward normal of the subdomain Ω^1.

Definition 1.1. *Let* $\Omega \equiv \Omega_\gamma$. *Let* $\tau^1 \in H(\mathrm{Div}; \Omega^1)$ *and* $\tau^2 \in H(\mathrm{Div}; \Omega^2)$. *Then the stress vector* $\gamma_\nu \tau$ *is said to be continuous at* γ *provided*

$$\langle \gamma_\nu \tau^1, w \rangle_\gamma = -\langle \gamma_\nu \tau^2, w \rangle_\gamma , \tag{1.16}$$

where w *is the common trace of functions* $v^1 \in (V^0(\Omega^1))^2$ *and* $v^2 \in (V^0(\Omega^2))^2$ *on* γ. □

The interface conditions on γ for the Airy stress function are established in

Theorem 1.4. *Let* Ω_γ *be given and let* $\tau^k \in H(\mathrm{Div}; \Omega^k)$, $\mathrm{Div}\, \tau^k = 0$ *on* Ω^k, $k = 1, 2$. *Then there exist representatives* U^1 *and* U^2 *of the classes of equivalence from the quotient spaces* $H^2(\Omega^1)/P_1(\Omega^1)$ *and* $H^2(\Omega^2)/P_1(\Omega^2)$ *such, that*

$$\langle \gamma_\nu \tau^1, w \rangle_\gamma = -\langle \gamma_\nu \tau^2, w \rangle_\gamma \Leftrightarrow \begin{cases} \partial_j U^1|_\gamma = \partial_j U^2|_\gamma , \; j = 1, 2, \\ \;\;U^1|_\gamma = \;\;U^2|_\gamma . \end{cases} \tag{1.17}$$

Proof. We choose $\Omega^* \subset \Omega' \subset \bar{\Omega}' \subset \Omega_\gamma$, $\Omega^*, \Omega' \in \mathfrak{M}$, such that both $\Omega^* \cap \Omega^1$ and $\Omega^* \cap \Omega^2$ are non-empty. Let $u_i^1 \in V^0(\Omega^1)$, $u_i^2 \in V^0(\Omega^2)$ and let w_i be the common trace of these functions on γ, $i = 1, 2$. Then $u_i \in L^2(\Omega_\gamma)$ and $u_i^k = u_i|\Omega^k$, $k = 1, 2$. Let us regularize $u_i \in L^2(\Omega_\gamma)$ by means of a kernel $\omega_h(x - y)$, where $A = \mathrm{const}$.

$$A\omega_h(z) = \begin{cases} \exp\left(|z|^2/(|z|^2 - h^2)\right) & \text{for} \quad |z| < h , \\ 0 & \text{for} \quad |z| \geqq h , \end{cases}$$

$h < \mathrm{dist}\,(\partial\Omega^*, \partial\Omega')$. We obtain $(u_i)_h\,(x) \in C_0^\infty(\Omega_\gamma)$,

$$(u_i)_h\,(x) = \int_{\Omega'} \omega_h(x - y)\, u_i(y)\, dy , \quad i = 1, 2 .$$

Then $(u_i^k)_h = (u_i)_h \mid \Omega^k$, $k = 1, 2$ and $(w_i)_h \in C_0^\infty(\gamma)$ (where $C_0^\infty(\gamma) = \{\varphi|\gamma| \; \varphi \in C_0^\infty(\Omega_\gamma)\}$) is the common trace of the functions $(u_i^1)_h$ and $(u_i^2)_h$ on γ, $i = 1,2$. The function $(u_i^k)_h$ are dense in $V^0(\Omega^k)$, $k = 1, 2$. Let $\langle \gamma_\nu \tau^1, w_h \rangle_\gamma = -\langle \gamma_\nu \tau^2, w_h \rangle_\gamma$. Then by (1.15) we have

$$(\tau^1, \varepsilon(u_h^1))_{0,\Omega^1} + (\mathrm{Div}\, \tau^1, u_h^1)_{0,\Omega^1} = -(\tau^2, \varepsilon(u_h^2))_{0,\Omega^2} - (\mathrm{Div}\, \tau^2, u_h^2)_{0,\Omega^2} .$$

Next, let us choose in particular $(u_1^k)_h \neq 0$, $(u_2^k)_h = 0$, $k = 1, 2$. Then a direct calculation from $(q_1^1, \mathrm{grad}\,(u_1^1)_h)_{0,\Omega^1} = -(q_2^1, \mathrm{grad}\,(u_1^2)_h)_{0,\Omega^2}$, if we make use of (1.14) again, yields

$$\int_\gamma (\partial_2 U^1 - \partial_2 U^2)\, \partial_\tau(w_1)_h \, ds = 0 \quad \forall (w_1)_h \in C_0^\infty(\gamma) .$$

This implies that $\partial_\tau(\partial_2 U^1|_\gamma - \partial_2 U^2|_\gamma)$ exists in the sense of distributions and is equal to zero. Interchanging the subscripts we obtain the same assertion for $\partial_\tau(\partial_1 U^1|_\gamma - \partial_1 U^2|_\gamma)$. Hence $(\partial_j U^1|_\gamma - \partial_j U^2|_\gamma) = a_j$, $a_j \in \mathbb{R}$, $j = 1, 2$. Because the mapping $\tau \to U$, for $\operatorname{Div} \tau = 0$, is unique modulo P_1, then $U^{k,*} = U^k + p^k \ \forall p^k \in \in P_1(\Omega^k)$, $k = 1, 2$ is also pertaining to the given stress tensor τ^k on Ω^k, $k = 1, 2$. Now we can express

$$\partial_\tau(U^1|_\gamma - U^2|_\gamma) = -(\partial_1 U^1|_\gamma - \partial_1 U^2|_\gamma)\, v_2 + (\partial_2 U^1|_\gamma - \partial_2 U^2|_\gamma)\, v_1 =$$
$$= -a_1 v_2 + a_2 v_1 \,.$$

Hence we obtain, making use of the parametrical expression of the boundary curve with the length of arc of the curve γ for the parameter,

$$U^1(s)|_\gamma - U^2(s)|_\gamma = a_1\, x_1(s) + a_2\, x_2(s) + a_0 \,.$$

Let us denote

$$p^1 - p^2 \equiv p^3 = a_1^3 x_1 + a_2^3 x_2 + a_0^3 \quad \text{in} \quad \bar{\Omega}^1 \cup \bar{\Omega}^2 \,.$$

Therefore, we obtain

$$U^{1,*}|_\gamma - U^{2,*}|_\gamma = (a_1 + a_1^3)\, x_1(s) + (a_2 + a_2^3)\, x_2(s) + (a_0 + a_0^3)$$

and

$$(\partial_j U^{1,*}|_\gamma - \partial_j U^{2,*}|_\gamma) = a_j + a_j^3 \,, \quad j = 1, 2 \,.$$

From the last two formulas the assertion follows, for we can choose p^3 so, that $a_j + a_j^3 = 0, j = 0, 1, 2$.

Conversely, let $U^1 \in H^2(\Omega^1)$ and $U^2 \in H^2(\Omega^2)$ satisfy the right-hand part of the equivalence (1.17). Let us take into consideration the functions $(u_i^k)_h$ from the proof of the first implication. Then the definition of the derivative in the sense of distributions implies that

$$\int_\gamma \partial_1 U^1\, \partial_\tau(w_2)_h \,\mathrm{d}s = \int_\gamma \partial_1 U^2\, \partial_\tau(w_2)_h \,\mathrm{d}s \,.$$

Making use of (1.14), we obtain $(q_1^2, \operatorname{grad}\, (u_2^1)_h)_{0,\Omega^1} = -(q_2^2, \operatorname{grad}\, (u_2^2)_h)_{0,\Omega^2}$. This together with (1.13) and (1.9) yields $\langle \gamma_v q_1^2, (w_2)_h \rangle_\gamma = -\langle \gamma_v q_2^2, (w_2)_h \rangle_\gamma$. From the density of the functions $(u_2^k)_h$ in $V^0(\Omega^k)$, $k = 1, 2$, we conclude that $\gamma_v q_1^2|_\gamma = -\gamma_v q_2^2|_\gamma$ in the sense of distributions. In fact, it is sufficient to verify, that $C_0^\infty(\gamma)$ is dense in $H^{1/2}(\gamma)$ with respect to the L^2-norm. This, however, follows from the density of the set $V^0(\Omega^k) \cap C^\infty(\bar{\Omega}_\gamma)$ in $V^0(\Omega^k)$, $k = 1, 2$, if we employ the trace theorem.

Interchanging the subscripts we obtain the same result for $\gamma_v q_1^k|_\gamma$, $k = 1, 2$. □

Theorem 1.5. *Let $\Omega \equiv \Omega_\gamma$, $U^1 \in H^2(\Omega^1)$, $U^2 \in H^2(\Omega^2)$ and let the right-hand part of the equivalence* (1.17) *holds.*

Then $U \in H^2(\Omega_\gamma)$ for any function U such that $U|_{\Omega^1} = U^1$ and $U|_{\Omega^2} = U^2$. The proof follows from Green's theorem. □

2. SOME SPACES OF CONFORMING EQUILIBRIUM STRESS FINITE ELEMENTS

In this section we shall study conforming equilibrium stress finite elements conjugated, in the sense of slab analogy, with compatible elements used for solving an approximation of the biharmonic problem (the so-called C^1 elements). Throughout the paper we shall use *a compatible finite element* $(K, \boldsymbol{P}_K, \Sigma_K)$ defined in [2].

With regard to Theorem 1.3 let us define the space $\mathscr{M}_K$ by

$$\mathscr{M}_K = \varrho \boldsymbol{P}_K . \tag{2.1}$$

Clearly, $\dim \mathscr{M}_K = \dim \boldsymbol{P}_K - 3$ and for $\tau \in \mathscr{M}_K$ the three overall equilibrium conditions

$$\int_{\partial K} t_k(\tau)\,\mathrm{d}s = 0\,, \quad k = 1, 2\,, \tag{2.2}$$

$$\int_{\partial K} (x_1\, t_2(\tau) - x_2\, t_1(\tau))\,\mathrm{d}s = 0$$

are fulfilled.

We shall start from the general definition of such an element.

Definition 2.1. *A conforming equilibrium stress finite element is a triple* $(K, \mathscr{M}_K, \Sigma_K^*)$, *where*

- $K \subset \mathbb{R}^2$ *is either a triangle or a rectangle;*
- $\mathscr{M}_K$ *is a finite-dimensional space of stress fields* τ *defined over* K *and satisfying* $\operatorname{Div} \tau = 0$ *in* K;
- Σ_K^* *is a set of linear functionals* T_i, $i = 1, \ldots, q$, *defined on* $\mathscr{M}_K$ *and determining a distribution of the stress vector on the boundary* ∂K *of* K, *from which* q-3 *are linearly independent; the functionals which are linear combinations of the others are determined by the three conditions of total equilibrium.*

Let us note that $\operatorname{card} \Sigma_K^* = \operatorname{card} \Sigma_K$. □

Definition 2.2. *We shall say that the set* Σ_K^* *is* $\mathscr{M}_K$-*unisolvable, if for an arbitrary stress vector satisfying the conditions of total equilibrium, the distribution of which on the boundary* ∂K *of* K *is determined by a* q-*tuple of real numbers* $\alpha_1, \ldots, \alpha_q$, *there exists precisely one tensor* $\tau \in \mathscr{M}_K$ *such that*

$$T_i(\tau) = \alpha_i\,, \quad i = 1, \ldots, q\,.$$ □

Let us denote by $\hat{K}$ a reference domain which can be either the unit square with the vertices $\hat{a}_1 = (0, 0)^{\mathrm{T}}$, $\hat{a}_2 = (1, 0)^{\mathrm{T}}$, $\hat{a}_3 = (1, 1)^{\mathrm{T}}$, $\hat{a}_4 = (0, 1)^{\mathrm{T}}$ or the triangle with the vertices $\hat{a}_1 = (0, 0)^{\mathrm{T}}$, $\hat{a}_2 = (1, 0)^{\mathrm{T}}$, $\hat{a}_3 = (0, 1)^{\mathrm{T}}$. Let l_i denote the length of the side $\hat{a}_i\hat{a}_{i+1}$ and let $\hat{n}^i$ denote an outward normal to the side $\hat{a}_i\hat{a}_{i+1}$. Denote by $(\hat{\tau}\hat{n}^i)_j$ the j-th component of the stress vector on the side $\hat{a}_i\hat{a}_{i+1}$. Introduce the affine one-to-one mapping $F : \hat{K} \to K$ by

(2.3) $$x \equiv F(\hat{x}) = B\hat{x} + b$$

where

$$B = \begin{pmatrix} x_1^2 - x_1^1, & x_1^r - x_1^1 \\ x_2^2 - x_2^1, & x_2^r - x_2^1 \end{pmatrix},$$

$$b = (x_1^1, x_2^1)^{\mathsf{T}}$$

and (x_1^i, x_2^i), $i = 1, \ldots, r$, (where $r \in \{3, 4\}$) are the vertices of K. We note that the mapping F maps the gravity center of $\hat{K}$ onto the gravity center of K and maintains the dividing rates of a point on a straight line.

