Ivan Hlaváček

Optimization of the domain in elliptic problems by the dual finite element method

Aplikace matematiky, Vol. 30 (1985), No. 1, 50–72

OPTIMIZATION OF THE DOMAIN IN ELLIPTIC PROBLEMS BY THE DUAL FINITE ELEMENT METHOD

IVAN HLAVÁČEK

Dedicated to Professor Miloš Zlámal on the occasion of the sixtieth anniversary of his birthday

(Received March 9, 1984)

INTRODUCTION

The problem of optimal domain in elliptic boundary value problems has been studied thoroughly on a simple model by Begis and Glowinski [1]. It is the aim of the present paper to extend their results to two further types of cost functionals, namely to those involving the gradient of the solution of the state problem. Thus we minimize (i) the internal energy (i.e., the Dirichlet integral) and (ii) the norm of the outward flux.

A dual variational formulation of the state problem (in terms of gradients) is used for the numerical solution and finite element subspaces of divergence-free (solenoidal) piecewise linear functions are employed (see [2], [4]). The existence of an optimal domain is proved and an analysis of the convergence of piecewise linear approximations presented.

Let us mention that the state problem with unilateral boundary conditions has been studied by Nečas and the author in [6] and by Haslinger and coauthors in [5], [8] (see also [9]).

1. FORMULATION OF THE OPTIMIZATION PROBLEMS

Let us consider the following model problems: Let $\Omega(v) \subset \mathbb{R}^2$ be the domain (see fig. 1)

$$\Omega(v) = \{0 < x_1 < v(x_2), 0 < x_2 < 1\},$$

where the function v is to be determined from one of the two problems

$$(1.0) \qquad \mathscr{J}_i(y(v)) = \min \quad (i = 1, 2)$$

over the set of $v \in \mathscr{U}_{ad}$.

Here

$$\mathscr{U}_{ad} = \Big\{ v \in C^{(0),1}([0, 1]) \quad \text{(i.e. Lipschitz function)}\,,$$

$$0 < \alpha \leqq v \leqq \beta\,, \quad |\mathrm{d}v/\mathrm{d}x_2| \leqq C_1\,, \quad \int_0^1 v(x_2)\,\mathrm{d}x_2 = C_2 \Big\}$$

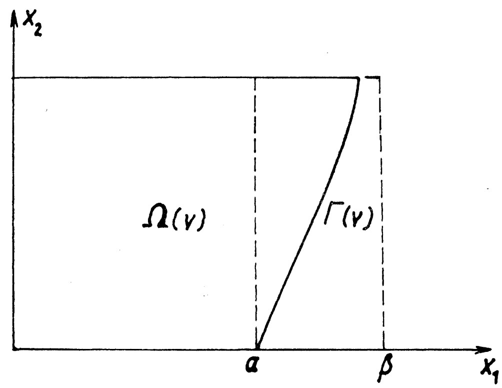

Fig. 1.

with given constants α, β, C_1. C_2;

(1.1) $$\mathscr{J}_1(y(v)) = \int_{\Omega(v)} |\nabla y(v)|^2 \,\mathrm{d}x\,,$$

(1.2) $$\mathscr{J}_2(y(v)) = \|\partial y(v)/\partial v\|^2_{-1/2,\Gamma(v)}\,,$$

and $y(v)$ denotes the solution of the following boundary value problem:

(1.3) $$\begin{aligned} -\Delta y &= f \quad \text{in} \quad \Omega(v)\,, \\ y &= 0 \quad \text{on} \quad \Gamma(v)\,, \\ \frac{\partial y}{\partial v} &= 0 \quad \text{on} \quad \partial\Omega(v) - \Gamma(v)\,. \end{aligned}$$

The function $f \in L^2(\Omega_\beta)$ is given, $\Omega_\beta = (0, \beta) \times (0, 1)$, $\partial y/\partial v$ denotes the derivative with respect to the outward normal to $\Gamma(v)$ and the norm in (1.2) will be defined later. In the following, we denote by $H^k(\Omega)$ the Sobolev space $W_2^{(k)}(\Omega)$ with the usual norm $\|\cdot\|_{k,\Omega}$ and the scalar product $(.,.)_{k,\Omega}$, $H^0 \equiv L^2$. For vector-functions, the notation

$$\|\boldsymbol{q}\|_{k,\Omega} = \Big(\sum_{i=1}^{2} \|q_i\|^2_{k,\Omega}\Big)^{1/2}\,;$$

$$(\boldsymbol{q}, \boldsymbol{p})_{k,\Omega} = \sum_{i=1}^{2} (q_i, p_i)_{k,\Omega}$$

will be used.

It is well-known that the state problem (1.3) can be formulated in the following variational way. Let us introduce the subspace

$$V(v) = \{w \in H^1(\Omega(v)) : \gamma w = 0 \text{ on } \Gamma(v)\},$$

where γ is the trace operator, $\gamma: H^1(\Omega(v)) \to H^{1/2}(\partial\Omega(v))$.

The weak solution of (1.3) is the function $y = y(v) \in V(v)$ such that

$$\int_{\Omega(v)} \nabla y \,.\, \nabla w \,\mathrm{d}x = \int_{\Omega(v)} f w \,\mathrm{d}x \quad \forall w \in V(v).$$

There exists a unique weak solution for any $v \in \mathscr{U}_{ad}$.

Next we explain the sense of the functional $\mathscr{J}_2$ in (1.2).

