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ON PERIODIC AUTOREGRESSION WITH UNKNOWN MEAN

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The periodic autoregression is a model for seasonal time series. It is assumed that the autoregressive parameters are periodic functions with the period corresponding to the seasonal behaviour of the given series. The mean value can also be a periodic function. Bayes approach is used in the paper for estimating parameters and for testing hypotheses. Two models are investigated, one with constant variances of the innovation process and the other with periodically changing variances.

1. INTRODUCTION

Let $\{Y_t\}$ be an innovation process with vanishing expectation and with $\operatorname{Var} Y_t = \sigma^2 > 0$. A process $\{X_t\}$ is called autoregressive, if it is generated by the relation

$$X_t = b_1 X_{t-1} + \ldots + b_n X_{t-n} + Y_t\,,$$

where $b = (b_1, \ldots, b_n)'$ is a vector of autoregressive parameters.

The process $\{X_t\}$ is stationary, if

$$z^n - b_1 z^{n-1} - \ldots - b_n \neq 0 \quad \text{for} \quad |z| \geqq 1\,.$$

If we analyze a seasonal time series having a period p, it is quite natural to assume that the elements of the autoregressive vector b are also periodic functions with the same period p. This can be formulated more precisely as follows.

Let $X_1, \ldots, X_n$ be given variables. Consider vectors $b_1 = (b_{11}, \ldots, b_{1n})', \ldots, b_p = (b_{p1}, \ldots, b_{pn})'$. Let X_t for $t > n$ be defined by the formula

$$X_{n+(j-1)p+k} = \sum_{i=1}^{n} b_{ki} X_{n+(j-1)p+k-i} + Y_{n+(j-1)p+k}\,, \tag{1.1}$$

where $k = 1, \ldots, p$ and $j = 1, 2, \ldots$. Denote $b = (b'_1, \ldots, b'_p)'$.

There are two important cases of model (1.1). If $\operatorname{Var} Y_t = \sigma^2$ does not depend on t, we have the model with constant variances. If $\operatorname{Var} Y_{n+(j-1)p+k} = \sigma_k^2$, $k = 1, \ldots,$

$\ldots, p$, where $\sigma_1^2, \ldots, \sigma_p^2$ do not coincide, we come to the model with periodic variances. In the latter case we denote $\sigma = (\sigma_1, \ldots, \sigma_p)'$.

The analysis of such periodic models was started by Gladyshev [3] and [4]. Pagano [5] investigated properties of estimators of parameters in process (1.1). A detailed statistical analysis of model (1.1) is given by Anděl [1], where also the related references concerning the problem of periodic autoregressive processes can be found.

In the present paper we generalize model (1.1) to the case where X_t have non-vanishing expectations. Assume that

$$\mathsf{E}X_{n+(j-1)p+k} = \xi_k \quad \text{for} \quad k = 1, \ldots, p$$

and that model (1.1) can be used for differences from the means. Then we have

$$X_{n+(j-1)p+k} - \xi_k = \sum_{i=1}^{n} b_{ki}(X_{n+(j-1)p+k-i} - \xi_{k-i}) + Y_{n+(j-1)p+k}\,.$$

Of course, in this formula we put $\xi_{k-i} = \xi_{p+k-i}$, if $k - i \leqq 0$.

Rearranging the terms, we come to our model

$$X_{n+(j-1)p+k} = \mu_k + \sum_{i=1}^{n} b_{ki}X_{n+(j-1)p+k-i} + Y_{n+(j-1)p+k}\,, \tag{1.2}$$

where

$$\mu_k = \xi_k - \sum_{i=1}^{n} b_{ki}\xi_{n+(j-1)p+k-i}\,.$$

We denote

$$\mu = (\mu_1, \ldots, \mu_p)'\,.$$

In this paper we shall assume that b, μ and σ are random vectors with a vague prior density. This approach has been successfully used by many authors, e.g. by Zellner [6] and by Anděl [1]. An interesting argumentation for this procedure can be found also in Box and Tiao [2]. The authors point out that under very general conditions this Bayes approach leads to results which are asymptotically the same as those obtained by the maximum-likelihood method. It should be emphasized, however, that Bayes approach is substantially easier in our case than the direct asymptotics of maximum-likelihood estimators.

For analyzing model (1.2) we use methods quite analogous to those which were applied in [1]. The presence of new parameters μ leads to certain complications and, therefore, these new results seem to be worth publishing separately.

2. PRELIMINARIES

To keep this paper self-contained, we recall here some general assertions which will be used in the statistical analysis of model (1.2). At the beginning we would like to stress that the symbol c will denote a constant. It will be used in this sense throughout all the paper. We point out explicitly that c in any two formulas need not be the same constant.