Given the stress tensor τ defined on K we define the stress tensor $\hat{\tau}$ defined on $\hat{K}$ by

(2.4) $$\hat{\tau}(\hat{x}) = B^{-1}\tau(F(\hat{x}))\,(B^{-1})^{\mathsf{T}},$$

(i.e. a correspondence between contravariant tensors). If $\hat{n}$ is a normal to the side $\hat{S}$ of $\hat{K}$, then $n = (B^{-1})^{\mathsf{T}}\,\hat{n}$ (i.e. a correspondence between covariant vectors) is a normal to the corresponding side S of K. Making use of (2.3) we have $\hat{t} = B^{-1}t$ (i.e. a correspondence between contravariant vectors) and the relation

$$\tau \in \mathcal{M}_K \Leftrightarrow \hat{\tau} \in \mathcal{M}_{\hat{K}}$$

can be verified by direct calculation.

C—E—S—I ELEMENT

This conforming equilibrium stress element (known as Watwood-Hartz's element) is conjugated, in the sense of slab analogy, with the Hsieh-Clough-Tocher (H—C—T) compatible element defined in [2].

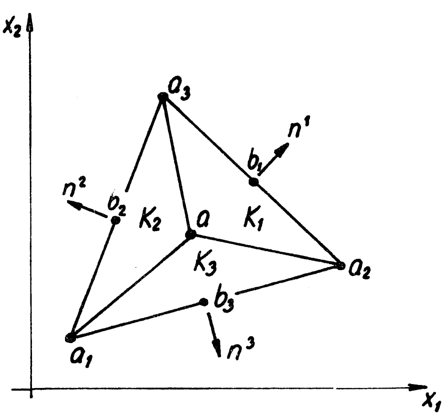

Fig. 1.

Let us recall that for an H–C–T element, K is a triangle (see Fig. 1) with vertices a_i, $1 \leqq i \leqq 3$, which is decomposed into three subtriangles K_i $1 \leqq i \leqq 3$. The point $a = \bigcap_{i=1}^{3} K_i$ is a point lying inside the triangle K, usually the center of gravity of K.

With the triangle K we associate the space

$$(2.5) \qquad \boldsymbol{P}_K = \{U \in C^1(K) \,|U|_{K_i} \equiv U^i \in P_3(K_i)\,, \ 1 \leqq i \leqq 3\}$$

and the set of degrees of freedom

$$(2.6) \qquad \Sigma_K = \{U(a_i), \partial_1 U(a_i), \partial_2 U(a_i), \partial_\nu U(b_i), 1 \leqq i \leqq 3\}\,,$$

where b_i is the mid-point of the side $a_{i+1}a_{i+2}$. The subscripts are calculated modulo 3.

The connecting line of the vertex a_i, $1 \leqq i \leqq 3$ with the gravity center will be called *the internal side* of the *triangular "building block"* K.

Let the restriction $U|_{K_i} \equiv U^i \in P_3(K_i)$, $1 \leqq i \leqq 3$, of the Airy stress function $U \in \boldsymbol{P}_K$ be given in the form

$$(2.7) \qquad U^i = \beta^i_{10} + \beta^i_9 x_1 + \beta^i_8 x_2 + \tfrac{1}{2}\beta^i_1 x_2^2 + \tfrac{1}{2}\beta^i_2 x_1^2 - \beta^i_3 x_1 x_2 + \tfrac{1}{6}\beta^i_4 x_1^3 + \\ + \tfrac{1}{2}\beta^i_5 x_1^2 x_1 + \tfrac{1}{2}\beta^i_6 x_1 x_2^2 + \tfrac{1}{6}\beta^i_7 x_2^3\,,$$

where β^i_m, $m = 1, \ldots, 10$, are real constants.

By virtue of (2.1), on each K_i, $1 \leqq i \leqq 3$, we can define an auxiliary space

$$(2.8) \qquad \mathcal{N}(K_i) = \varrho\, P_3(K_i)\,.$$

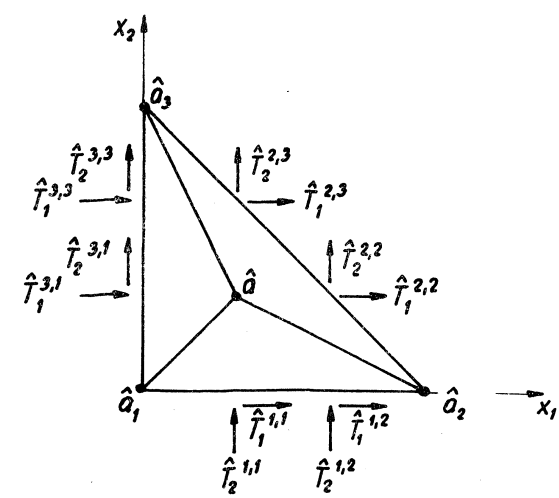

Fig. 2.

Thus the space $\mathcal{N}(K_i)$ is a seven-dimensional linear set. Obviously, the vectors $t_k(\tau^i)$, $k = 1, 2$, $\tau^i \in \mathcal{N}(K_i)$ are linear on each side j, $j = 1, 2, 3$ of the triangles K_i, $1 \leqq i \leqq 3$.

Now, due to Theorems 1.3, 1.4, 1.5 and (2.5), (2.8), we define *the self-equilibriated piecewise linear stress fields* over the triangle K:

$$\begin{aligned}(2.9)\qquad \mathcal{M}_K = \{\tau \in H(\mathrm{Div}; K) |\ \tau|_{K_i} \equiv \tau^i \in \mathcal{N}(K_i)\,.\quad 1 \leqq i \leqq 3\,,\\ \tau n \text{ is continuous across the internal sides of } K\}\,.\end{aligned}$$

Clearly, $\dim \mathcal{M}_K = 9$ and if we again employ (2.6), we can define a reference equilibrium triangular "building block" element by

Definition 2.3. *A reference conforming equilibrium stress triangular "building block" finite element is a triple* $(\hat{K}, \mathcal{M}_{\hat{K}}, \Sigma^*_{\hat{K}})$ (see *Fig.* 2), *where* $\hat{K}$ *is the reference triangle,* $\mathcal{M}_{\hat{K}}$ *is defined by* (2.9) *and* $\Sigma^*_{\hat{K}}$ *is the set of linear functionals defined on* $\mathcal{M}_{\hat{K}}$. *The functionals are defined in the following manner:*

For any external side $\hat{a}_i\hat{a}_{i+1} = \hat{K}_i \cap \partial\hat{K}$, $1 \leqq i \leqq 3$, *select an outward normal* $\hat{n}^i$. *Then, if* $\hat{\tau}$ *is any stress field of the space* $\mathcal{M}_{\hat{K}}$ *put*

$$\begin{aligned}(2.10)\qquad \hat{T}^{i,i}_k &= \tfrac{1}{2}\{(\hat{\tau}^i(\hat{a}_{i+1})\,\hat{n}^i)_k + (\hat{\tau}^i(\hat{a}_i)\,\hat{n}^i)_k\}\,,\\ \hat{T}^{i,i+1}_k &= \tfrac{1}{2}\{(\hat{\tau}^i(\hat{a}_{i+1})\,\hat{n}^i)_k - (\hat{\tau}^i(\hat{a}_i)\,\hat{n}^i)_k\}\,,\end{aligned}$$

$i = 1, 2, 3$; $i + 1 = 1$ *for* $i = 3$; $k = 1, 2$. □

Lemma 2.1. *Let* $\hat{\tau} \in \mathcal{M}_{\hat{K}}$ *and let twelve degrees of freedom be given by* (2.10). *Then the three conditions of the overall equilibrium hold*:

$$(2.11)\qquad \hat{T}^{1,1}_k + \sqrt{(2)}\,\hat{T}^{2,2}_k + \hat{T}^{3,3}_k = 0\,,\quad k = 1, 2\,,$$

(*the resultant forces vanish*)

$$\begin{aligned}(2.12)\qquad 3\hat{T}^{1,1}_2 + \hat{T}^{1,2}_2 + 3\sqrt{(2)}\,\hat{T}^{2,2}_2 - \sqrt{(2)}\,\hat{T}^{2,3}_2 - 3\sqrt{(2)}\,\hat{T}^{2,2}_1 -\\ - \sqrt{(2)}\,\hat{T}^{2,3}_1 - 3\hat{T}^{3,3}_1 + \hat{T}^{3,1}_1 = 0\end{aligned}$$

(*the resulting moment vanishes*).

Proof. On any side $\hat{S}$ of $\hat{K}$ we introduce the basic linear functions $\lambda^i_k \in P_1(\hat{S})$, $k = 1, 2$ such that

$$\begin{aligned}(2.13)\qquad \lambda^i_1(\hat{s}) &= 1\,,\\ \lambda^i_2(\hat{s}) &= \frac{2\hat{s} - l_i}{l_i}\,,\end{aligned}$$

where the parameter $\hat{s}$ has the starting point in $\hat{a}_i$ of any side $\hat{S}$.

Then (2.11) and (2.12) is a consequence of the equilibrium conditions and of the symmetry of $\hat{\tau}$ if we make use of the definitions of $\mathscr{M}_{\hat{K}}$ and $\mathscr{N}(\hat{K}_i)$ and insert $\hat{t}^i_k(\hat{\tau}^i) = \hat{T}^{i,i}_k \lambda^i_1 + \hat{T}^{i,i+1}_k \lambda^i_2$. □

Theorem 2.1. *Let twelve degrees of freedom $\hat{T}^{i,i}_k$, $\hat{T}^{i,i+1}_k$, $i = 1, 2, 3$, $k = 1, 2$, be given, which satisfy* (2.11) *and* (2.12).

Then there exists a unique stress tensor $\hat{\tau} \in \mathscr{M}_{\hat{K}}$ such, that the equations (2.10) *hold.*

Proof. As the dimension of the set $\mathscr{N}(\hat{K}_i)$ is seven, it is necessary to have 21 equations to determine the three functions $\hat{\tau}^i \equiv \hat{\tau}|_{\hat{K}_i}$. It is easy to see, that the set $\Sigma^*_{\hat{K}}$ of degrees of freedom generates twelve equations which are constrained by three equations (2.11) and (2.12). The conditions of continuity of the stress vector across internal sides of $\hat{K}$ give the remaining twelve equations. Due to these facts, it only remains to show regularity of the matrix of the linear system obtained.