Definition 1.1. *Let us introduce the subspace*

$$V^c = \{w \in H^1(\Omega): \gamma w = 0 \text{ on } \partial\Omega - \Gamma_0\}$$

where Γ_0 is an "extension" of Γ, such that $\bar{\Gamma} \subset \Gamma_0 \subset \partial\Omega$, Γ_0 is connected and open in $\partial\Omega$ *and denote*

$$H^{1/2}(\Gamma) = \gamma(V^c).$$

For $\varphi \in H^{1/2}(\Gamma)$ *we define the norm*

$$\|\varphi\|_{1/2,\Gamma} = \inf_{\substack{\gamma v = \varphi \\ v \in V^c}} \|v\|_{1,\Omega}. \tag{1.4}$$

For the linear continuous functionals $g \in H^{-1/2}(\Gamma) \equiv [H^{1/2}(\Gamma)]'$ *we define the usual norm*

$$\|g\|_{-1/2,\Gamma} = \sup_{\substack{\varphi \in H^{1/2}(\Gamma) \\ \varphi \neq 0}} \frac{\langle g, \varphi\rangle}{\|\varphi\|_{1/2,\Gamma}}. \tag{1.5}$$

Lemma 1.1. *Let* $g \in H^{-1/2}(\Gamma)$ *be given and let* $u \in V^c$ be the solution of the following problem

$$\int_\Omega (\nabla u \,.\, \nabla w + uw) \,\mathrm{d}x = \langle g, \gamma w\rangle \quad \forall w \in V^c. \tag{1.6}$$

Then it holds

$$\|g\|_{-1/2,\Gamma} = \|u\|_{1,\Omega}. \tag{1.7}$$

Proof. From the relation (1.6) we obtain, using (1.4)

$$\langle g, \varphi\rangle \leqq \|u\|_{1,\Omega} \inf_{\substack{\gamma w = \varphi \\ w \in V^c}} \|w\|_{1,\Omega} = \|u\|_{1,\Omega} \|\varphi\|_{1/2,\Gamma}.$$

Consequently, by means of (1.5) we can write

$$\|g\|_{-1/2,\Gamma} \leqq \|u\|_{1,\Omega}. \tag{1.8}$$

Inserting $w = u$ in (1.6) and using (1.4), we obtain

$$\|u\|^2_{1,\Omega} = \langle g, \gamma u\rangle \leqq \|g\|_{-1/2,\Gamma} \|\gamma u\|_{1/2,\Gamma} \leqq \|g\|_{-1/2,\Gamma} \|u\|_{1,\Omega} .$$

Cancelling and combining the result with (1.8), we are led to (1.7.) Q.E.D.

Definition 1.2. *Let $y \in H^1(\Omega)$ be such that $\Delta y \in L^2(\Omega)$ exists (in the sense of distributions).*

We define a functional $\partial y/\partial \nu \in H^{-1/2}(\Gamma)$ by the following relation

$$\left\langle \frac{\partial y}{\partial \nu}, w \right\rangle = \int_\Omega (\nabla y \,.\, \nabla \omega + \omega \,\Delta y)\, \mathrm{d}x , \tag{1.9}$$

where ω is any element of V^c such that $\gamma\omega = w$ on Γ_0.

Next let $y = y(v)$ be the weak solution of the state problem (1.3). Since $\Delta y = -f \in L^2(\Omega)$, we can apply the Definition 1.2 and Lemma 1.1 to define a function $u = u(y(v))$ as a solution of the following problem:

find $u \in V^c(v)$ such that

$$\int_{\Omega(v)} (\nabla u \,.\, \nabla w + uw)\, \mathrm{d}x = \int_{\Omega(v)} (\nabla y \,.\, \nabla w + w \,\Delta y)\, \mathrm{d}x \quad \forall w \in V^c(v) . \tag{1.10}$$

Using Lemma 1.1 we can write

$$\mathscr{J}_2(y(v)) = \|u(y(v))\|^2_{1,\Omega(v)} . \tag{1.11}$$

The latter relations will be used instead of (1.2) for the definition of the cost functional $\mathscr{J}_2$. Henceforth let us choose $\partial\Omega - \Gamma_0$ independent of v.

2. EXISTENCE OF THE OPTIMAL DOMAIN

In the present Section we shall prove that at least one solution of the problems (1.0) exists.

The solution $y(v)$ of the state problem for any $v \in \mathscr{U}_{ad}$ can be extended by zero to a rectangular domain

$$\Omega_\delta = (0, \delta) \times (0, 1) , \quad \delta > \beta .$$

The extended function will be again denoted by y and obviously $y \in H^1(\Omega_\delta)$ holds. The function f will be extended to Ω_δ by zero, as well.

Lemma 2.1. *Let $\{v_n\}$ be a sequence of $v_n \in \mathscr{U}_{ad}$ $\forall n$. Then a subsequence $\{v_k\}$ and element $v \in \mathscr{U}_{ad}$ exist such that*

$$y(v_n) \rightarrow y(v) \quad in \quad H^1(\Omega_\delta) , \tag{2.1}$$

$$v_k \rightarrow v \quad in \quad C([0, 1]) .$$

Proof. Let us denote $y(v_n) = y_n$, $V(v_n) = V_n$, $\Omega(v_n) = \Omega_n$. Since

$$(\nabla y_n, \nabla z)_{0,\Omega_n} = (f, z)_{0,\Omega_n} \quad \forall z \in V_n$$

follows from the definition of y_n, inserting $z = y_n$, we obtain

$$\|\nabla y_n\|^2_{0,\Omega_n} = (f, y_n)_{0,\Omega_n} \leqq \|f\|_{0,\Omega_\beta} \|y_n\|_{1,\Omega_n} .$$

For all $v \in \mathscr{U}_{ad}$ we have the generalized Friedrichs inequality

$$\|\nabla z\|^2_{0,\Omega(v)} \geqq C\|z\|^2_{1,\Omega(v)} \quad \forall z \in V(v) ,$$

with C independent of v.