Theorem 2.1. *Let A be an $n \times n$ symmetric regular matrix. Then for every n-dimensional vectors x and q the formula*

$$x'Ax - 2x'q = (x - A^{-1}q)' A(x - A^{-1}q) - q'A^{-1}q$$

holds.

Proof is clear. □

Theorem 2.2. *Let $m \geqq 2$. Then*

$$\int_{-\infty}^{\infty} (1 + a^2 + x^2)^{-m/2} \, dx = c(1 + a^2)^{-(m-1)/2} ,$$

where c does not depend on a.

Proof.

$$\int_{-\infty}^{\infty} (1 + a^2 + x^2)^{-m/2} \, dx = (1 + a^2)^{-m/2} \int_{-\infty}^{\infty} \left(1 + \frac{x^2}{1 + a^2}\right)^{-m/2} dx .$$

After the substitution $x = (1 + a^2)^{1/2} \, t$ we obtain the desired result. □

Theorem 2.3. *Let $Q_1, \ldots, Q_p$ be $n \times n$ symmetric positive definite matrices. Let $Q = Q_1 + \ldots + Q_p$.* If $p \geqq 2$, *then the matrix*

$$H = \begin{Vmatrix} Q_1, & 0, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & Q_{p-1} \end{Vmatrix} - \begin{Vmatrix} Q_1 Q^{-1} Q_1, & \ldots, & Q_1 Q^{-1} Q_{p-1} \\ \ldots & \ldots & \ldots \\ Q_{p-1} Q^{-1} Q_1, & \ldots, & Q_{p-1} Q^{-1} Q_{p-1} \end{Vmatrix}$$

is positive definite.

Proof. See [1], p. 366. □

Theorem 2.4. *Let V be an $n \times n$ symmetric positive definite matrix and let a random vector $X = (X_1, \ldots, X_n)'$ have the density*

(2.1) $$q(x) = c(1 - x'Vx)^{-m/2} ,$$

where $m \geqq n + 1$. Introduce a random vector

$$Z = (Z_1, \ldots, Z_s)' = (X_{i_1}, \ldots, X_{i_s})' ,$$

where $1 \leqq i_1 < i_2 < \ldots < i_s \leqq n$, $1 \leqq s < n$. Let W be the matrix arising from the rows $i_1, \ldots, i_s$ and from the columns $i_1, \ldots, i_s$ of the matrix V^{-1}. Then the marginal density of the vector Z is

$$q_1(z) = c(1 + z' W^{-1} z)^{-(m-n+s)/2} .$$

Proof. See [1], p. 367. □

Theorem 2.5. *Let a vector* $X = (X_1, \ldots, X_n)'$ *have the density* (2.1). *Then the random variable*

$$F = \frac{m-n}{n} X'VX$$

has the $F_{n,m-n}$ *distribution.*

Proof. See [1], p. 368. □

3. MODEL WITH EQUAL VARIANCES

We shall assume in this section that Y_t are independent $N(0, \sigma^2)$ variables with $\sigma^2 > 0$. Our analysis will be based on random variables $X_1, \ldots, X_N$, where $N > n + p + np + 1$. As noticed above, b and μ are random vectors and σ is a random variable. At the beginning, we introduce some notations which will be used till the end of this paper. Let

$$\alpha_k = \left[\frac{N-n-k}{p}\right] + 1\,, \quad k = 1, \ldots, p\,,$$

where [ ] denotes the integer part. Put

$$x_t^0 = (x_{t-1}, \ldots, x_{t-n})'\,, \quad t = n+1, \ldots, N\,,$$

$$\bar{x}_k = \alpha_k^{-1} \sum_{j=1}^{\alpha_k} x_{n+(j-1)p+k}\,, \quad \bar{x}_k^0 = \alpha_k^{-1} \sum_{j=1}^{\alpha_k} x^0_{n+(j-1)p+k}\,,$$

$$\Delta_{kj} = x_{n+(j-1)p+k} - \bar{x}_k\,, \quad \Delta_{kj}^0 = x^0_{n+(j-1)p+k} - \bar{x}_k^0\,,$$

$$T_k = \sum_{j=1}^{\alpha_k} \Delta_{kj}^2\,, \quad C_k = \sum_{j=1}^{\alpha_k} \Delta_{kj}\Delta_{kj}^0\,, \quad S_k = \sum_{j=1}^{\alpha_k} \Delta_{kj}^0 \Delta_{kj}^{0\prime}\,,$$

$$b_k^* = S_k^{-1} C_k\,, \quad R_k = T_k - b_k^{*\prime} S_k b_k^*\,,$$

$$T = T_1 + \ldots + T_p\,, \quad R = R_1 + \ldots + R_p\,, \quad S = S_1 + \ldots + S_p\,,$$