On any side $\hat{S}$ of $\hat{K}$ we express the stress vector in the form

$$\hat{t}^j_{i,k} = \hat{R}^{j,j}_{i,k}\lambda^j_1 + \hat{R}^{j,j+1}_{i,k}\lambda^j_2\,, \quad j, i = 1, 2, 3; \quad k = 1, 2\,, \tag{2.14}$$

where

$$\begin{aligned} \hat{R}^{j,j}_{i,k} &= \tfrac{1}{2}\{(\hat{\tau}^i(\hat{a}_{j+1})\,\hat{n}^j)_k + (\hat{\tau}^i(\hat{a}_j)\,\hat{n}^j)_k\}\,, \\ \hat{R}^{j,j+1}_{i,k} &= \tfrac{1}{2}\{(\hat{\tau}^i(\hat{a}_{j+1})\,\hat{n}^j)_k - (\hat{\tau}^i(\hat{a}_j)\,\hat{n}^j)_k\} \end{aligned} \tag{2.15}$$

and λ^j_k, $k = 1, 2$ are the basic linear functions from (2.13). The subscript i refers to the subtriangle $\hat{K}_i$, superscript j refers to each side of each $\hat{K}_i$. Then the continuity conditions for the stress vector on any side of $\hat{K}$ give

$$\begin{pmatrix} \hat{\mathbf{A}}^*_u \\ \hat{\mathbf{A}}^*_l \end{pmatrix} \hat{\mathbf{R}}^* = \begin{Bmatrix} \mathbf{0} \\ \hat{\mathbf{T}} \end{Bmatrix}, \tag{2.16}$$

where $\hat{\mathbf{A}}^*_u$ is a matrix of the type (12×36) and the continuity conditions are expressed across the internal sides of $\hat{K}$, and $\hat{\mathbf{A}}^*_l$ is a matrix of the type (12×36) and the continuity conditions are expressed across the external sides of $\hat{K}$. $\hat{\mathbf{R}}^*$ is the matrix of parameters $\hat{R}^{j,j}_{i,k}$, $\hat{R}^{j,j+1}_{i,k}$, $j, i = 1, 2, 3$; $k = 1, 2$, of the type (36×1). The column matrix on the right-hand side of (2.16) is of the type (24×1) and is partitioned between the 12th and 13th rows. $\hat{\mathbf{T}}$ denotes the submatrix of the degrees of freedom.

Express the stress vector on any side of $\hat{K}_i$ in terms of the stress field $\hat{\tau}^i$. To this end we have

$$\frac{1}{2d_{\hat{S}}}\,\hat{\mathbf{C}}^{i,j}\hat{\boldsymbol{\beta}}^i = \hat{\mathbf{R}}^j_i\,, \tag{2.17}$$

where $d_{\hat{S}}$ denotes the length of the j-th side of $\hat{K}_i$, $\hat{\boldsymbol{\beta}}^i = \{\hat{\beta}^i_1, \ldots, \hat{\beta}^i_7\}$, $\hat{\mathbf{R}}^j_i = \{\hat{R}^{j,j}_{i,1}, \hat{R}^{j,j}_{i,2}, \hat{R}^{j,j+1}_{i,1}, \hat{R}^{j,j+1}_{i,2}\}$ and

$$(2.18)\qquad \hat{\mathbf{C}}^{i,j} =$$

$$= \begin{pmatrix} -2(\hat{Y}_j - \hat{Y}_{j+1}), & 0, & 2(\hat{X}_j - \hat{X}_{j+1}), & -2(\hat{X}\hat{Y}_j - \hat{X}_{j+1}\hat{Y}_{j+1}), & 0, & \hat{X}_j^2 - \hat{X}_{i+1}^2, & -(\hat{Y}_j^2 - \hat{Y}_{j+1}^2) \\ 0, & 2(\hat{X}_j - \hat{X}_{j+1}), & -2(\hat{Y}_j - \hat{Y}_{j+1}), & \hat{Y}_j^2 - \hat{Y}_{j+1}^2, & \hat{X}_j^2 - \hat{X}_{j+1}^2, & -2(\hat{X}_j\hat{Y}_j - \hat{X}_{j+1}\hat{Y}_{j+1}), & 0 \\ 0, & 0, & 0, & 2(\hat{X}_j - \hat{X}_{j+1})(\hat{Y}_j - \hat{Y}_{j+1}), & 0, & -(X_j - \hat{X}_{j+1})^2, & (\hat{Y}_j - \hat{Y}_{j+1})^2 \\ 0, & 0, & 0, & -(\hat{Y}_j - \hat{Y}_{j+1})^2, & -(\hat{X}_j - \hat{X}_{j+1})^2, & 2(\hat{X}_j - \hat{X}_{j+1})(\hat{Y}_j - \hat{Y}_{j+1}), & 0 \end{pmatrix}$$

if we insert $\hat{\nu}_1^j = -(l_j)^{-1}(\hat{Y}_j - \hat{Y}_{j+1}), \hat{\nu}_2^j = (l_j)^{-1}(\hat{X}_j - \hat{X}_{j+1})$ for the components of the unit outward normal. $(\hat{X}_j, \hat{Y}_j)$ denotes the coordinates of the vertex $\hat{a}_j$, $j = 1, 2, 3$. Inserting (2.17) into (2.16) we obtain

$$(2.19)\qquad \begin{pmatrix} \hat{\mathbf{A}}_u \\ \hat{\mathbf{A}}_l \end{pmatrix} \hat{\boldsymbol{\beta}} = \begin{Bmatrix} \mathbf{0} \\ \hat{\mathbf{T}} \end{Bmatrix}.$$

As in [14], we use the continuity conditions for the stress vector across the internal sides of $\hat{K}$ to reduce the parameters $\hat{\boldsymbol{\beta}}$ of the stress field. The rank of the matrix $\hat{\mathbf{A}}_u$ is twelve and we have

$$(2.20)\qquad \hat{\mathbf{A}}_u \begin{pmatrix} (\hat{\mathbf{A}}_u^0)^{-1} & -(\hat{\mathbf{A}}_u^0)^{-1}\,\hat{\mathbf{A}}_u^d \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \hat{\mathbf{A}}_u \mathbf{Q} = (\mathbf{I} \vdots \mathbf{0}),$$

where $\hat{\mathbf{A}}_u^0$ is a matrix of the type (12×12) which is formed by the first twelve columns of the matrix $\hat{\mathbf{A}}_u$, while the matrix $\hat{\mathbf{A}}_u^d$ is of the type (12×9), I is the unit matrix of the type (9×9).

Then we introduce a transformation of the form

$$(2.21)\qquad \hat{\boldsymbol{\beta}} = \mathbf{Q}\hat{\boldsymbol{\beta}}' = (\mathbf{Q}_0 \vdots \mathbf{Q}_1)\begin{Bmatrix} \hat{\boldsymbol{\beta}}'_u \\ \hat{\boldsymbol{\beta}}'_l \end{Bmatrix},$$

where $\mathbf{Q}$ is partitioned between the 12th and 13th columns and $\hat{\boldsymbol{\beta}}'$ is partitioned accordingly.

The equation (2.21) is then inserted into the upper equations of (2.19) with the result:

$$(2.22)\qquad \hat{\mathbf{A}}_u \mathbf{Q} \hat{\boldsymbol{\beta}}' = \mathbf{0},$$

$$(\mathbf{I} \vdots \mathbf{0})\begin{Bmatrix} \hat{\boldsymbol{\beta}}'_u \\ \hat{\boldsymbol{\beta}}'_l \end{Bmatrix} = \mathbf{0}.$$

Hence the equation (2.21) takes the form:

$$(2.23)\qquad \hat{\boldsymbol{\beta}} = \mathbf{Q}_1 \hat{\boldsymbol{\beta}}'_l$$

and the lower equations of (2.19) produce

$$\hat{\mathbf{A}}_l \mathbf{Q}_1 \hat{\boldsymbol{\beta}}_l' = \hat{\boldsymbol{T}} . \tag{2.24}$$

The matrix product $\hat{\mathbf{A}}_l\mathbf{Q}_1$ is now considered as a matrix $\hat{\mathbf{C}}$ for a triangular "building block" and we have the desired system

$$\hat{\mathbf{C}} \hat{\boldsymbol{\beta}}_l' = \hat{\boldsymbol{T}} , \tag{2.25}$$

in details

$$\begin{pmatrix}
-1/6 & -1/6 & 0 & 0 & -1 & 0 & 0 & -1/3 & 1/6 \\
11/24 & 5/6 & 0 & -1 & 0 & 0 & -1/2 & -11/24 & -5/6 \\
1/4 & 1/2 & 0 & . & . & . & 0 & -3/4 & -1/2 \\
7/8 & 3/2 & 0 & . & . & 0 & -1/2 & -7/8 & -3/2 \\
\sqrt{2}/12 & \sqrt{2}/12 & \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 & 0 & \sqrt{2}/6 & \sqrt{2}/6 \\
-11\sqrt{2}/48 & -5\sqrt{2}/12 & 0 & \sqrt{2}/2 & \sqrt{2}/2 & -\sqrt{2}/4 & \sqrt{2}/4 & 11\sqrt{2}/48 & 5\sqrt{2}/12 \\
-\sqrt{2}/8 & \sqrt{2}/4 & 0 & . & 0 & -\sqrt{2}/2 & 0 & -\sqrt{2}/8 & \sqrt{2}/2 \\
5\sqrt{2}/16 & 3\sqrt{2}/4 & 0 & . & 0 & -\sqrt{2}/4 & -\sqrt{2}/4 & -13\sqrt{2}/16 & -3\sqrt{2}/4 \\
0 & 0 & -1 & 0 & . & . & . & 0 & -1/2 \\
0 & . & . & 0 & -1 & 1/2 & 0 & . & 0 \\
0 & . & . & . & . & . & . & 0 & 1/2 \\
0 & . & . & . & 0 & -1/2 & 0 & . & 0
\end{pmatrix} .$$

$$\cdot \begin{pmatrix} \hat{\beta}_1' \\ \hat{\beta}_2' \\ \hat{\beta}_3' \\ \hat{\beta}_4' \\ \hat{\beta}_5' \\ \hat{\beta}_6' \\ \hat{\beta}_7' \\ \hat{\beta}_8' \\ \hat{\beta}_9' \end{pmatrix} = \begin{pmatrix} \hat{T}_1^{1,1} \\ \hat{T}_2^{1,1} \\ \hat{T}_1^{1,2} \\ \hat{T}_2^{1,2} \\ \hat{T}_1^{2,2} \\ \hat{T}_2^{2,2} \\ \hat{T}_1^{2,3} \\ \hat{T}_2^{2,3} \\ \hat{T}_1^{3,3} \\ \hat{T}_2^{3,3} \\ \hat{T}_1^{3,1} \\ \hat{T}_2^{3,1} \end{pmatrix}$$

The three conditions (2.11), (2.12) imply that we can omit three equations of the system, e.g. the equations 5, 6, 7.

Then the remaining system has the form

$$\mathscr{C} \hat{\boldsymbol{\beta}}' = \hat{\boldsymbol{T}}' ,$$

where

$$|\det \mathscr{C}| = \frac{7 \,.\, \sqrt{2}}{3 \,.\, 2^9} .$$

Consequently, the system has a unique solution $\hat{\boldsymbol{\beta}}' \in \mathbb{R}^9$. □

C—E—S—II ELEMENT

This element is conjugated, in the sense of slab analogy, with Ahlin's compatible element defined in [2] (as Bogner-Fox-Schmit's element).

Let us recall that for Ahlin's element, K is a rectangle (see Fig. 3), the sides of which are parallel to the coordinate axis x_1, x_2 respectively, with vertices a_i, $1 \leqq i \leqq 4$.