Consequently,

$$\|y_n\|^2_{1,\Omega_n} \leqq C^{-1}\|f\|_{0,\Omega_\beta} \|y_n\|_{1,\Omega_n} \tag{2.2}$$

so that the sequence $\{y_n\}$ (extension) is bounded in $H^1(\Omega_\delta)$. Hence there exist a subsequence $\{y_k\}$ and an element $y^* \in H^1(\Omega_\delta)$ such that

$$y_k \rightharpoonup y^* \quad \text{(weakly) in} \quad H^1(\Omega_\delta) . \tag{2.3}$$

Since $\mathscr{U}_{ad}$ is compact in $C([0, 1])$, a subsequence $\{v_m\}$ of $\{v_k\}$ and $v \in \mathscr{U}_{ad}$ exist such that

$$v_m \to v \quad \text{in} \quad C([0, 1]) .$$

We shall prove that y^* is the weak solution of (1.3), i.e., $y^* = y^*(v)$.

First we prove that $y^* = 0$ a.e. in $\Omega_\delta - \overline{\Omega(v)}$. In fact, let $y^* \neq 0$ on a set $E \subset \Omega_\delta - \overline{\Omega(v)}$, the measure of E being positive.

Let Ω_ε denote the domain bounded by the graph of $v + \varepsilon$. Obviously, there exists $\varepsilon > 0$ such that

$$\text{mes}\,(E \cap \Omega_\delta - \Omega_\varepsilon) > 0 .$$

We have $\bar{\Omega}_m \subset \Omega_\varepsilon$ for m great enough and therefore

$$\int_{\Omega_\delta} (y_m - y^*)^2 \,\mathrm{d}x \geqq \int_{E\cap(\Omega_\delta - \Omega_\varepsilon)} (y_m - y^*)^2 \,\mathrm{d}x = \int_{E\cap(\Omega_\delta - \Omega_\varepsilon)} (y^*)^2 \,\mathrm{d}x > 0 . \tag{2.4}$$

On the other hand,

$$y_k \to y^* \quad \text{in} \quad L^2(\Omega_\delta) , \quad k \to \infty \tag{2.5}$$

follows from (2.3) and the Rellich's Theorem for a subsequence $\{y_k\}$. We arrive at a contradiction with (2.4).

Consequently, we have

$$y^*|_{\Omega(v)} \in V(v) .$$

Let a $w \in V(v)$ be given. There exists a sequence $\{\varphi_n\}$, $\varphi_n \in C^\infty(\overline{\Omega(v)})$ such that $\text{supp}\, \varphi_n \cap \Gamma(v) = \emptyset$,

$$\varphi_n \to w \quad \text{in} \quad H^1(\Omega(v))$$

(or in $H^1(\Omega_\delta)$ if the extensions of w and φ_n are considered).

We may write

$$(\nabla y_m, \nabla \varphi_n)_{0,\Omega_m} = (f, \varphi_n)_{0,\Omega_m}$$

for all m great enough. Passing to the limit with m, we obtain

$$(\nabla y^*, \nabla \varphi_n)_{0,\Omega_\delta} = (f, \varphi_n)_{0,\Omega_\delta} .$$

Passing to the limit with n, we are led to the relation

$$(\nabla y^*, \nabla w)_{0,\Omega(v)} = (f, w)_{0,\Omega(v)} ;$$

consequently, $y^* = y^*(v)$ holds.

Since the definition of y_m implies

$$\|\Delta y_m\|^2_{0,\Omega_m} = (f, y_m)_{0,\Omega_m} ,$$

using the extensions and the weak convergence, we obtain

$$\|\nabla y_m\|^2_{0,\Omega_\delta} = (f, y_m)_{0,\Omega_\delta} \to (f, y^*)_{0,\Omega_\delta} = \|\nabla y^*\|^2_{0,\Omega_\delta} . \tag{2.6}$$

From (2.5) and (2.6) the convergence of norms in $H^1(\Omega_\delta)$ follows. Consequently, the strong convergence (2.1) holds. Q.E.D.

Theorem 2.1. *There exists at least one solution of the optimisation problems* (1.0), $i = 1, 2$.

Proof. Let us consider a minimizing sequence for $\mathscr{J}_1$, i.e., $v_n \in \mathscr{U}_{ad}$,

$$\mathscr{J}_1(y(v_n)) \to \inf_{v \in \mathscr{U}_{ad}} \mathscr{J}_1(y(v)) , \quad n \to \infty . \tag{2.7}$$

Let us apply Lemma 2.1 to obtain a uniformly converging subsequence $\{v_k\}$ with a limit v, such that

$$\mathscr{J}_1(y(v_k)) = \|\nabla y_k\|^2_{0,\Omega_k} \to \|\nabla y(v)\|^2_{0,\Omega(v)} = \mathscr{J}_1(y(v)) . \tag{2.8}$$

Combining (2.7) and (2.8) we arrive at the following relation

$$\inf_{v \in \mathscr{U}_{ad}} \mathscr{J}_1(y(v)) = \mathscr{J}_1(y(v)) .$$

Consequently, v is a solution of (1.0) for $i = 1$.