$$v_k = \mu_k - \bar{x}_k + b_k' \bar{x}_k^0\,, \quad \mu_k^* = \bar{x}_k - b_k^{*\prime} \bar{x}_k^0\,,$$

$$q_k = \alpha_k[1 - \alpha_k \bar{x}_k^{0\prime}(S_k + \alpha_k \bar{x}_k^0 \bar{x}_k^{0\prime})^{-1} \bar{x}_k^0]\,, \quad q = q_1 + \ldots + q_p\,,$$

$$b^* = (b_1^{*\prime}, \ldots, b_p^{*\prime})'\,, \quad v = (v_1, \ldots, v_p)'\,,$$

$$\mu = (\mu_1, \ldots, \mu_p)'\,, \quad \mu^* = (\mu_1^*, \ldots, \mu_p^*)'\,,$$

$$W_k = b_k - b_k^*\,, \quad v_k = \mu_k - \bar{x}_k + b_k^{*\prime} \bar{x}_k^0 = \mu_k - \mu_k^*\,,$$

$$\tilde{b}_k = b_k^* - \alpha_k v_k (S_k + \alpha_k \bar{x}_k^0 \bar{x}_k^{0\prime})^{-1} \bar{x}_k^0\,, \quad \text{for} \quad k = 1, \ldots, p\,.$$

It is clear that S_k are symmetric positive definite matrices with probability one.

Theorem 3.1. *Given $X_1 = x_1, \ldots, X_n = x_n$, b, μ and σ, the conditional density of $X_{n+1}, \ldots, X_N$ is given by the formula*

$$f(x_{n+1}, \ldots, x_N \mid x_1, \ldots, x_n, b, \mu, \sigma) = (2\pi)^{-(N-n)/2} \sigma^{-N+n} \times$$
$$\times \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} \left[R_k + \alpha_k v_k^2 + (b_k - b_k^*)' S_k (b_k - b_k^*)\right]\right\}.$$

Proof. Our assumptions on Y_t immediately yield that the conditional density is

$$(2\pi)^{-(N-n)/2} \sigma^{-N+n} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} \sum_{j=1}^{\alpha_k} z_{kj}^2\right\},$$

where

$$z_{kj} = x_{n+(j-1)p+k} - \mu_k - \sum_{i=1}^{n} b_{ki} x_{n+(j-1)p+k-i}.$$

However, since

$$\sum_{i=1}^{n} b_{ki} x_{n+(j-1)p+k-i} = b_k' x_{n+(j-1)p+k}^0,$$

we have

$$z_{kj} = \Delta_{kj} - v_k - b_k' \Delta_{kj}^0.$$

Because

$$\sum_{j=1}^{\alpha_k} \Delta_{kj} = 0, \quad \sum_{j=1}^{\alpha_k} \Delta_{kj}^0 = 0,$$

we obtain

$$\sum_{j=1}^{\alpha_k} z_{kj}^2 = T_k - b_k' C_k - C_k' b_k + b_k' S_k b_k + \alpha_k v_k^2.$$

Finally, it follows from Theorem 2.1 that

$$\sum_{j=1}^{\alpha_k} z_{kj}^2 = R_k + \alpha_k v_k^2 + (b_k - b_k^*)' S_k (b_k - b_k^*).$$ □

It should be noticed for further purposes that v_k depend on μ_k and b_k.

Theorem 3.2. *Let the prior density of b, μ and σ be σ^{-1} for $\sigma > 0$ and zero otherwise, independently of $X_1, \ldots, X_n$. Then the posterior density of b, μ and σ is*

$$g(b, \mu, \sigma \mid x) = c\,\sigma^{-N+n-1} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} \left[R_k + \alpha_k v_k^2 + (b_k - b_k^*)' S_k (b_k - b_k^*)\right]\right\}$$

for $\sigma > 0$ and zero otherwise, where $x = (x_1, \ldots, x_N)'$ stands for $X_1 = x_1, \ldots, X_N = x_N$.

Proof follows from the Bayes theorem. □

Theorem 3.3. *The modus of the posterior density is*

$$b = b^*\,, \quad \mu = \mu^*\,, \quad \sigma^2 = \sigma^{*2} = (N - n + 1)^{-1} R\,.$$

Proof. Since S_k are positive definite, $g(b, \mu, \sigma \mid x)$ for any fixed σ reaches its maximum for $b_k = b_k^*$ and $v_k = 0$, i.e. $\mu_k = \mu_k^*$. Therefore,

$$g(b, \mu, \sigma \mid x) \leqq g(b^*, \mu^*, \sigma \mid x) = c\sigma^{-N+n-1} \exp\left\{-\frac{R}{2\sigma^2}\right\} = g_0(\sigma)\,.$$

The function $g_0(\sigma)$ is maximized when $\sigma^2 = (N - n + 1)^{-1} R$. □

The modus can be used as a point estimator of the parameters b, μ and σ.