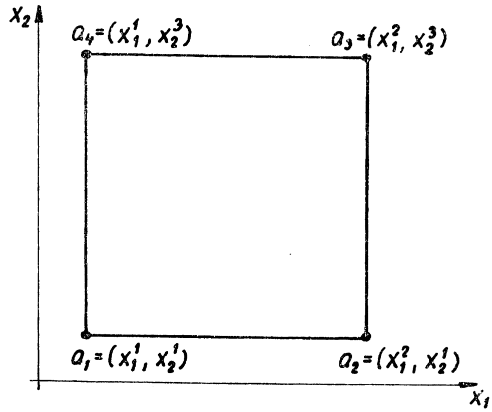

Fig. 3.

With the rectangle K we associate the space

$$P_K = Q_3\,, \tag{2.26}$$

where

$$Q_3 = \sum_{\substack{0 \leqq \alpha_i \leqq 3 \\ i=1,2}} \beta_{\alpha_1\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}\,,$$

and the set of degrees of freedom

$$\Sigma_K = \{U(a_i), \partial_1 U(a_i), \partial_2 U(a_i), \partial_{12} U(a_i), 1 \leqq i \leqq 4\}\,. \tag{2.27}$$

Let the Airy stress function $U \in P_K$ be given in the form

$$\begin{aligned} U = {} & \beta_{16} + \beta_{15}x_1 + \beta_{14}x_2 + \tfrac{1}{2}\beta_1 x_2^2 + \tfrac{1}{2}\beta_2 x_1^2 - \beta_3 x_1 x_2 + \\ & + \tfrac{1}{2}\beta_4 x_1^2 x_2^2 + \tfrac{1}{6}\beta_5 x_1^3 + \tfrac{1}{6}\beta_6 x_2^3 + \tfrac{1}{2}\beta_7 x_1^2 x_2 + \tfrac{1}{2}\beta_8 x_1 x_2^2 + \tfrac{1}{6}\beta_9 x_1^3 x_2 + \\ & \tfrac{1}{6}\beta_{10} x_1 x_2^3 - \tfrac{1}{2}\beta_{11} x_1^3 x_2^2 - \tfrac{1}{2}\beta_{12} x_1^2 x_2^3 - \tfrac{1}{6}\beta_{13} x_1^3 x_2^3\,, \end{aligned} \tag{2.28}$$

where β_m, $m = 1, \ldots, 16$, are real constants.

From (2.1) we have

$$\mathscr{M}_K = \{\tau \in H(\mathrm{Div}; K) \mid \tau \in P^s(K), \mathrm{Div}\, \tau = 0\}\,, \tag{2.29}$$

where

$$P^s(K) = \{\tau \in P_{11}(K) \times P_{22}(K) \times (P_{12}(K))^2 \mid \tau = \tau^{\mathrm{T}}\}$$

and

$P_{11} = \varrho_{11}\boldsymbol{P}_K$ is the set of functions cubic in x_1 and linear in x_2 .

$P_{22} = \varrho_{22}\boldsymbol{P}_K$ is the set of functions cubic in x_2 and linear in x_1 ,

$P_{12} \equiv Q_2 = \varrho_{12}\boldsymbol{P}_K$ is the set of functions of the type $\sum_{\substack{0 \leqq \alpha_i \leqq 2 \\ i=1,2}} \beta_{\alpha_1\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$ and the operators ϱ_{ij} $(i, j = 1, 2)$ are the entries of the matrix ϱ (see (1.10)). Clearly, the space $\mathscr{M}_K$ is a thirteen-dimensional linear set and if we employ (2.27) we can define a reference equilibrium square element (the matrix B from (2.3) is diagonal) by

Definition 2.4. *A reference conforming equilibrium square stress element* (see Fig. 4) *is a triple* $(\hat{K}, \mathscr{M}_{\hat{K}}, \Sigma_{\hat{K}}^*)$, *where* $\hat{K}$ *is the reference square,* $\mathscr{M}_{\hat{K}}$ *is defined by* (2.29) *and* $\Sigma_{\hat{K}}^*$ *is the set of linear functional defined on* $\mathscr{M}_{\hat{K}}$ *as follows:*

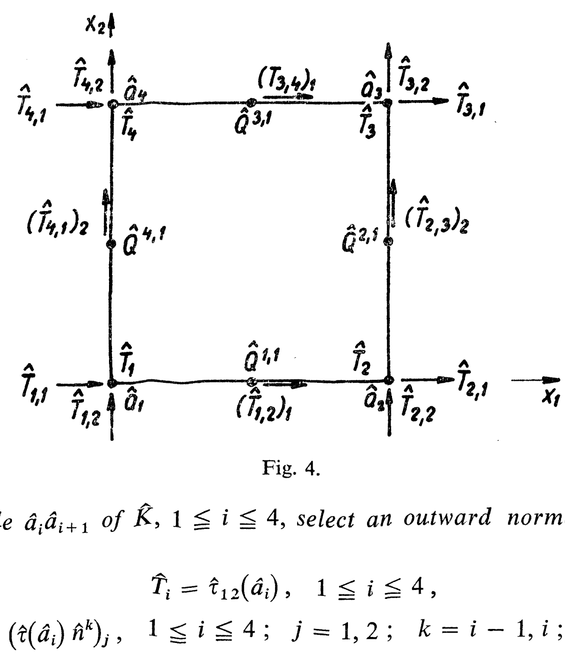

Fig. 4.

For any side $\hat{a}_i\hat{a}_{i+1}$ *of* $\hat{K}$, $1 \leqq i \leqq 4$, *select an outward normal* $\hat{n}^i$. *Then for* $\hat{\tau} \in \mathscr{M}_{\hat{K}}$ *put*

(2.30)
$$\hat{T}_i = \hat{\tau}_{12}(\hat{a}_i)\,, \quad 1 \leqq i \leqq 4\,,$$

$$\hat{T}_{ij} = (\hat{\tau}(\hat{a}_i)\,\hat{n}^k)_j\,, \quad 1 \leqq i \leqq 4\,; \quad j = 1, 2\,; \quad k = i - 1, i\,; \quad j = 1$$

$$\text{for} \quad k = i - 1\,, \quad \textit{where} \quad i - 1 = 4 \quad \textit{for} \ i = 1\,,$$

$$(\hat{T}_{i,i+1})_j = (\hat{\tau}(\hat{Q}_{i,i+1})\,\hat{n}^k)_j\,, \quad 1 \leqq i \leqq 4\,; \quad j = 1, 2\,; \quad k = i\,;$$

$i + 1 = 1$ for $i = 4$; $j = 1$ for $k = i$ *odd*, $j = 2$ for $k = i$ *even*,

where $\hat{Q}_{i,i+1}$ *denotes the mid-point of the side* $\hat{a}_i\hat{a}_{i+1}$. □

Lemma 2.2. *Let $\hat{\tau} \in \mathcal{M}_{\hat{K}}$ and let sixteen degrees of freedom be given by* (2.30). *Then the following three conditions of the overall equilibrium hold;*

$$(2.31)\qquad -\hat{T}_1 + 4(\hat{T}_{1,2})_1 - \hat{T}_2 + 3\hat{T}_{2,1} + 3\hat{T}_{3,1} + \hat{T}_3 + 4(\hat{T}_{3,4})_1 + \\ + \hat{T}_4 + 3\hat{T}_{4,1} + 3\hat{T}_{1,1} = 0\,,$$

$$(2.32)\qquad 3\hat{T}_{1,2} + 3\hat{T}_{2,2} + \hat{T}_2 + 4(\hat{T}_{2,3})_2 + \hat{T}_3 + 3\hat{T}_{3,2} + 3\hat{T}_{4,2} - \hat{T}_4 + \\ + 4(\hat{T}_{4,1})_2 - \hat{T}_1 = 0\,,$$

$$(2.33)\qquad \hat{T}_{1,2} + 2\hat{T}_{2,2} + \hat{T}_2 + 4(\hat{T}_{2,3})_2 - \hat{T}_{2,1} - 2\hat{T}_{3,1} + 2\hat{T}_{3,2} + \hat{T}_{4,2} - \\ - 4(\hat{T}_{3,4})_1 - \hat{T}_4 - 2\hat{T}_{4,1} - \hat{T}_{1,1} = 0\,.$$

Proof. On any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$, $1 \leqq i \leqq 4$; $i + 1 = 1$ for $i = 4$, $l_i = 1$, we introduce two systems of basic functions $\lambda_k^i \in P_1(\hat{a}_i\hat{a}_{i+1})$, $k = 1,2$ and $\mu_k^i \in P_2(\hat{a}_i\hat{a}_{i+1})$ $k = 1, 2, 3$ such, that

$$\lambda_1^i(\hat{s}) = 1 - \hat{s}\,,$$
$$\lambda_2^i(\hat{s}) = \hat{s}\,,$$
$$\mu_1^i(\hat{s}) = 1 - 3\hat{s} + 2\hat{s}^2\,,$$
$$\mu_2^i(\hat{s}) = 4\hat{s}(1 - \hat{s})\,,$$
$$\mu_3^i(\hat{s}) = \hat{s}(2\hat{s} - 1)\,,$$

where the parameter $\hat{s}$ has the starting point in $\hat{a}_i$ of the side $\hat{a}_i\hat{a}_{i+1}$.

Then (2.31) to (2.33) is a consequence of the equilibrium conditions and of the symmetry of $\hat{\tau}$ if we make use of the definition of $\mathcal{M}_{\hat{K}}$ and insert the stress vector on any side $\hat{a}_i\hat{a}_{i+1}$ expressed by the degrees of freedom and the basic functions λ_k^i, μ_k^i, and if we take into consideration the fact, that the stress vector $\hat{t}$ for $\hat{\tau} \in \mathcal{M}_{\hat{K}}$ is

i) $\hat{x}_1 = \text{const.}$: $\hat{t}_1$ – linear function,
$\hat{t}_2$ – quadratic function;

ii) $\hat{x}_2 = \text{const.}$: $\hat{t}_1$ – quadratic function,
$\hat{t}_2$ – linear function

As an outward normal $\hat{n}^i$ we choose the unit outward normal $\hat{\nu}^i$ for any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$, and any side of $\hat{K}$ is expressed parametrically as $\hat{x}_k = \hat{x}_k(\hat{a}_i)\,\lambda_1^i + \hat{x}_k(\hat{a}_{i+1})\,\lambda_2^i$, $k = 1, 2$. □

Theorem 2.2. *Let sixteen degrees of freedom be given, which satisfy* (2.31), (2.32) *and* (2.33).

Then there exists a unique stress field $\hat{\tau} \in \mathcal{M}_{\hat{K}}$ such that the equations (2.30) *hold.*

Proof. As in the proof of Theorem 2.1, it is sufficient to show the regularity of the matrix of the linear system obtained. We write the system (2.30) and insert the equations for $\hat{\tau} \in \mathcal{M}_K$ into it. We obtain the system of 16 equations for 13 parameter $\hat{\beta}_l, 1 \leqq l \leqq 13$, i.e

$$\hat{\mathbf{C}}\hat{\boldsymbol{\beta}} = \hat{\mathbf{T}} . \tag{2.34}$$

The three conditions (2.31) to (2.33) imply that we can omit three equations of the system.