The case $i = 2$. Let us denote by u_k the solution of the problem (1.10) on the domain $\Omega(v_k) \equiv \Omega_k$, where $y = y_k$ is substituted on the right-hand side. Inserting $w = u_k$ in (1.10), we obtain

$$\|u_k\|^2_{1,\Omega_k} = \int_{\Omega_k} (\nabla y_k \cdot \nabla u_k + \Delta y_k \cdot u_k) \, dx . \tag{2.9}$$

Every function u_k will be extended onto the rectangle Ω_δ as follows: we define for $v_k(x_2) < x_1 < 2v_k(x_2)$

(2.10) $$\tilde{u}_k(x_1, x_2) = u_k(2v_k(x_2) - x_1, x_2).$$

Thus the extension $\tilde{u}_k$ is "symmetric" with respect to the graph of v_k. It is easy to derive the following estimate for $x \in \Omega(2v_k) - \Omega_k$:

(2.11) $$|\nabla \tilde{u}_k(x)|^2 \leqq (2 + 4(v_k')^2) |\nabla u_k(x^s)|^2,$$

where x^s is the "symmetric" point with the coordinates

$$x_1^s = 2v_k(x_2) - x_1, \quad x_2^s = x_2.$$

For points where $x_1 > 2v_k(x_2)$, we define $\tilde{u}_k = 0$. Then using (2.11) we can write

$$\int_{\Omega_\delta - \Omega_k} |\nabla \tilde{u}_k|^2 \, dx \leqq \int_{\Omega(2v_k) - \Omega_k} |\nabla \tilde{u}_k|^2 \, dx \leqq C_0 \int_{\Omega_k} |\nabla u_k|^2 \, dx,$$

where C_0 does not depend on k and u_k, if $v_k \in \mathcal{U}_{ad}$.

Since

$$\int_{\Omega_\delta - \Omega_k} \tilde{u}_k^2 \, dx \leqq \int_{\Omega_k} u_k^2 \, dx,$$

we arrive at the estimate

(2.12) $$\|\tilde{u}_k\|^2_{1,\Omega_\delta - \Omega_k} \leqq C_0 \|u_k\|^2_{1,\Omega_k}.$$

Inserting $\Delta y_m = -f$ in (2.9), we obtain

$$\|u_k\|^2_{1,\Omega_k} \leqq \|y_k\|_{1,\Omega_k} \|u_k\|_{1,\Omega_k} + \|f\|_{0,\Omega_k} \|u_k\|_{0,\Omega_k}.$$

Using the boundedness of norms of y_k (cf. (2.2)), we deduce

(2.13) $$\|u_k\|_{1,\Omega_k} \leqq C \quad \forall k.$$

Combining (2.12) and (2.13), we obtain

(2.14) $$\|\tilde{u}_k\|^2_{1,\Omega_\delta} \leqq C^2(1 + C_0).$$

Consequently, a subsequence of $\{\tilde{u}_k\}$ (which will be denoted by the same symbol) and a function $u \in H^1(\Omega_\delta)$ exist such that

(2.15) $$\tilde{u}_k \rightharpoonup u \quad \text{(weakly) in} \quad H^1(\Omega_\delta), \quad k \to \infty.$$

Next let us seek the limit of the right-hand side of (2.9) for $k \to \infty$. We can write

(2.16) $$(\nabla y_k, \nabla u_k)_{0,\Omega_k} = (\nabla y_k, \nabla \tilde{u}_k)_{0,\Omega\delta} \to (\nabla y^*, \nabla u)_{0,\Omega\delta} = (\nabla y^*, \nabla u)_{0,\Omega(v)},$$

where the strong convergence of $\{y_k\}$ and the weak convergence (2.15) has been used.

Moreover, one easily finds that

$$\Delta y_k \rightharpoonup \Delta y^* \quad \text{(weakly) in} \quad L^2(\Omega_\delta)\,. \tag{2.17}$$

In fact, for any $w \in L^2(\Omega_\delta)$ we have

$$-(\Delta y_k, w)_{0,\Omega_\delta} = (f, w)_{0,\Omega_k} \to (f, w)_{0,\Omega(v)} = -(\Delta y^*, w)_{0,\Omega(v)}\,.$$

From (2.15) and the Rellich's Theorem it follows that

$$\tilde{u}_k \to u \quad \text{in} \quad L^2(\Omega_\delta)\,, \quad k \to \infty\,. \tag{2.18}$$

Thus we may write

$$(\Delta y_k, u_k)_{0,\Omega_k} = (\Delta y_k, \tilde{u}_k)_{0,\Omega_\delta} \to (\Delta y^*, u)_{0,\Omega_\delta} = (\Delta y^*, u)_{0,\Omega(v)}\,. \tag{2.19}$$

If we substitute (2.16) and (2.19) into (2.9), we obtain

$$\|u_k\|^2_{1,\Omega_k} \to \int_{\Omega(v)} (\nabla y^* \,.\, \nabla u + \Delta y^* u)\,\mathrm{d}x\,. \tag{2.20}$$

Let us verify that

$$u|_{\Omega(v)} = u(y^*)$$

in the sense of the definition (1.10).

First we shall prove that

$$\int_{\Omega(v)} (\nabla u \,.\, \nabla w + uw)\,\mathrm{d}x = \int_{\Omega(v)} (\nabla y^* \,.\, \nabla w + \Delta y^* w)\,\mathrm{d}x \quad \forall w \in V^c(v)\,. \tag{2.21}$$

In fact, let a $w \in V^c(v)$ be given. Let $\tilde{w} \in H^1(\Omega_\delta)$ be the extension of w constructed in the same way as in (2.10). Then

$$\tilde{w}|_{\Omega_k} \in V^c(v_k) \quad \forall k\,.$$

By virtue of (1.6) and (1.9), we have

$$\int_{\Omega_k} (\nabla u_k \,.\, \nabla \tilde{w} + u_k \tilde{w})\,\mathrm{d}x = \int_{\Omega_k} (\nabla y_k \,.\, \nabla \tilde{w} + \Delta y_k \tilde{w})\,\mathrm{d}x\,. \tag{2.22}$$