Theorem 3.4. *The marginal posterior densities of σ, b and μ are given by the formulas*

(i) $$g_1(\sigma \mid x) = c\sigma^{-N+n+p+np-1} \exp\left\{-\frac{R}{2\sigma^2}\right\}, \quad \sigma > 0\,,$$

(ii) $$g_2(b \mid x) = c\left[1 + R^{-1} \sum_{k=1}^{p} (b_k - b_k^*)' S_k(b_k - b_k^*)\right]^{-(N-n-p)/2},$$

(iii) $$g_3(\mu \mid x) = c\left[1 + R^{-1} \sum_{k=1}^{p} q_k(\mu_k - \mu_k^*)^2\right]^{-(N-n-np)/2}.$$

Proof. The simultaneous posterior density of b and σ is

$$h_1(b, \sigma \mid x) = \int_{\mathbf{R}_p} g(b, \mu, \sigma \mid x)\, \mathrm{d}\mu =$$

$$= c\sigma^{-N+n-1} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} [R_k + (b_k - b_k^*)' S_k(b_k - b_k^*)\right\} . J\,,$$

where

$$J = \int_{\mathbf{R}_p} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} \alpha_k(\mu_k - \bar{x}_k + b_k'\bar{x}_k^0)^2\right\} \mathrm{d}\mu\,.$$

Using the substitutions

$$\frac{1}{\sigma}(\mu_k - \bar{x}_k + b_k'\bar{x}_k^0) = u_k\,, \quad k = 1, \ldots, p\,,$$

we get

$$J = \sigma^p \int_{\mathbf{R}_p} \exp\left\{-\tfrac{1}{2} \sum_{k=1}^{p} \alpha_k u_k^2\right\} \mathrm{d}u = c\sigma^p\,,$$

which gives

$$h_1(b, \sigma \mid x) = c\sigma^{-N-n+p-1} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} [R_k + (b_k - b_k^*)' S_k(b_k - b_k^*)]\right\}.$$

Further,

$$g_1(\sigma \mid x) = \int_{\mathbf{R}_{np}} h_1(b, \sigma \mid x)\, \mathrm{d}b = c^{-N+n+p-1} \exp\left\{-\frac{R}{2\sigma^2}\right\} . J_1 ,$$

where

$$J_1 = \int_{\mathbf{R}_{np}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{p} (b_k - b_k^*)' S_k(b_k - b_k^*)\right\} \mathrm{d}b .$$

We make the substitution

$$t_k = \sigma^{-1} S_k^{1/2}(b_k - b_k^*) , \quad k = 1, \ldots, p ,$$

the Jacobian of which is

$$\prod_{k=1}^{p} |\sigma^{-1} S_k^{1/2}|^{-1} = \sigma^{np} \prod_{k=1}^{p} |S_k|^{-1/2} = c\sigma^{np} .$$

Therefore,

$$J_1 = c\sigma^{np} \int_{\mathbf{R}_{np}} \exp\left\{-\tfrac{1}{2} \sum_{k=1}^{p} t_k' t_k\right\} \mathrm{d}t = c\sigma^{np}$$

and

$$g_1(\sigma \mid x) = c\sigma^{-N+n+p+np-1} \exp\left\{-\frac{R}{2\sigma^2}\right\} .$$

Further we have

$$g_2(b \mid x) = \int_0^\infty h_1(b, \sigma \mid x)\, \mathrm{d}\sigma .$$

Denote

$$A = \sum_{k=1}^{p} [R_k + (b_k - b_k^*)' S_k(b_k - b_k^*)]$$

for this part of the proof. From the substitution

$$A^{1/2} \sigma^{-1} = z$$

we get

$$g_2(b \mid x) = c[R + \sum_{k=1}^{p} (b_k - b_k^*)' S_k(b_k - b_k^*)]^{-(N-n-p)/2} .$$

Since R does not depend on b, it can be taken away as a constant. This gives the second assertion of the theorem.