Then the remaining system has the form

$$\mathscr{C}\hat{\boldsymbol{\beta}} = \hat{\mathbf{T}}'$$

where

$$|\det \mathscr{C}| = \frac{9}{2^{11}} .$$

Consequently, the system has a unique solution $\hat{\boldsymbol{\beta}} \in \mathbb{R}^{13}$. □

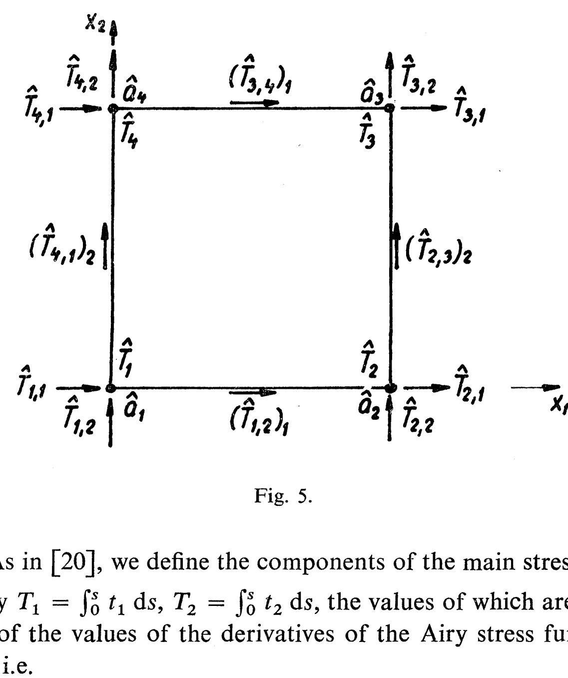

Fig. 5.

Remark 2.1. As in [20], we define the components of the main stress vector acting on the arc $\widehat{0s}$ by $T_1 = \int_0^s t_1 \,ds$, $T_2 = \int_0^s t_2 \,ds$, the values of which are determined as the differences of the values of the derivatives of the Airy stress function U at the points 0 and s, i.e.

$$T_1 = \partial_2 U(s) - \partial_2 U(0) ,$$
$$-T_2 = \partial_1 U(s) - \partial_1 U(0) .$$

Now, we can define the set Σ_K^* of linear functionals, defined on $\mathcal{M}_K$ (see Fig. 5), as follows:

For any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$, $1 \leqq i \leqq 4$, *select an outward normal* $\hat{n}^i$. *Then for* $\hat{\tau} \in \mathcal{M}_{\hat{K}}$ *put*

(2.35)
$$\hat{T}_i = \hat{\tau}_{12}(\hat{a}_i)\,, \quad 1 \leqq i \leqq 4\,,$$
$$\hat{T}_{i,j} = (\hat{\tau}(\hat{a}_i)\,\hat{n}^k)_j\,, \quad 1 \leqq i \leqq 4\,; \quad j = 1, 2\,; \quad k = i - 1, i\,;$$
$$j = 1 \quad \textit{for} \quad k = i - 1\,, \quad \textit{where} \quad i - 1 = 4 \quad \textit{for} \quad i = 1\,;$$
$$j = 2 \quad \textit{for} \quad k = i\,,$$
$$(\hat{T}_{i,i+1})_j = \int_{\hat{a}_i\hat{a}_{i+1}} (\hat{\tau}(\hat{s})\,\hat{n}^k)_j\,\mathrm{d}\hat{s}\,, \qquad 1 \leqq i \leqq 4\,; \quad i + 1 = 1 \quad \textit{for} \quad i = 4\,;$$
$$j = 1 \quad \textit{for} \quad k = 1 \quad \textit{odd}\,; \quad j = 2 \quad \textit{for} \quad k = i \quad \textit{even}\,.$$

It is easy to verify the $\mathcal{M}_{\hat{K}}$-unisolvability of $\Sigma^*_{\hat{K}}$ in a way analogous to that used in the proof of Theorem 2.2.

C—E—S—III ELEMENT

This coforming equilibrium stress element is conjugated, in the sense of slab analogy, with Fellipa's compatible element defined in [2] (as Argyris' element).

Let us recall that for Fellipa's element K is a triangle (see Fig. 6) with the vertices a_i, $1 \leqq i \leqq 3$.

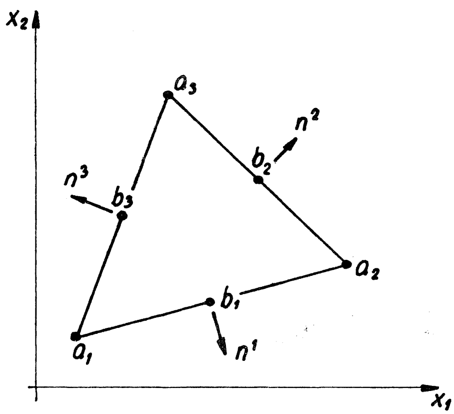

Fig. 6.

With the triangle K we associate the space

(2.36)
$$P_K = P_5$$

and the set of degrees of freedom

(2.37)
$$\Sigma_K = \{\mathrm{D}^\alpha\,U(a_i), |\alpha| \leqq 2,\ \partial_\nu U(b_i), 1 \leqq i \leqq 3\}\,,$$

where b_i is the mid-point of the side $a_i a_{i+1}$.

Let the Airy stress function $U \in \boldsymbol{P}_K$ be given in the form:

$$(2.38)\quad U = \beta_{21} + \beta_{20}x_1 + \beta_{19}x_2 + \tfrac{1}{2}\beta_1 x_2^2 + \tfrac{1}{2}\beta_2 x_1^2 - \beta_3 x_1 x_2 + \tfrac{1}{6}\beta_4 x_1^3 + \\ + \tfrac{1}{2}\beta_5 x_1^2 x_2 - \tfrac{1}{2}\beta_6 x_1 x_2^2 + \tfrac{1}{6}\beta_7 x_2^3 + \tfrac{1}{12}\beta_8 x_1^4 + \tfrac{1}{6}\beta_9 x_1^3 x_2 - \tfrac{1}{2}\beta_{10} x_1^2 x_2^2 - \\ - \tfrac{1}{6}\beta_{11} x_1 x_2^3 + \tfrac{1}{12}\beta_{12} x_2^4 + \tfrac{1}{20}\beta_{13} x_1^5 - \tfrac{1}{12}\beta_{14} x_1^4 x_2 - \tfrac{1}{6}\beta_{15} x_1^3 x_2^2 - \\ - \tfrac{1}{6}\beta_{16} x_1^2 x_2^3 - \tfrac{1}{12}\beta_{17} x_1 x_2^4 + \tfrac{1}{20}\beta_{18} x_2^5 ,$$

where β_m, $m = 1, \ldots, 21$, are real constants.

From (2.1) we have

$$(2.39)\quad \mathscr{M}_K = \{\tau \in H(\mathrm{Div}; K) \mid \tau \in (P_3(K))^4 ,\ \tau = \tau^{\mathrm{T}}\} .$$

Clearly, the space $\mathscr{M}_K$ is an eighteen-dimensional linear set.

Definition 2.5. *A reference conforming equilibrium triangle stress element is a triple* $(\hat{K}, \mathscr{M}_{\hat{K}}, \Sigma^*_{\hat{K}})$ *(see Fig. 7), where* $\mathscr{M}_{\hat{K}}$ *is defined by (2.39) and* $\Sigma^*_{\hat{K}}$ *is the set of linear functionals defined on* $\mathscr{M}_{\hat{K}}$ *as follows:*

For any side $\hat{a}_i\hat{a}_{i+1}$, $1 \leqq i \leqq 3$, *select an outward normal* $\hat{n}^i$. *Then for* $\hat{\tau} \in \mathscr{M}_{\hat{K}}$ *put*

$$(2.40)\quad \hat{T}_{i,j} = \hat{\tau}_{sm}(\hat{a}_i) ,\quad 1 \leqq i \leqq 3 ,\quad 1 \leqq j \leqq 3 ;\quad s, m = 1, 2 ; \\ j = s = m \quad for \quad j = 1, 2 ;\quad j = 3 \quad for \quad s < m , \\ (\hat{T}_{i,r})_j = (\hat{\tau}(\hat{Q}^{i,r})\, \hat{n}^i)_j ,\quad 1 \leqq i \leqq 3 ;\quad r = 1, 2 ;\quad j = 1, 2 ,$$

where $\hat{Q}^{i,r}$, $r = 1, 2$, *denote the points which partition the side* $\hat{a}_i\hat{a}_{i+1}$ *into three equal parts.* □

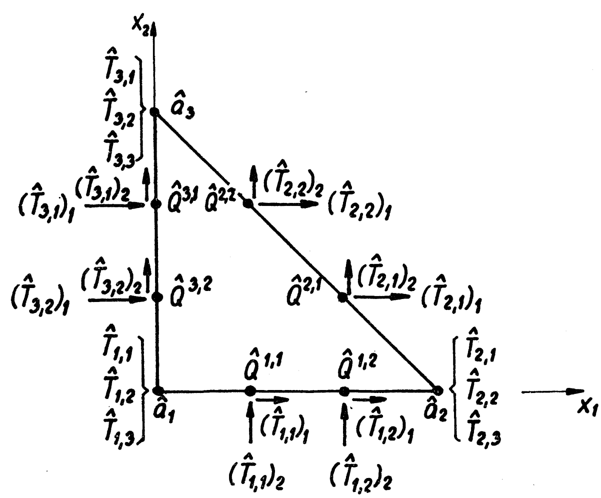

Fig. 7.

Lemma 2.3 *Let $\hat{\tau} \in \mathscr{M}_K$ and let twenty-one degrees of freedom be given by* (2.40). *Then the following three conditions of the overall equilibrium hold:*

$$(2.41)\qquad -\hat{T}_{1,1} - \hat{T}_{1,3} + 3(\hat{T}_{1,1})_1 + \hat{T}_{2,1} + 3(\hat{T}_{1,2})_1 + 3\sqrt{(2)}\,(\hat{T}_{2,1})_1 + \\ + 3\sqrt{(2)}\,(\hat{T}_{2,2})_1 + \hat{T}_{3,3} + 3(\hat{T}_{3,1})_1 + 3(\hat{T}_{3,2})_1 = 0\,,$$

$$(2.42)\qquad -\hat{T}_{1,3} - \hat{T}_{1,2} + 3(\hat{T}_{1,1})_2 + 3(\hat{T}_{1,2})_2 + \hat{T}_{2,3} + 3\sqrt{(2)}\,(\hat{T}_{2,1})_2 + \\ + 3\sqrt{(2)}\,(\hat{T}_{2,2})_2 + \hat{T}_{3,2} + 3(\hat{T}_{3,1})_2 + 3(\hat{T}_{3,2})_2 = 0\,,$$

$$(2.43)\qquad 2\hat{T}_{1,1} - 2\hat{T}_{1,2} + 9(\hat{T}_{1,1})_2 + 36(\hat{T}_{1,2})_2 - 2\hat{T}_{2,1} + 11\hat{T}_{2,3} - \\ - 9\sqrt{(2)}\,(\hat{T}_{2,1})_1 + 36\sqrt{(2)}\,(\hat{T}_{2,1})_2 - 36\sqrt{(2)}(\hat{T}_{2,2})_1 + 9\sqrt{(2)}\,(\hat{T}_{2,2})_2 + 2\hat{T}_{3,2} - \\ - 11\hat{T}_{3,3} - 36(\hat{T}_{3,1})_1 - 9(\hat{T}_{3,2})_1 = 0\,.$$