Using the convergence (2.1) and (2.17), we obtain that the right-hand side tends to

$$\int_{\Omega(v)} (\nabla y^* \,.\, \nabla w + \Delta y^* w)\,\mathrm{d}x\,.$$

For the left-hand side of (2.22) we may write

$$\left|\int_{\Omega_k} (\nabla \tilde{u}_k \,.\, \nabla \tilde{w} + \tilde{u}_k \tilde{w})\,\mathrm{d}x - \int_{\Omega(v)} (\nabla u \,.\, \nabla w + uw)\,\mathrm{d}x\right| \leqq$$

$$\leqq \left| \int_{\Omega_k} (\nabla \tilde{u}_k \, . \, \nabla \tilde{w} + \tilde{u}_k \tilde{w}) \, \mathrm{d}x - \int_{\Omega(v)} (\nabla \tilde{u}_k \, . \, \nabla \tilde{w} + \tilde{u}_k \tilde{w}) \, \mathrm{d}x \right| +$$

$$+ \left| \int_{\Omega(v)} [\nabla \tilde{u}_k \, . \, \nabla w + \tilde{u}_k w - (\nabla u \, . \, \nabla w + uw)] \, \mathrm{d}x \right| = I_{1k} + I_{2k} \, .$$

On the basis of (2.15) and of the uniform convergence of $\{v_k\}$, it holds

$$I_{1k} \leqq \left|(\tilde{u}_k, \tilde{w})_{1,\Delta(\Omega_k,\Omega(v))}\right| \leqq \|\tilde{u}_k\|_{1,\Omega_\delta} \, \|\tilde{w}\|_{1,\Delta(\Omega_k,\Omega(v))} \to 0 \, ,$$

where

$$\Delta(A, B) = (A - B) \cup (B - A)$$

denotes the symmetric difference of the sets A and B. The weak convergence (2.15) implies that also

$$I_{2k} \to 0 \, .$$

Consequently, passing to the limit in (2.22) we obtain (2.21).

The subspace $V^c(\Omega_\delta)$ is weakly closed in $H^1(\Omega_\delta)$. Since $\tilde{u}_k \in V^c(\Omega_\delta)$, the weak limit $u \in V^c(\Omega_\delta)$ and therefore

$$u|_{\Omega(v)} \in V^c(v) \, .$$

From the uniqueness of the solution of (1.10) we conclude that

$$u|_{\Omega(v)} = u(y^*) \, .$$

Inserting $w = u(y^*)$ into (2.21) we obtain

$$\|u(y^*)\|^2_{1,\Omega(v)} = \int_{\Omega(v)} (\nabla y^* \, . \, \nabla u(y^*) + \Delta y^* \, u(y^*)) \, \mathrm{d}x \, .$$

From (2.20) and (1.11) it follows that

$$\mathscr{J}_2(y(v_k)) = \|u_k\|^2_{1,\Omega_k} \to \|u(y^*)\|^2_{1,\Omega(v)} = \mathscr{J}_2(y(v)) \, .$$

Since $\{v_k\}$ is a minimizing sequence, v is a solution of the optimization problem for $i = 2$. Q.E.D.

3. DUAL FORMULATION OF THE STATE PROBLEM

Since the cost functionals are expressed in terms of the gradient ∇y and not in terms of the function y itself, it seems to be advantageous to employ the dual variational formulation of the state problem. Thus we shall calculate the gradient ∇y directly. To this aim we have to introduce the space of solenoidal (divergence-free) vector functions

$$Q_0(v) = \{\boldsymbol{q} \in [L^2(\Omega(v))]^2 : \operatorname{div} \boldsymbol{q} = 0 \quad \text{in} \quad \Omega(v) \, ,$$
$$\boldsymbol{q} \, . \, \nu = 0 \quad \text{on} \quad \partial\Omega(v) - \Gamma(v)\} \, .$$

We shall use also the following equivalent definition

$$Q_0(v) = \left\{ \mathbf{q} \in [L^2(\Omega(v))]^2 : \int_{\Omega(v)} \mathbf{q} \,.\, \nabla w \,\mathrm{d}x = 0 \quad \forall w \in V(v) \right\}.$$

Let us construct the vector field $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)$,

$$\text{(3.1)} \qquad \bar{\lambda}_1(x_1, x_2) = -\int_0^{x_1} f(t, x_2)\,\mathrm{d}t\,, \quad \bar{\lambda}_2 \equiv 0\,,$$

assuming that the integral has sense for $x_2 = 0$, $x_2 = 1$ and almost all $x_2 \in (0, 1)$.

It is readily seen that

$$\operatorname{div} \bar{\lambda} = \partial\bar{\lambda}_1/\partial x_1 = -f \quad \text{in} \quad \Omega_\beta\,,$$

$$\bar{\lambda} \,.\, \nu = \bar{\lambda}_1 \nu_1 = 0 \quad \text{on} \quad \partial\Omega_\beta - \Gamma_\beta\,,$$

where $\Gamma_\beta = \{(x_1, x_2) : x_1 = \beta\,, \quad x_2 \in (0, 1)\}$.

Then a suitable dual formulation of the problem (1.3) is: to find $\mathbf{q}(v) \in Q_0(v)$ such that

$$\text{(3.2)} \qquad (\mathbf{q}(v), \mathbf{p})_{0,\Omega(v)} = -(\bar{\lambda}, \mathbf{p})_{0,\Omega(v)} \quad \forall \mathbf{p} \in Q_0(v)\,.$$

There exists a unique solution of (3.2) and

$$\text{(3.3)} \qquad \bar{\lambda} + \mathbf{q}(v) = \nabla y(v)$$

holds. (Henceforth $\bar{\lambda}$ denotes everywhere the restriction of the vector field (3.1) onto the domain under considerations and $y(v)$ is the weak solution of (1.3)).