For the last part of the proof we put

$$B = \sum_{k=1}^{p} [R_k + \alpha_k v_k^2 + (b_k - b_k^*)' S_k(b_k - b_k^*)] .$$

Then

$$g(b, \mu, \sigma \mid x) = c\sigma^{-N+n-1} \exp\left\{-\frac{B}{2\sigma^2}\right\}$$

and

$$h_2(b, \mu \mid x) = \int_0^\infty g(b, \mu, \sigma \mid x)\, \mathrm{d}\sigma = cB^{-(N-n)/2} =$$

$$= c\left\{\sum_{k=1}^{p} [R_k + \alpha_k(\mu_k - \bar{x}_k + b_k'\bar{x}_k^0)^2 + (b_k - b_k^*)'\, S_k(b_k - b_k^*)]\right\}^{-(N-n)/2}.$$

Since

$$\alpha_k(\mu_k - \bar{x}_k + b_k'\bar{x}_k^0)^2 + (b_k - b_k^*)'\, S_k(b_k - b_k^*) =$$

$$= \alpha_k[\mu_k - \bar{x}_k + (b_k - b_k^*)'\, \bar{x}_k^0 + b_k^{*\prime}\bar{x}_k^0]^2 + (b_k - b_k^*)'\, S_k(b_k - b_k^*) =$$

$$= \alpha_k(v_k + W_k'\bar{x}_k^0)^2 + W_k'S_kW_k =$$

$$= W_k'(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})\, W_k + 2\alpha_k v_k W_k'\bar{x}_k^0 + \alpha_k v_k^2 =$$

$$= [W_k + \alpha_k v_k(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{-1}\, \bar{x}_k^0]'\, (S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})\, [W_k + \alpha_k v_k(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{-1}\, \bar{x}_k^0] +$$

$$+ \alpha_k v_k^2 - \alpha_k^2 v_k^2 \bar{x}_k^{0\prime}(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{-1}\, \bar{x}_k^0 =$$

$$= (b_k - \tilde{b}_k)'\, (S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})\, (b_k - \tilde{b}_n) + q_k v_k^2\,,$$

we have

$$h_2(b, \mu \mid x) =$$

$$= c\left[1 + R^{-1}\sum_{k=1}^{p}(b_k - \tilde{b}_k)'\, (S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})\, (b_k - \tilde{b}_k) + R^{-1}\sum_{k=1}^{p} q_k v_k^2\right]^{-(N-n)/2}.$$

The integral

$$g_3(\mu \mid x) = \int_{\mathbf{R}_{np}} h_2(b, \mu \mid x)\, \mathrm{d}b$$

can be calculated by using the substitution

$$u_k = R^{-1/2}(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{1/2}\, (b_k - \tilde{b}_k)\,, \quad k = 1, \ldots, p\,,$$

the Jacobian of which is a constant. Thus

$$g_3(\mu \mid x) = c\int_{\mathbf{R}_{np}} (1 + R^{-1}\sum_{k=1}^{p} q_k v_k^2 + \sum_{k=1}^{p} u_k'u_k)^{-(N-n)/2}\, \mathrm{d}u\,.$$

Now, we use Theorem 2.2 p-times and this leads to the formula for $g_3(\mu \mid x)$ which is given as the last assertion of Theorem 3.4. □

Theorem 3.5. *Let*

$$\lambda_b = \frac{N - n - np - p}{npR}\sum_{k=1}^{p}(b_k - b_k^*)'\, S_k(b_k - b_k^*)\,,$$

$$\lambda_\mu = \frac{N - n - np - p}{pR} \sum_{k=1}^{p} q_k(\mu_k - \mu_k^*)^2 .$$

Then the posterior distributions of λ_b and λ_μ are

$$\lambda_b \sim F_{np,N-n-np-p}, \quad \lambda_\mu \sim F_{p,N-n-np-p} .$$

Proof. The assertion follows from Theorem 3.4 (i), (ii) and from Theorem 2.5. □

The variables λ_b and λ_μ can be used for tests of fit. If a hypothesis specifies some special value of b or of μ, we insert it in the formulas for λ_b or λ_μ. If the result exceeds the critical value of the corresponding F distribution, we reject the hypothesis.

Theorem 3.6. *The posterior distribution of the variable R/σ^2 is $\chi^2_{N-n-p-np}$.*

Proof follows from Theorem 3.4 (iii) by direct calculation. □

This result can be used for constructing confidence intervals for σ^2.

Theorem 3.7. *Denote by $s^{(k)ij}$ the elements of the matrix S_k^{-1}. Then the posterior distribution of the variable*

$$T_{ki} = (N - n - p - np)^{1/2} \left(Rs^{(k)ii}\right)^{-1/2} \left(b_{ki} - b_{ki}^*\right)$$

is the Student $t_{N-n-np-p}$ distribution.

Proof. The assertion follows from Theorem 3.4 (ii) and from Theorem 2.4. □

The most important question is whether our model can be reduced to the classical autoregressive model or not. A test of such hypothesis can be based on the following two theorems.

Theorem 3.8. *Denote*

$$H = \operatorname{Diag}\{S_1, \ldots, S_{p-1}\} - (S_1, \ldots, S_{p-1})' \, S^{-1}(S_1, \ldots, S_{p-1}),$$
$$\Delta_k = (b_k - b_k^*) - (b_p - b_p^*), \quad \Delta = (\Delta_1', \ldots, \Delta_{p-1}')' .$$

Then the variable

$$F_\Delta = \frac{N - p - np - n}{n(p-1)\,R} \Delta' H \Delta .$$

has the posterior $F_{n(p-1),N-p-np-n}$ distribution.