Proof. On any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$ we introduce a sys:em of basic functions $\lambda_k^i \in \in P_3(\hat{a}_i\hat{a}_{i+1})$ $k = 1, 2, 3, 4$ such, that

$$\lambda_1^i(\hat{s}) = 1 - \tfrac{11}{2}\hat{s}(l_i)^{-1} + 9\hat{s}^2(l_i)^{-2} - \tfrac{9}{2}\hat{s}^3(l_i)^{-3}\,,$$
$$\lambda_2^i(\hat{s}) = 9\hat{s}(l_i)^{-1} - \tfrac{45}{12}\hat{s}^2(l_i)^{-2} + \tfrac{27}{2}\hat{s}^3(l_i)^{-3}\,,$$
$$\lambda_3^i(\hat{s}) = \tfrac{9}{2}\hat{s}(l_i)^{-1} + 18\hat{s}^2(l_i)^{-2} - \tfrac{27}{2}\hat{s}^3(l_i)^{-3}\,,$$
$$\lambda_4^i(\hat{s}) = \hat{s}(l_i)^{-1} - \tfrac{9}{2}\hat{s}^2(l_i)^{-2} + \tfrac{9}{2}\hat{s}^3(l_i)^{-3}\,,$$

where the parameter $\hat{s}$ has the starting point in $\hat{a}_i$ of the side $\hat{a}_i\hat{a}_{i+1}$. As an outward normal $\hat{n}^i$ we choose the unit outward normal $\hat{v}^i$ for any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$. On any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$, $i = 1, 2, 3$, let us express the stress vector in the following manner:

$$\hat{t}_1^i(\hat{s}) = (\hat{T}_{i,1}\hat{v}_1^i + \hat{T}_{i,3}\hat{v}_2^i)\,\lambda_1^i(\hat{s}) + (\hat{T}_{i,1})_1\,\lambda_2^i(\hat{s}) + (\hat{T}_{i,2})_1\,\lambda_3^i(\hat{s}) + \\ + (\hat{T}_{i+1,1}\hat{v}_1^i + \hat{T}_{i+1,3}\hat{v}_2^i)\,\lambda_4^i(\hat{s})$$

$$\hat{t}_2^i(\hat{s}) = (\hat{T}_{i,3}\hat{v}_1^i + \hat{T}_{i,2}\hat{v}_2^i)\,\lambda_1^i(\hat{s}) + (\hat{T}_{i,1})_2\,\lambda_2^i(\hat{s}) + (\hat{T}_{i,2})_2\,\lambda_3^i(\hat{s}) + \\ + (\hat{T}_{i+1,3}\hat{v}_1^i + \hat{T}_{i+1,2}\hat{v}_2^i)\,\lambda_4^i(\hat{s})$$

where $i + 1 = 1$ for $i = 3$.

Now the proof is analogous to that of Lemma 2.2. □

Theorem 2.3. *Let twenty-one degrees of freedom be given, which satisfy* (2.41), (2.42) *and* (2.43).

Then there exists a unique stress field $\hat{\tau} \in \mathscr{M}_K$ such that the equations (2.40) *hold.*

Proof. To prove the regularity of the matrix of the linear system obtained we write the system (2.40) for any $i = 1, 2, 3$ and insert the equations for $\hat{\tau} \in \mathscr{M}_K$ into it.

We obtain a system of 21 equations for 18 parameters $\hat{\beta}_l$, $1 \leqq l \leqq 18$, i.e.

$$\hat{\mathbf{C}}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{T}}\,. \tag{2.44}$$

Due to the equations (2.41), (2.42) and (2.43) we can omit three equations of the system. The remaining system has the form

$$\mathscr{C}\hat{\boldsymbol{\beta}} = \hat{\mathbf{T}}'$$

where

$$\left|\det \mathscr{C}\right| = \frac{2^6}{3^{27}}\,.$$

Consequently, the system has a unique solution $\hat{\boldsymbol{\beta}} \in \mathbb{R}^{18}$. □

Remark 2.2 As in Remark 2.1, we define the components of the main stress vector acting on the arc $\widehat{0s}$. Thus

$$\partial_\nu U = \partial_1 U \nu_1 + \partial_2 U \nu_2 = -T_2 \nu_1 + T_1 \nu_2 + c \sim T_\tau\,,$$
$$\partial_\tau U = -\partial_1 U \nu_2 + \partial_2 U \nu_1 = T_2 \nu_2 + T_1 \nu_1 + c \sim T_\nu\,,$$

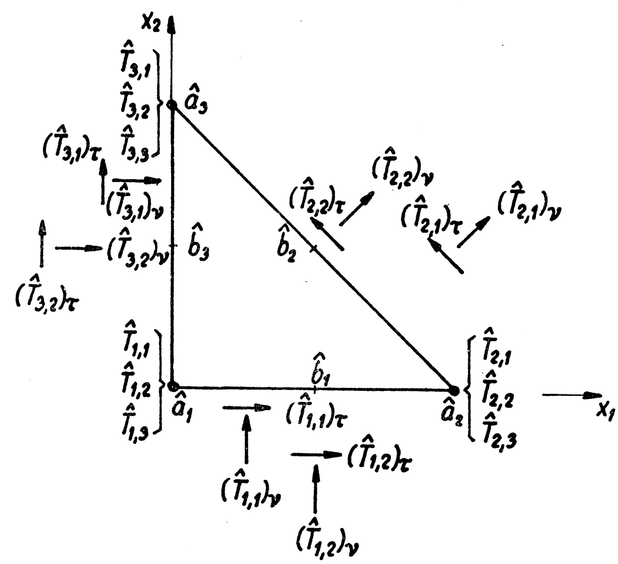

Fig. 8.

Now, using Theorem 1.4, we deduce that the tangent and normal components of the main stress vector are continuous across any side $\hat{a}_i\hat{a}_{i+1}$ of $\hat{K}$ and we can define the set $\Sigma_{\hat{K}}^*$ of linear functionals defined on $\mathscr{M}_{\hat{K}}$ (see Fig. 8) as follows:

For any side $\hat{a}_i\hat{a}_{i+1}$, $1 \leqq i \leqq 3$, select an outward normal $\hat{n}^i$. Then for $\hat{\tau} \in \mathscr{M}_K$ put

$$(2.45) \qquad \hat{T}_{i,j} = \hat{\tau}_{sm}(\hat{a}_i)\,, \quad 1 \leqq i \leqq 3\,; \quad 1 \leqq j \leqq 3\,; \quad s, m = 1, 2\,;$$

$$j = s = m \quad for \quad j = 1, 2\,; \quad j = 3 \quad for \quad s < m\,,$$

$$(\hat{T}_{i,1})_\nu = \int_{\hat{a}_i\hat{b}_i} \Big(\sum_{k=1}^{2} (\hat{\tau}\hat{n}^i)_k\, \hat{n}^i_k\Big)\, \mathrm{d}\hat{s}\,,$$

$$(\hat{T}_{i,1})_\tau = \int_{\hat{a}_i\hat{b}_i} \Big(\sum_{\substack{k,l=1\\k\neq l}}^{2} (-1)^k\, (\hat{\tau}\hat{n}^i)_k\, n^i_l\Big)\, \mathrm{d}\hat{s}\,,$$

$$(\hat{T}_{i,2})_\nu = \int_{\hat{a}_i\hat{a}_{i+1}} \Big(\sum_{k=1}^{2} (\hat{\tau}\hat{n}^i)_k\, \hat{n}^i_k\Big)\, \mathrm{d}\hat{s}\,,$$

$$(\hat{T}_{i,2})_\tau = \int_{\hat{a}_i\hat{a}_{i+1}} \Big(\sum_{\substack{k,l=1\\k\neq l}}^{2} (-1)^k\, (\hat{\tau}\hat{n}^i)_k\, \hat{n}^i_l\Big)\, \mathrm{d}\hat{s}\,,$$

$i = 1, 2, 3$; $i + 1 = 1$ for $i = 3$, *$\hat{b}_i$ denotes the mid-point of the side $\hat{a}_i\hat{a}_{i+1}$.*

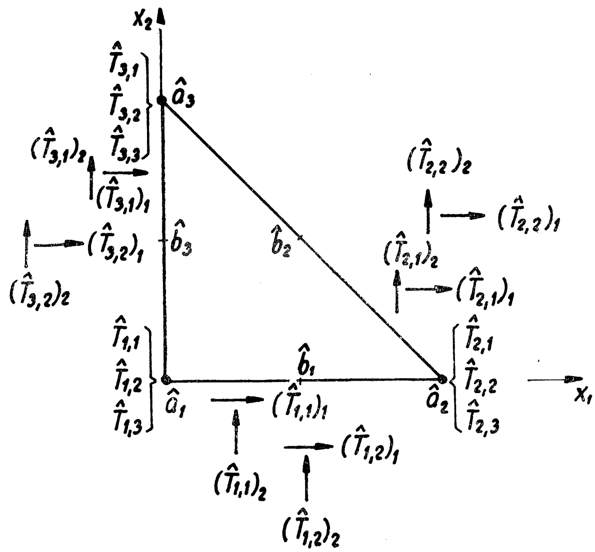

Fig. 9.

If we take into consideration the cartesian's components of the main stress vector, we can define the set Σ_K^* of linear functionals defined on $\mathscr{M}_K$ (see Fig. 9) as follows:

For any side $\hat{a}_i\hat{a}_{i+1}$, $1 \leqq i \leqq 3$, select an outward normal $\hat{n}^i$. Then for $\hat{\tau} \in \mathscr{M}_K$ put

$$(2.46) \qquad \hat{T}_{i,1} = \hat{\tau}_{sm}(\hat{a}_i)\,, \quad 1 \leqq i \leqq 3\,; \quad 1 \leqq j \leqq 3\,; \quad s, m = 1, 2\,;$$

$$j = s = m \quad for \quad j = 1, 2, ; \quad j = 3 \quad for \quad s < m \,,$$

$$(\hat{T}_{i,1})_k = \int_{\hat{a}_i \hat{b}_i} (\hat{\tau}\hat{n}^i)_k \, d\hat{s} \,, \quad 1 \leqq i \leqq 3 \,; \quad k = 1, 2 \,,$$

$$(\hat{T}_{i,2})_k = \int_{\hat{a}_i \hat{a}_{i+1}} (\hat{\tau}\hat{n}^i)_k \, d\hat{s} \,, \quad 1 \leqq i \leqq 3 \,; \quad k = 1, 2 \,,$$

where $i + 1 = 1$ *for* $i = 3$, $\hat{b}_i$ *denotes the mid-point of the side* $\hat{a}_i \hat{a}_{i+1}$.

It is easy to verify the $\mathscr{M}_K$-unisolvability of both sets Σ_K^*, defined in (2.45) and (2.46), in a way analogous to that used in the proof of Theorem 2.3.

3. APPROXIMATION PROPERTIES OF EQUILIBRIUM STRESS ELEMENTS

From now on, let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with a Lipschitz boundary Γ. Let the boundary Γ be divided into mutually disjoint parts Γ^0, Γ_u, Γ_σ such that

(3.1) $$\Gamma^0 \cup \Gamma_u \cup \Gamma_\sigma = \Gamma \,,$$

where Γ^0 is the union of a finite number of points and Γ_u and Γ_σ are open in Γ. Let $\mathscr{T}_h$ be a decomposition of $\bar{\Omega}$ into *convex polygons* K_i, i.e., we write $\bar{\Omega} = \bigcup\limits_i K_i$. For their mutual position precisely one of the following relations holds:

– $K_i \cap K_j = \emptyset, \quad i \neq j \,,$

– $K_i \cap K_j = K'$, *where* K' *is either a common side or a common vertex of the elements* K_i, K_j.

Definition 3.1. *Let* Γ^0, Γ_u, Γ_σ *of the boundary* Γ *satisfy* (3.1). *Then a decomposition* $\mathscr{T}_h$ *of* $\bar{\Omega}$ *is said to be consistent with* Γ_u *and* Γ_σ *if the interior of any side of any* $K \in \mathscr{T}_h$ *is disjoint with* Γ^0. □

Henceforth, we shall suppose that any decomposition $\mathscr{T}_h$ is consistent with Γ_u and Γ_σ and the family of decompositions $\{\mathscr{T}_h\}$ *is regular* (see [2]).

The approximation properties of the C^1-elements for the Airy stress function U, generated by the elements from Section 2, are characterized in [2], Chap. 6.