The cost functional $\mathscr{J}_1$ can be rewritten as follows

$$\mathscr{J}_1(y(v)) = \|\bar{\lambda} + \mathbf{q}(v)\|^2_{0,\Omega(v)} = \mathscr{J}_1^*(\mathbf{q}(v))\,.$$

Using (3.3) in (1.10), (1.11), the cost functional $\mathscr{J}_2$ can be transformed into

$$\mathscr{J}_2^*(\mathbf{q}(v)) = \|u(\mathbf{q}(v))\|^2_{1,\Omega(v)}\,,$$

where $u \equiv u(\mathbf{q}(v))$ is the solution of the following problem

$$\text{(3.4)} \quad \int_{\Omega(v)} (\nabla u \,.\, \nabla w + uw)\,\mathrm{d}x = \int_{\Omega(v)} [(\bar{\lambda} + \mathbf{q}(v)) \,.\, \nabla w - wf]\,\mathrm{d}x \quad \forall w \in V^c(v)\,.$$

Obviously, Theorem 2.1 yields the existence of a solution of the equivalent optimization problem

$$\text{(3.5)} \qquad \mathscr{J}_i^*(\mathbf{q}(v)) = \min \quad (i = 1, 2)$$

over the set of $v \in \mathscr{U}_{ad}$.

In fact, for all $w \in \mathscr{U}_{ad}$ we may write

$$\mathscr{J}_i^*(\mathbf{q}(v)) = \mathscr{J}_i(y(v)) \leqq \mathscr{J}_i(y(w)) = \mathscr{J}_i^*(\mathbf{q}(w))\,.$$

4. APPROXIMATION OF THE DUAL STATE PROBLEM

Let N be a positive integer and $h = 1/N$. We denote by e_j, $j = 1, 2, \ldots, N$, the subintervals $[(j - 1)\, h, jh]$ and introduce the set

$$\mathscr{U}^h_{ad} = \{v_h \in \mathscr{U}_{ad} : v_h|_{e_j} \in P_1(e_j)\ \forall_j\},$$

where P_1 denotes the set of linear polynomials. Let Ω_h denote the domain bounded by the graph Γ_h of the function $v_h \in \mathscr{U}^h_{ad}$, i.e. $\Omega_h = \Omega(v_h)$.

The domain Ω_h will be carved into triangles by the following way (see fig. 2).

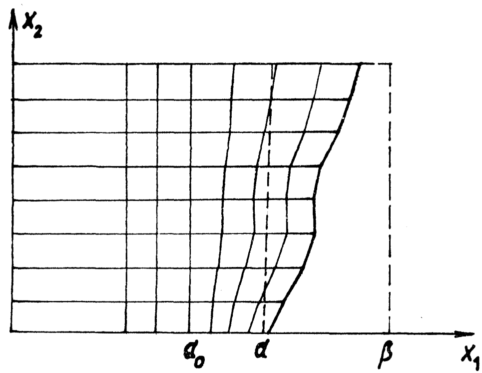

Fig. 2.

We choose $\alpha_0 \in (0, \alpha)$ and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, independent of v_h if h is fixed.

In the remaining part $\Omega_h - \mathscr{R}$ let the nodal points divide the intervals $[\alpha_0, v_h(jh)]$ into M uniform segments, where

$$M = 1 + [(\beta - \alpha_0)\, N]$$

and the square brackets denote the integer part of the number inside. One can find easily, that then the segments parallel with the x_1-axis are not longer than h and shorter than $h(\alpha - \alpha_0)/(\beta - \alpha_0)$.

One also deduces the following estimate for the interior angles ω of the triangulation

$$\operatorname{tg} \omega \geqq \frac{\alpha - \alpha_0}{\beta - \alpha_0} (1 + C_1 + C_1^2)^{-1}.$$

Consequently, one obtains a *regular family* $\{\mathscr{T}_h(v_h)\}$ *of triangulations*, with

$$\max_{K \in \mathscr{T}_h(v_h)} (\operatorname{diam} K) \leqq h/\sin \omega_0,$$

$$\omega_0 = \operatorname{arctg} \left(\frac{\alpha - \alpha_0}{\beta - \alpha_0} (1 + C_1 + C_1^2)^{-1}\right).$$

Let us consider the space $\mathcal{N}_h(v_h)$ of piecewise linear solenoidal (divergence-free) functions on the triangulation $\mathcal{T}_h$ (see [2], [4]) and define

$$S_h = \mathcal{N}_h(v_h) \cap Q_0(v_h) = \{\boldsymbol{q}_h \in \mathcal{N}_h(v_h) : \boldsymbol{q}_h \cdot v = 0 \text{ on } \partial\Omega_h - \Gamma_h\} .$$

Instead of the problem (3.2) we shall solve the following approximate problem: to find $\boldsymbol{q}^h(v_h) \in S_h$ such that

$$(4.1) \qquad (\boldsymbol{q}^h(v_h), \boldsymbol{p}^h)_{0,\Omega_h} = -(\bar{\lambda}, \boldsymbol{p}^h)_{0,\Omega_h} \quad \forall \boldsymbol{p}^h \in S_h .$$

There exists a unique solution of (4.1) for any h and any $v_h \in \mathcal{U}^h_{ad}$.

Lemma 4.1. *Let $\{v_h\}$, $h \to 0$, be a sequence of $v_h \in \mathcal{U}^h_{ad}$, converging uniformly to a function v.*
Then

$$\boldsymbol{q}^{0h}(v_h) \to \boldsymbol{q}(v) \quad \text{in} \quad [L^2(\Omega_\delta)]^2 \quad \text{for} \quad h \to 0 ,$$

where $\boldsymbol{q}^{0h}$ is the solution of (4.1) extended by zero to the domain $\Omega_\delta - \bar{\Omega}_h$ and $\boldsymbol{q}(v)$ is the solution of (3.2), extended by zero to $\Omega_\delta - \overline{\Omega(v)}$.