Proof. First of all we notice that the matrix H is constructed from $S_1, \ldots, S_p$ in the same way as in Theorem 2.3. This ensures that H is positive definite. The proof of Theorem 3.8 is the same as that of Theorem 3.5 in [1] (only N must be replaced by $N - p$), and thus we sketch it very briefly. We put $\Delta_p = b_p - b_p^*$ and calculate the posterior density of $\Delta_1, \ldots, \Delta_{p-1}, \Delta_p$. After that we find the marginal density of $\Delta_1, \ldots, \Delta_{p-1}$, which have the same form as the density $q(x)$ in Theorem 2.4. This enables us to apply Theorem 2.5. □

Theorem 3.9. *Let*

$$K = \mathrm{Diag}\{q_1, \ldots, q_{p-1}\} - q^{-1}(q_1, \ldots, q_{p-1})'(q_1, \ldots, q_{p-1}),$$

$$\delta_k = (\mu_k - \mu_k^*) - (\mu_p - \mu_p^*) \quad \text{for} \quad k = 1, \ldots, p-1, \quad \delta = (\delta_1, \ldots, \delta_{p-1})'.$$

Then the variable

$$F_\delta = \frac{N - n - p - np}{(p-1)R} \delta' K \delta$$

has the posterior $F_{p-1, N-n-p-np}$ *distribution.*

Proof is analogous to that of Theorem 3.8. □

If the hypothesis H_0: $b_1 = \ldots = b_p$ is true, then $\Delta_k = b_p^* - b_k^*$. We calculate F_Δ with $\Delta = (\Delta'_1, \ldots, \Delta'_{p-1})'$ and in the case that F_Δ exceeds the critical value of the corresponding F distribution, we reject H_0. Similar procedure can be used for testing H'_0: $\mu_1 = \ldots \mu_p$. In this case we have $\delta_k = \mu_p^* - \mu_k^*$ and H'_0 is rejected if F_δ exceeds its critical value.

4. MODEL WITH PERIODIC VARIANCES

In this model we assume that Y_t are independent variables such that $Y_{n+(j-1)p+k} \sim N(0, \sigma_k^2)$. We shall keep the notations introduced at the beginning of Section 3.

Theorem 4.1. *Let the prior density of* b, μ *and* σ *be* $\sigma_1^{-1} \ldots \sigma_p^{-1}$ *for* $\sigma_1 > 0, \ldots$ $\ldots, \sigma_p > 0$ *and zero otherwise, independently* of $X_1, \ldots, X_n$. *Then the posterior density of* b, μ *and* σ *is*

$$g(b, \mu, \sigma \mid x) = c \prod_{k=1}^{p} \sigma_k^{-\alpha_k - 1} \exp\left\{-\frac{1}{2\sigma_k^2}\left[R_k + \alpha_k v_k^2 + (b_k - b_k^*)' S_k (b_k - b_k^*)\right]\right\}.$$

Proof is analogous to that of Theorem 3.2. □

Theorem 4.2. *The modus of the posterior density is*

$$b = b^*, \quad \mu = \mu^*, \quad \sigma_k^2 = \sigma_k^{*2} = \frac{R_k}{\alpha_k + 1} \quad \textit{for} \quad k = 1, \ldots, p.$$

Proof is analogous to that of Theorem 3.3. □

If we use the modus as an estimator of the parameters, we can see that the estimators for b and μ are the same in the model with equal variances as in the model with periodic variances. We get different estimators only for σ_k^2.

Theorem 4.3. *The marginal posterior densities of* σ, b and μ *are*:

(i) $$g_1(\sigma \mid x) = c \prod_{k=1}^{p} \sigma_k^{-\alpha_k + n} \exp\left\{-\frac{R_k}{2\sigma_k^2}\right\},$$

(ii) $$g_2(b \mid x) = c \prod_{k=1}^{p} [1 + R_k^{-1}(b_k - b_k^*)' S_k(b_k - b_k^*)]^{-(\alpha_k - 1)/2},$$

(iii) $$g_3(\mu \mid x) = c \prod_{k=1}^{p} [1 + q_k R_k^{-1}(\mu_k - \mu_k^*)^2]^{-(\alpha_k - n)/2}.$$