Lemma 3.1. *Let the operator* Π_K *of the* $\boldsymbol{P}_K$*-interpolation on an element* K *be such that* $\Pi_K p = p \;\forall p \in P_k \subset \boldsymbol{P}_K$. *Then there exists a constant* c, *independent of the element* K *and such, that for the elements of class* C^1 *from Section* 2 *and a regular family of decomposition* $\{\mathscr{T}_h\}$ *the following inequality holds*:

(3.2) $$\|U - \Pi_K U\|_{m,K} \leqq c h^{k+1-m} |U|_{k+1,K} \quad \forall U \in H^{k+1}(K)$$

and for all $0 \leqq m \leqq k + 1$ *for which* $\boldsymbol{P}_K \subset H^m(K)$ *holds*; $k = 3$ *for* H–C–T *and Ahlin's elements and* $k = 5$ *for Fellipa's element.* □

Let us denote

$$(3.3) \qquad Q(K) = \{\tau \in H(\mathrm{Div}; K) \mid \mathrm{Div}\, \tau = 0\}.$$

Then for any $\tau \in Q(K) \cap (H^{k-1}(K))^4$ we can define the matrix function $\tilde{\Pi}_K \tau$, the $\mathscr{M}_k$-interpolation of the function τ, by

$$(3.4) \qquad \tilde{\Pi}_K \tau = \varrho(\Pi_K U),$$

where $\Pi_K U$ is the $\boldsymbol{P}_K$-interpolation of the function $U \in H^{k+1}(K)$ and U corresponds to τ by Theorem 1.3.

Then the approximation properties of the conforming equilibrium stress elements, discussed in Section 2, are characterized by

Theorem 3.1. *Let the operator Π_K of the $\boldsymbol{P}_K$-interpolation be the operator from Lemma 3.1. and let the operator $\tilde{\Pi}_K$ be the operator from (3.4). Then there exists a constant $c > 0$, independent of $K \in \mathscr{T}_h$ and such, that for all elements $C-E-S-I$ to III and a regular family of decomposition $\{\mathscr{T}_h\}$ the following inequality holds;*

$$(3.5) \qquad \|\tau - \tilde{\Pi}_K \tau\|_{0,K} \leqq c h^{k-1} |\tau|_{k-1,K} \quad \forall \tau \in Q(K) \cap (H^{k-1}(K))^4,$$

where $k = 3$ for $C-E-S-I$ and II elements and $k = 5$ for a $C-E-S-III$ element.

Proof. First we note, that by virtue of Lemma 3.1 and Theorems 1.3, 1.4, the interpolation matrix function $\tilde{\Pi}_K \tau$ is fully determined by the inclusion $(H^{k-1}(K))^4 \subset$ $\subset (C(K))^4$, where $k = 3$ for C–E–S–I and II elements and $k = 5$ for a C–E–S–III element. On the other hand, we cannot have the inequality of the type (3.5) for the left hand side for the $\|\cdot\|_{m,\Omega}$-norm, $m \geqq 1$, because we have only the inclusion $\mathscr{M}_K \subset (L^2(K))^4$ for all types of the finite elements mentioned above.

Now, from Theorem 1.3, definition (3.4) and Lemma 3.1, we have

$$\|\tau - \tilde{\Pi}_K \tau\|_{0,K} = \|\varrho U - \varrho(\Pi_K U)\|_{0,K} \leqq \|U - \Pi_K U\|_{2,K} \leqq c h^{k-1} |U|_{k+1,K} =$$
$$= c h^{k-1} |\tau|_{k-1,K}. \qquad \square$$

4. DUAL VARIATIONAL FORMULATION OF THE LINEAR BOUNDARY VALUE PROBLEMS OF ELASTOSTATICS

Making use of the superposition of a particular solution of the equations of total equilibrium and the general solution of the homogeneous equations, we may assume, that the body forces are zero. Let a surface load vector $T \in (L^2(\Gamma_\sigma))^2$ and a displacement vector $u_0 \in (H^1(\Omega))^2$ be given. In the case $\Gamma_u = \emptyset$ (the so called first basic boundary value problem) we always assume that the conditions of total equilibrium

$$(4.1) \qquad \int_\Gamma T_i \, ds = 0, \quad \int_\Gamma (x \times T) \, ds = 0, \quad i = 1, 2,$$

are satisfied.

Let us consider the generalized Hook's law in the form

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}\,, \tag{4.2}$$

where c_{ijkl} are measurable bounded functions in Ω, mostly constant or piecewise constant in practice, and a repeated index means summation over the range 1,2. Assume that

$$c_{ijkl} = c_{jikl} = c_{klij} \tag{4.3}$$

and that the corresponding quadratic form is uniformly positive definite in Ω, i.e. there exists a constant $c_0 > 0$ such, that

$$c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqq c_0\varepsilon_{ij}\varepsilon_{ij} \tag{4.4}$$

is valid for all symmetric tensors ε (see (1.7)) and almost every $x \in \Omega$. It is known that the generalized Hook's law can be inverted, i.e.

$$\varepsilon_{ij} = A_{ijkl}\tau_{kl}\,, \tag{4.5}$$

where the coefficients A_{ijkl} are bounded and measurable in Ω and satisfy the conditions analogous to (4.4).

Let us introduce the bilinear form on $\boldsymbol{H} \times \boldsymbol{H}$ (for $\boldsymbol{H}$ see Section 1)

$$\boldsymbol{a}(\tau', \tau'') = \int_\Omega A_{ijkl}\tau'_{ij}\tau''_{kl}\,\mathrm{d}x\,. \tag{4.6}$$

From the properties of the coefficients of the generalized Hook's law we conclude, that this form is symmetric and uniformly positive definite. The form $\boldsymbol{a}(\tau', \tau'')$ is a scalar product on $\boldsymbol{H}$. Now let us define *the set of statically admissible stress fields*

$$\boldsymbol{Q}_T = \left\{\tau \in H(\mathrm{Div}; \Omega) \mid (\tau, \varepsilon(v))_{0,\Omega} = \int_{\Gamma_\sigma} T_i v_i\,\mathrm{d}s \ \ \forall v \in \boldsymbol{V}\right\}, \tag{4.7}$$

where

$$\boldsymbol{V} = \{v \in (H^1(\Omega))^2 \mid \gamma_0 v \equiv 0 \text{ on } \Gamma_u\}\,, \tag{4.8}$$

It is known, that $\tau \in \boldsymbol{Q}_T$ iff $\mathrm{Div}\,\tau = 0$ in Ω and $\tau v = T$ on Γ_σ.

Definition 4.1. *The dual variational problem of the linear elastostatics consists in finding σ which minimizes the functional of the complementary energy ψ: $\boldsymbol{H} \to \mathbb{R}$ defined by*

$$\psi(\tau) = \tfrac{1}{2}\boldsymbol{a}(\tau, \tau) - \langle\gamma_v\tau, \gamma_0 u_0\rangle_\Gamma\,, \quad \tau \in \boldsymbol{H}$$

over the set $\boldsymbol{Q}_T$. □

It is known [1], that this problem has a unique solution. As in [7], the variational problem from Definition 4.1 can be transformed into the following one:

Given $\bar{\sigma} \in \boldsymbol{Q}_T$ fixed [11], *find σ which minimizes the functional*

$$\Phi(\tau) = \tfrac{1}{2}\boldsymbol{a}(\tau, \tau) + \boldsymbol{a}(\tau, \bar{\sigma}) - \langle \gamma_\nu \tau, \gamma_0 u_0 \rangle_\Gamma \tag{4.9}$$

over the space

$$\boldsymbol{Q}_0 = \{\tau \in H(\mathrm{Div}; \Omega) \mid (\tau, \varepsilon(v))_{0,\Omega} = 0 \ \forall v \in \boldsymbol{V}\}. \tag{4.10}$$

The stress tensor $\sigma + \bar{\sigma}$ is considered to be the solution of the dual problem of linear elastostatics and for any $\bar{\sigma} \in \boldsymbol{Q}_T$ there exists precisely one solution σ.

Then, as in [7], we may replace the minimum problem from Definition 4.1 by an equivalent problem:

$$\text{find} \quad \tau^0 \in \boldsymbol{Q}_0 \quad \text{such that} \tag{4.11}$$

$$\Phi(\tau^0) \leqq \Phi(\tau) \quad \forall \tau \in \boldsymbol{Q}_0 .$$

Now, let $\{\boldsymbol{Q}_h\}$ be a family of finite-dimensional subspaces of $\boldsymbol{Q}_0$. We define the following approximate problem:

$$\text{find} \quad \tau_h^0 \in \boldsymbol{Q}_h \quad \text{such that} \tag{4.12}$$

$$\Phi(\tau_h^0) \leqq \Phi(\tau) \quad \forall \tau \in \boldsymbol{Q}_h .$$

In [7] the following lemma (analogous to Cea's lemma-see [2]) was proved.

Lemma 4.1. *There exists h_0 such, that for any $h \in (0, h_0)$ exists precisely one solution of the problem* (4.12). *Moreover, the following inequality holds;*

$$\|\tau^0 - \tau_h^0\|_{\boldsymbol{H}} \leqq \inf_{\tau \in \boldsymbol{Q}_h} \|\tau^0 - \tau\|_{\boldsymbol{H}} . \tag{4.13}$$

□

5. ORDER OF CONVERGENCE IN THE L^2-NORM OF THE DUAL FINITE ELEMENT ANALYSIS

Let us consider the variational problem (4.11). Let us define the set of admissible functions of the variational problem for the Airy stress function, dual to (4.11), by:

$$\boldsymbol{W} = \{U \in H^2(\Omega) \mid U = \partial_\nu U = 0 \ \text{on} \ \Gamma_\sigma\} . \tag{5.1}$$

Now, the following theorem generalizes Theorem 1.3 to the case, when the stress vector is zero only on part of the boundary of the domain investigated.

Theorem 5.1. *Let Γ_u and Γ_σ be connected. Then*

$$\boldsymbol{Q}_0 = \varrho \boldsymbol{W} . \tag{5.2}$$

For the proof see [17] p. 50. □

In an approximate problem for the Airy stress function the spaces of finite elements, without boundary conditions, denoted by X_h, are defined via the following suppositions;

i) *the corresponding family of decompositions* $\{\mathcal{T}_h\}$ *is regular*;
ii) *the finite elements are of the class* C^1;
iii) $P_K \subset H^2(K)$.

Then i) to iii) imply $X_h \subset C^1(\bar{\Omega}) \cap H^2(\Omega)$ (see [2]) and we define the finite – dimensional subspace of admissible functions for the Airy stress function by

$$W_h = X_h \cap W = \{U_h \in X_h \mid U_h = \partial_\nu U_h = 0 \text{ on } \Gamma_\sigma\} . \tag{5.3}$$

Next, the operator r_h of the X_h-interpolation is defined for $\mathcal{T}_h \in \{\mathcal{T}_h\}$ by

$$(r_h U)|_K = \Pi_K(U|_K) \quad \forall K \in \mathcal{T}_h , \tag{5.4}$$

where $\Pi_K(U|_K)$ is the P_K-interpolation $U|_K$ on K and P_K is the space of polynomial (or piecewise polynomial) functions on K from Section 2. Let

$$\operatorname{dom} r_h = W \cap H^{k+1}(\Omega) \subset C^{k-1}(\bar{\Omega}) , \quad k = 2, 3 , \tag{5.5}$$

be the domain of the operator r_h.

The operator r_h of the X_h-interpolation corresponding to an arbitrary element of the class C^1, mentioned in Section 2, satisfies the implication

$$U \in \operatorname{dom} r_h \Rightarrow r_h U \in W_h . \tag{5.6}$$

Now, due to Theorems 5.1, 1.4 and 1.5, we are justified to choose in the approximate problem (4.12)

$$Q_h = \varrho W_h . \tag{5.7}$$

We say, that *the space* Q_h *is conjugated with the space* W_h *in the sense of slab analogy if* (5.7) *holds*.