Proof. We find easily that the limit v belongs to $\mathcal{U}_{ad}$. It follows from (4.1) that

$$\|\boldsymbol{q}^h\|^2_{0,\Omega_h} \leqq \|\bar{\lambda}\|_{0,\Omega_\beta} \cdot \|\boldsymbol{q}^h\|_{0,\Omega_h} ,$$

consequently,

$$\|\boldsymbol{q}^h\|_{0,\Omega_h} \leqq \|\bar{\lambda}\|_{0,\Omega_\beta} = C$$

and

$$(4.2) \qquad \|\boldsymbol{q}^{0h}\|_{0,\Omega_\delta} \leqq C \quad \forall h$$

holds for the extension $\boldsymbol{q}^{0h}$.

Therefore a subsequence of $\{\boldsymbol{q}^{0h}\}$ exists (and we will denote it by the same symbol) such that

$$(4.3) \qquad \boldsymbol{q}^{0h} \to \boldsymbol{q} \quad \text{(weakly) in} \quad [L^2(\Omega_\delta)]^2 \quad \text{for} \quad h \to 0 .$$

We can show that

$$(4.4) \qquad \boldsymbol{q}|_{\Omega(v)} \in Q_0(v) .$$

In fact, let us consider a $w \in V(v)$ and denote by $\tilde{w}$ its extension to Ω_δ by means of zero. A sequence $\{w_\varkappa\}$, $\varkappa \to 0$, exists such that

$$w_\varkappa \in C^\infty(\bar{\Omega}_\delta) , \quad w_\varkappa = 0 \quad \text{on} \quad \bar{\Omega}_\delta - \Omega(v) ,$$

$$\operatorname{supp} w_\varkappa \cap \Gamma(v) = \emptyset ,$$

$$(4.5) \qquad w_\varkappa \to \tilde{w} \quad \text{in} \quad H^1(\Omega_\delta) .$$

There exist a $h_0(\varkappa)$ such that $w_\varkappa$ vanishes on $\Gamma(v_h)$ for $h < h_0(\varkappa)$, so that

$$w_\varkappa|_{\Omega_h} \in V(v_h) \quad \forall h < h_0(\varkappa) .$$

Since $\boldsymbol{q}^h \in S_h \subset Q_0(v_h)$, we have

$$(\boldsymbol{q}^h, \nabla w_\varkappa)_{0,\Omega_h} = 0 .$$

Using (4.3), we obtain

$$0 = (\boldsymbol{q}^{0h}, \nabla w_\varkappa)_{0,\Omega_\delta} \to (\boldsymbol{q}, \Delta w_\varkappa)_{0,\Omega_\delta} .$$

Passing to the limit for $\varkappa \to 0$ and uding (4.5), we arrive at

$$(\boldsymbol{q}, \nabla w)_{0,\Omega(v)} = (\boldsymbol{q}, \nabla \tilde{w})_{0,\Omega_\delta} = 0$$

and (4.4) is verified.

Next we show that

$$\boldsymbol{q} = 0 \quad \text{a.e. in} \quad \Omega_\delta - \Omega(v) . \tag{4.6}$$

In fact, let $\boldsymbol{q} \neq 0$ on a set $E \subset \Omega_\delta - \Omega(v)$, mes $E > 0$.

Let χ_E be the characteristic function of the set E. From (4.3) it follows for $h \to 0$ that

$$(\boldsymbol{q}^{0h}, \chi_E \boldsymbol{q})_{0,\Omega_\delta} \to (\boldsymbol{q}, \chi_E \boldsymbol{q})_{0,\Omega_\delta} = \|\boldsymbol{q}\|_{0,E}^2 > 0 .$$

On the other hand, we may write

$$(\boldsymbol{q}_h^0, \chi_E \boldsymbol{q})_{0,\Omega_\delta} = (\tilde{\boldsymbol{q}}^h, \boldsymbol{q})_{0,\Omega_h \cap E} \leqq \|\boldsymbol{q}^{0h}\|_{0,\Omega_h} \|\boldsymbol{q}\|_{0,\Omega_h \cap E} \to 0 ,$$

since (4.2) holds and

$$\text{mes}\,(\Omega_h \cap E) \to 0 .$$

Thus we come to a contradiction.

Let us show that $\boldsymbol{q}$ solves the dual problem (3.2). Let us consider a $\boldsymbol{p} \in Q_0(v)$. From Theorem 3 in [3] and from its proof we deduce that a sequence $\{\boldsymbol{p}^\varkappa\}$, $\varkappa \to 0$, exists such that

$$\boldsymbol{p}^\varkappa \in [C^\infty(\bar{\Omega}_\delta)]^2, \ \boldsymbol{p}^\varkappa|_{\Omega(v)} \in Q_0(v) ,$$

$$\text{supp}\, \boldsymbol{p}^\varkappa \cap (\partial\Omega(v) - \Gamma(v)) = \emptyset ,$$

$$\boldsymbol{p}^\varkappa|_{\Omega_h} \in Q_0(v_h) \quad \forall h < h_1(\varkappa) ,$$

$$\boldsymbol{p}^\varkappa \to \boldsymbol{p} \quad \text{in} \quad [L^2(\Omega(v))]^2 \quad \text{for} \quad \varkappa \to 0 . \tag{4.7}$$