Proof is similar to that of Theorem 3.4 and thus we introduce only its main points. First of all we calculate

$$h_1(b, \sigma|x) = \int_{\mathbf{R}_p} g(b, \mu, \sigma \mid x) \, d\mu =$$

$$= c \prod_{k=1}^{p} \sigma_k^{-\alpha_k} \exp\left\{-\frac{1}{2\sigma_k^2}[R_k + (b_k - b_k^*)' S_k(b_k - b_k^*)]\right\}.$$

From here we easily get the marginal densities $g_1(\sigma \mid x)$ and $g_2(b \mid x)$. Further we derive

$$h_2(b, \mu \mid x) = \int_0^\infty \dots \int_0^\infty g(b, \mu, \sigma \mid x) \, d\sigma =$$

$$= c \prod_{k=1}^{p} [R_k + (b_k - b_k^*)' S_k(b_k - b_k^*) + \alpha_k(\mu_k - \bar{x}_k + b_k' \bar{x}_k^0)^2]^{-\alpha_k/2} =$$

$$= c \prod_{k=1}^{p} [R_k + (b_k - \tilde{b}_k)' (S_k + \alpha_k \bar{x}_k^0 \bar{x}_k^{0\prime}) (b_k - \tilde{b}_k) + q_k v_k^2]^{-\alpha_k/2}.$$

The formula for $g_3(\mu \mid x)$ follows from

$$g_3(\mu \mid x) = \int_{\mathbf{R}_{np}} h_2(b, \mu \mid x) \, db. \quad \square$$

Theorem 4.4. *Let*

$$F_k = \frac{\alpha_k - 1 - n}{n R_k} (b_k - b_k^*)' S_k (b_k - b_k^*), \quad k = 1, \dots, p.$$

Then the posterior distribution of F_k is $F_{n, \alpha_k - 1 - n}$ and, given x, the variables $F_1, \dots$ $\dots, F_p$ are independent.

Proof. From Theorem 4.3 (ii) it is clear that $b_1, \dots, b_p$ are conditionally independent and that the density of b_k is

$$g_{2,k}(b_k \mid x) = c[1 + R_k^{-1}(b_k - b_k^*)' S_k(b_k - b_k^*)]^{-(\alpha_k - 1)/2}.$$

We apply Theorem 2.5, which gives the assertion about the $F_{n, \alpha_k - 1 - n}$ distribution. $\square$

Each variable F_k can be used for a test of fit that the k th vector of the autoregressive parameters is b_k. If $F_k \geqq F_{n, \alpha_k - 1 - n}(\alpha)$, we reject this hypothesis on the level α. A simultaneous test of fit for the whole vector b can be based on the following result.

Theorem 4.5. *Let H_k be the distribution function of the F_{n,α_k-1-n} distribution. Put $\pi_k = 1 - H_k(F_k)$. Then the posterior distribution of*

$$\varrho = -2\sum_{k=1}^{p} \ln \pi_k$$

is χ^2_{2p}.

Proof. It is well known that $H_k(F_k)$ has the rectangular distribution $R(0, 1)$. Then π_k has the same rectangular distribution and $-2 \ln \pi_k$ has the χ^2_2 distribution. Since π_k are independent, given x, ϱ has the χ^2_{2p} distribution. □

We have the following application of Theorem 4.5. If $\varrho \geqq \chi^2_{2p}(x)$, we reject the hypothesis that the vector of all autoregressive parameters is b.

Theorem 4.6. *Let*

$$T_k = [(\alpha_k - n - 1)\, q_k/R_k]^{1/2}\,(\mu_k - \mu_k^*)\,, \quad k = 1, \ldots, p\,.$$

Then the posterior distribution of T_k is the Student t_{α_k-n-1} distribution and, given x, $T_1, \ldots, T_p$ are independent.

Proof. The marginal distribution of μ_k can be calculated either by Theorem 2.2 or by Theorem 2.4. From here we derive the density of T_k, which coincides with the density of the t_{α_k-n-1} distribution. □

The result given in Theorem 4.6 can be used for constructing a test of fit about the true value of μ_k. If $|T_k| \geqq t_{\alpha_k-n-1}(\alpha)$, we reject the hypothesis that μ_k is the true value. The simultaneous test of fit can be constructed as follows. If $N \to \infty$, then also $\alpha_k \to \infty$ for all k. Since T_k has asymptotically the $N(0, 1)$ distribution, T_k^2 has asymptotically the χ^2_1 distribution, and from the conditional independence we get that

$$T = T_1^2 + \ldots + T_p^2$$

has asymptotically the χ^2_p distribution. If $T \geqq \chi^2_p(\alpha)$, we reject the hypothesis that $\mu_1, \ldots, \mu_p$ are true values of the model. This procedure enables us to decide whether the vector μ is the zero vector or not, because under the hypothesis $\mu_1 = \ldots = \mu_p = 0$, the variable

$$T = \sum_{k=1}^{n} [(\alpha_k - n - 1)\, q_k/R_k]\, \mu_k^{*2}$$

has asymptotically the χ^2_p distribution.