Then for any $\tau \in Q_0 \cap (H^{k-1}(\Omega))^4$ we can define the matrix function $\tilde{r}_h \tau$ by

$$(\tilde{r}_h \tau)|_K = \tilde{\Pi}_K(\tau|_K) \quad \forall K \in \mathcal{T}_h , \tag{5.8}$$

where $\tilde{\Pi}_K(\tau|_K)$ is the $\mathcal{M}_K$-interpolation of the function $\tau|_K$ on K, the operator $\tilde{\Pi}_K$ is from (3.4) and $\mathcal{M}_K$ is a finite-dimensional space of functions defined on K (see Section 2 for individual elements). The function $\tilde{r}_h \tau$ will be called *the* Q_h*-interpolation of the function* τ.

Lemma 5.1. *Let the operator* $\tilde{r}_h$ *be given by* (5.8). *Then*

$$\tilde{r}_h \tau \in Q_h . \tag{5.9}$$

Proof. Due to the suppositions (5.5), (5.6), the definitions (5.8), (3.2) and (5.4) we can write for $v \in \boldsymbol{V} \cap (C^\infty(\bar{\Omega}))^2$

$$\int_\Omega (\tilde{r}_h \tau)\, \varepsilon(v)\, \mathrm{d}x = \sum_{K \in \mathcal{T}_h} \int_K \tilde{\Pi}_K(\tau|_K)\, \varepsilon(v)\, \mathrm{d}x = \sum_{K \in \mathcal{T}_h} \int_K \varrho(\Pi_K(U|_K))\, \varepsilon(v)\, \mathrm{d}x =$$

$$= \sum_{K \in \mathcal{T}_h} \int_K \varrho(r_h U)|_K\, \varepsilon(v)\, \mathrm{d}x = \int_\Omega \varrho(r_h U)\, \varepsilon(v)\, \mathrm{d}x = \int_\Gamma (-\partial_2(r_h U)\, \partial_\tau v_1 +$$

$$+ \partial_1(r_h U)\, \partial_\tau v_2)\, \mathrm{d}s = 0$$

for $\partial_\tau v_j = 0$ on Γ_u and $\partial_j(r_h U) = 0$ on Γ_σ, $j = 1, 2$. $\square$

The next theorem yields the order of convergence in the L^2-norm of the dual finite element analysis.

Theorem 5.2. *Let the supposition of Theorem* 5.1 *and the relations* (5.8), (3.5) *be satisfied. Then, if the solution* $\tau^0 \in \boldsymbol{Q}_0$ *of the problem* (4.11) *belongs to* $\boldsymbol{Q}_0 \cap (H^{k-1}(\Omega))^4$, *for* $k = 3$ *for C–E–S–I and II elements and* $k = 5$ *for a C–E–S–III element, there exists a constant* $c > 0$, *independent of h and such that*

$$\|\tau^0 - \tau_h^0\|_{0,\Omega} \leqq c h^{k-1} |\tau^0|_{k-1,\Omega}, \tag{5.10}$$

where $\tau_h^0 \in \boldsymbol{Q}_h$ *is a solution of the approximate problem* (4.12).

Proof. Making use of Lemmas 4.1, 5.1, equivalence of the norms $\|\cdot\|_{\boldsymbol{H}}$ and $\|\cdot\|_{0,\Omega}$, the relations (5.8) and (3.5) for $k = 3$ for C–E–S–I and II elements and $k = 5$ for a C–E–S–III element, we obtain

$$\|\tau^0 - \tau_h^0\|_{0,\Omega} \leqq c' \inf_{\tau \in \boldsymbol{Q}_h} \|\tau^0 - \tau\|_{\boldsymbol{H}} \leqq c \|\tau^0 - \tilde{r}_h \tau^0\|_{0,\Omega} =$$

$$= c\Big(\sum_{K \in \mathcal{T}_h} \|\tau^0 - \tilde{\Pi}_K \tau^0\|_{0,K}^2\Big)^{1/2} \leqq c h^{k-1} \Big(\sum_{K \in \mathcal{T}_h} |\tau^0|_{k-1,K}^2\Big)^{1/2} = c h^{k-1} |\tau^0|_{k-1,\Omega}. \quad \square$$

The following result concerns all the types of finite elements taken into account:

Theorem 5.5. (on the convergence). *Let the supposition of Theorem* 1.3, *the relations* (5.5), (5.6), (5.7) *and the suppositions* i) *to iii) of this section be satisfied. Let the set* $\mathcal{V} = \boldsymbol{W} \cap C^\infty(\bar{\Omega})$ *be dense in* $\boldsymbol{W}$ *and let* Γ_σ *be connected.*

Then

$$\lim_{h \to 0+} \|\tau^0 - \tau_h^0\|_{0,\Omega} = 0\,. \tag{5.11}$$

Proof. From Lemma 4.2, equivalence of the norms on $\boldsymbol{H}$ and the suppositions of this theorem we have

$$\|\tau^0 - \tau_h^0\|_{0,\Omega} \leqq c \inf_{\tau_h \in \boldsymbol{Q}_h} \|\tau^0 - \tau_h\|_{0,\Omega} = c \inf_{U_h \in \boldsymbol{W}_h} \|\varrho U - \varrho U_h\|_{0,\Omega} = c \inf_{U_h \in \boldsymbol{W}_h} |U - U_h|_{2,\Omega}\,.$$

The density of $\mathscr{V}$ in W implies that there exists $\tilde{U} \in W$ such that $\|U - \tilde{U}\|_{2,\Omega} \leqq \frac{1}{2}\varepsilon$. Due to $\mathscr{V} \subset \operatorname{dom} r_h$ it follows from Lemma 3.1 (for $m = 2$) that there exists a constant $c > 0$ independent of h and such that $\|\tilde{U} - r_h\tilde{U}\|_{2,\Omega} \leqq ch^{k-1}|\tilde{U}|_{k+1,\Omega}$, then $\forall\varepsilon > 0\ \exists h_0(\varepsilon)$: $\|U - r_h\tilde{U}\|_{2,\Omega} \leqq \frac{1}{2}\varepsilon\ \forall h \leqq h_0(\varepsilon)$. Due to the supposition (5.6) we have $r_h\tilde{U} \in W_h$ and thus $\|U - r_h\tilde{U}\|_{2,\Omega} \leqq \|U - \tilde{U}\|_{2,\Omega} + \|\tilde{U} - r_h\tilde{U}\|_{2,\Omega}$ for sufficiently small h. Then, finally, we have

$$\|\tau^0 - \tau_h^0\|_{0,\Omega} \leqq c\|U - r_h\tilde{U}\|_{2,\Omega} \to 0 \quad \text{for} \quad h \to 0_+ . \qquad \square$$

Acknowledgments. The author is indebted to Ing. I. Hlaváček, CSc., for valuable advice concerning this article.

References

[1] *J. Nečas, I. Hlaváček:* Mathematical theory of elastic and elasto-plastic bodies: An introduction. Elsevier Sci. Publ. Comp., Amsterdam, Oxford, New York, 1981.

[2] *P. G. Ciarlet:* The finite element method for elliptic problems. North-Holland Publ. Comp., Amsterdam, New York, Oxford, 1978.

[3] *B. M. Fraeijs de Veubeke:* A course in elasticity. Springer Verlag, New York, Heidelberg, Berlin, 1979.

[4] *V. Girault, P. A. Raviart:* Finite element approximation of the Navier-Stokes equations. Springer-Verlag, Berlín, Heidelberg, New York, 1979.

[5] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague, 1967.

[6] *I. Hlaváček:* Variational principles in the linear theory of elasticity for general boundary conditions. Apl. Mat. 12 (1967), 425—448.

[7] *I. Hlaváček:* Convergence of an equilibrium finite element model for plane elastostatics. Apl. Mat. 24 (1979), 427—457.

[8] *I. Hlaváček:* Some equilibrium and mixed model in the finite element method. Banach Center Publ., Vol. 3, 1975, 147—165.

[9] *J. Haslinger, I. Hlaváček:* Convergence of a finite element method based on the dual variational formulations. Apl. Mat. 21 (1976), 43—65.

[10] *J. Haslinger, I. Hlaváček:* Contact between elastic bodies — III. Dual finite Element Analysis. Apl. Mat. 26 (1981), 321—344.

[11] *M. Křížek:* Equilibrium elements for the linear elasticity problem. Variational — difference methods in math. phys., Moscow, 1984, 81—92.

[12] *G. Sander:* Applications of the dual analysis principle. Proceedings of the IUTAM Symp. on High Speed Computing of Elastic Structures, Congrés et Colloques Do l'Université de Liege (1971), 167—207.

[13] *B. F. Veubeke, G. Sander:* An equilibrium model for plate bending. Internat. J. Solids and Structures 4 (1968), 447—468.

[14] *V. B. Watwood, B. J. Hartz:* An equilibrium stress field model for the finite element solutions of two-dimensional elastostatic problems. Internat. J. Solides and Structures 4 (1968), 857—873.

[15] *C. Johnson, B. Mercier:* Some equilibrium finite element methods for two-dimensional elasticity problems. Numer. Math. 30 (1978), 103—116.

[16] *M. Křížek:* An equilibrium finite element method in three-dimensional elasticity. Apl. Mat. 27 (1982), 46—75.

[17] *M. Křížek:* Conforming equilibrium finite element methods for some elliptic plane problems, R.A.I.R.O. Analyse numérique, vol. 17, n^0 1, 1983, 35–65.
[18] *D. J. Allman:* On compatible and equlibrium models with linear stresses for stretching of elastic plates. Energy methods in finite element analysis. John Wiley & Sons Ltd. Chichester, New York, Brisbane, Toronto, 1979.
[19] *B. F. de Veubeke, O. C. Zienkiwicz:* Strain energy bounds in finite element analysis by slab analogy. J. of Strain Analysis, 1967, vol. 2, No. 4.
[20] *I. Babuška, K. Rektorys, F. Vyčichlo:* Mathematische Elastizitätstheorie der ebenen Probleme, Akademieverlag, Berlin 1960.
[21] *J. H. Michell:* On the direct determination of stress in an elastic solid with application to the theory of plates. Proc. Lond. math. Soc. 1899, 31, 100.

Souhrn

DESKOVÁ ANALOGIE V TEORII A PRAXI KONFORMNÍCH ROVNOVÁŽNÝCH MODELŮ POLÍ NAPĚTÍ PRO ŘEŠENÍ ROVINNÉ PRUŽNOSTI METODOU KONEČNÝCH PRVKŮ

Miroslav Vondrák

Pro jednotlivé trojúhelníkové resp. obdélníkové prvky lze uplatnit metodu Airyho funkce napětí, odtud vyplývá, že apriorní odhady známé z teorie kompatibilních prvků pro rovnice čtvrtého řádu zároveň poskytují odhady v L^2-normě pro aproximace pole napětí. Je odůvodněn pojem „desková analogie" a podán rozbor souvislosti Airyho funkce – pole napětí ve vztahu k přechodovým podmínkám na styku dvou prvků. Jsou navrženy unisolventní množiny stupňů volnosti pro prvky sdružené ve smyslu deskové analogie s některými kompatibilními prvky a odvozeny aproximační vlastnosti těchto rovnovážných prvků.

Pro všechny tyto typy prvků je dokázána konvergence v L^2-normě pro dostatečně hladká řešení.

Author's address: Ing. *Miroslav Vondrák*, CSc., Vývojová konstrukce letadel n. p. Rudý Letov, Beranových 65, 199 02 Praha 9-Letňany.