Also the procedure described in Theorem 4.5 can be simplified if we use asymptotic results. If an n-dimensional random vector X has the density

$$q(x) = c(1 + x'Vx)^{-m/2}\,,$$

where V is a positive definite matrix and $m \geqq n + 1$, then $Y = m^{1/2}X$ has the density

$$q_1(y) = c(1 + y'Vy/m)^{-m/2}\,.$$

If $m \to \infty$, then

$$q_1(y) \to c \exp\{-\tfrac{1}{2}y'Vy\},$$

i.e. Y has asymptotically the $N(0, V^{-1})$ distribution. If m is sufficiently large, we can approximate the distribution of X by $N(0, m^{-1}V^{-1})$. In particular, we have approximately

$$mX'VX \approx \chi^2_n .$$

If we apply these considerations to the density $g_2(b \mid x)$ given in Theorem 4.3 (ii), we get that

$$\varrho^* = \sum_{k=1}^{p} [(\alpha_k - 1)/R_k]\,(b_k - b_k^*)'\,S_k(b_k - b_k^*) \approx \chi^2_{np}$$

approximately holds. Thus we can use a testing procedure which is based on ϱ^* instead of that based on ϱ in Theorem 4.5.

Let us approximate the density $g_2(b \mid x)$ from Theorem 4.3 (ii) by

$$g_2^*(b \mid x) = c \exp\{-\tfrac{1}{2}\sum_{k=1}^{p} [(\alpha_k - 1)/R_k]\,(b_k - b_k^*)'\,S_k(b_k - b_k^*)\} .$$

If we define

$$\Delta_k = (b_k - b_k^*) - (b_p - b_p^*) \quad \text{for} \quad k = 1, \ldots, p-1,$$

$$\Delta = (\Delta_1', \ldots, \Delta_{p-1}')', \quad U_k = R_k^{-1}(\alpha_k - 1)\,S_k, \quad U = U_1 + \ldots + U_p,$$

$$L = \operatorname{Diag}(U_1, \ldots, U_{p-1}) -$$
$$- (U_1, \ldots, U_{p-1})'\,U^{-1}\,(U_1, \ldots, U_{p-1}),$$

then

$$r_b = \Delta' L \Delta$$

has approximately the $\chi^2_{n(p-1)}$ distribution. The derivation of this result from $g_2^*(b \mid x)$ is analogous to the proof of Theorem 3.8. Similarly, the density $g_3(\mu \mid x)$ from Theorem 4.3 (iii) can be approximated by

$$g_3^*(\mu \mid x) = c \exp\{-\tfrac{1}{2}\sum_{k=1}^{p} u_k(\mu_k - \mu_k^*)^2\},$$

where

$$u_k = R_k^{-1}(\alpha_k - n)\,q_k, \quad k = 1, \ldots, p .$$

If we put

$$\delta_k = (\mu_k - \mu_k^*) - (\mu_p - \mu_p^*) \quad \text{for} \quad k = 1, \ldots, p-1,$$

$$\delta = (\delta_1', \ldots, \delta_{p-1}')', \quad u = u_1 + \ldots + u_p,$$

$$M = \operatorname{Diag}\{u_1, \ldots, u_{p-1}\} - u^{-1}(u_1, \ldots, u_{p-1})\,(u_1, \ldots, u_{p-1})',$$

then

$$r_\mu = \delta' M \delta$$

has approximately the χ^2_{p-1} distribution.

Using r_b and r_μ we can test the hypotheses H_0: $b_1 = \ldots = b_p$ and H'_0: $\mu_1 = \ldots = \mu_p$, respectively. If H_0 holds, then $\Delta_k = b_p^* - b_k^*$ and when $r_k \geqq \chi^2_{n(p-1)}(\alpha)$, we reject H_0. Similarly, if H'_0 holds, then $\delta_k = \mu_p^* - \mu_k^*$ and when $r_\mu \geqq \chi^2_{p-1}(\alpha)$, we reject H'_0.

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Souhrn

O PERIODICKÉ AUTOREGRESI S NEZNÁMOU STŘEDNÍ HODNOTOU

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Periodická autoregrese je model pro sezónní časové řady. Předpokládá se, že autoregresní parametry jsou periodické funkce s periodou, která odpovídá sezónnímu charakteru řady. Střední hodnota řady může být rovněž periodická funkce. V práci je pro odhad parametrů a pro testování hypotéz použit bayesovský přístup. Jsou vyšetřovány dva modely. Jeden se týká případu, kdy inovační proces má konstantní rozptyl, druhý model odpovídá inovačnímu procesu s periodicky se měnícími rozptyly.

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