In the paper [2] (see also [4]) a projection operator

$$r_h \colon [C^\infty(\bar{\Omega}_\delta)]^2 \cap Q_0(v_h) \to \mathcal{N}_h(v_h)$$

has been introduced. The properties of $\boldsymbol{p}^\varkappa$ and r_h imply that

$$r_h \boldsymbol{p}^\varkappa \in Q_0(v_h) \quad \forall h < h_1(\varkappa) .$$

By virtue of (4.1) we have

$$(\boldsymbol{q}^h, r_h \boldsymbol{p}^\varkappa)_{0,\Omega_h} = -(\bar{\lambda}, r_h \boldsymbol{p}^\varkappa)_{0,\Omega_h} . \tag{4.8}$$

Let us extend $r_h p^{\varkappa}$ by means of zero and denote the extension by the same symbol.
We may write

$$\left|(q^{0h}, r_h p^{\varkappa})_{0,\Omega_\delta} - (q, p^{\varkappa})_{0,\Omega_\delta}\right| \leqq \left|(q^{0h}, r_h p^{\varkappa})_{0,\Omega_\delta} - (q^{0h}, p^{\varkappa})_{0,\Omega_\delta}\right| + $$
$$+ \left|(q^{0h}, p^{\varkappa})_{0,\Omega_\delta} - (q, p^{\varkappa})_{0,\Omega_\delta}\right| .$$

The second term tends to zero by virtue of (4.3). The first term can be estimated as follows:

$$\left|(q^{0h}, r_h p^{\varkappa} - p^{\varkappa})_{0,\Omega_\delta}\right| \leqq \left\|q^{0h}\right\|_{0,\Omega_\delta} \left\|r_h p^{\varkappa} - p^{\varkappa}\right\|_{0,\Omega_h} \to 0 ,$$

where (4.2) and the following result (see Theorem 2.2 in [4]) has been used:

$$\left\|r_h p^{\varkappa} - p^{\varkappa}\right\|_{0,\Omega_h} \leqq C h^2 \left\|p^{\varkappa}\right\|_{2,\Omega_\delta} . \tag{4.9}$$

Consequently, using also (4.6), we obtain

$$(q^{0h}, r_h p^{\varkappa})_{0,\Omega_\delta} \to (q, p^{\varkappa})_{0,\Omega(v)} , \quad h \to 0 . \tag{4.10}$$

Furthermore, we can write

$$\left|(\bar{\lambda}, r_h p^{\varkappa})_{0,\Omega_h} - (\bar{\lambda}, p^{\varkappa})_{0,\Omega(v)}\right| \leqq \left|(\bar{\lambda}, r_h p^{\varkappa} - p^{\varkappa})_{0,\Omega_h}\right| + \left|(\bar{\lambda}, p^{\varkappa})_{0,\Omega_h} - \right. \tag{4.11}$$
$$\left. - (\bar{\lambda}, p^{\varkappa})_{0,\Omega(v)}\right| \to 0$$

for $h \to 0$, if we make use of (4.9) and

$$\text{mes}\, \Delta(\Omega_h, \Omega(v)) \to 0 .$$

Passing to the limit in the equation (4.8) and using (4.10), (4.11), we obtain

$$(q, p^{\varkappa})_{0,\Omega(v)} = -(\bar{\lambda}, p^{\varkappa})_{0,\Omega(v)} .$$

From the convergence (4.7) the equation (3.2) follows. Since the solution of (3.2) is unique, we arrive at $q = q(v)$.

On the basis of (4.1), we have

$$\left\|q^h\right\|^2_{0,\Omega_h} = -(\bar{\lambda}, q^h)_{0,\Omega_h} . \tag{4.12}$$

Consequently, using the weak convergence (4.3) and (3.2), we obtain

$$\left\|q^{0h}\right\|^2_{0,\Omega_\delta} = -(\bar{\lambda}, q^{0h})_{0,\Omega_\delta} \to -(\bar{\lambda}, q)_{0,\Omega_\delta} = -(\bar{\lambda}, q)_{0,\Omega(v)} =$$
$$= \left\|q(v)\right\|^2_{0,\Omega(v)} = \left\|q\right\|^2_{0,\Omega_\delta} .$$

Combining the weak convergence and the convergence of norms, we arrive at the strong convergence in $[L^2(\Omega_\delta)]^2$.

Since $q(v)$ is the unique solution of (3.2), the whole sequence $q^{0h}(v_h)$ converges to $q(v)$.

Lemma 5.1. *Let $\{v_h\}$, $h \to 0$, be a sequence of $v_h \in \mathscr{U}^h_{ad}$, converging uniformly to a function v. Let $q^h(v_{\hat{h}})$ be the solution of (4.1).*
Then

$$\mathscr{J}_1^*(\mathbf{q}^h(v_h)) \to \mathscr{J}_1^*(\mathbf{q}(v)) \quad \textit{for} \quad h \to 0\,,$$

where $\mathbf{q}(v)$ is the solution of (3.2).

Proof. By virtue of (4.12) and Lemma 4.1, we have

$$\mathscr{J}_1^*(\mathbf{q}^h(v_h)) = \|\bar{\lambda}\|^2_{0,\Omega_h} + (\bar{\lambda}, \mathbf{q}^h)_{0,\Omega_h} \to \|\bar{\lambda}\|^2_{0,\Omega(v)} + (\bar{\lambda}, \mathbf{q}(v))_{0,\Omega(v)} = \mathscr{J}_1^*(\mathbf{q}(v))\,,$$

where the last equation is a consequence of (3.2).

Theorem 5.1. *Let $\{\omega_h\}$, $h \to 0$, be a sequence of solutions of the following approximate problem*

$$\mathscr{J}_1^*(\mathbf{q}^h(\omega_h)) = \min\,, \quad \omega_h \in \mathscr{U}^h_{ad}\,. \tag{5.1}$$

Then a subsequence $\{\omega_{\hat{h}}\}$ exists such that for $\hat{h} \to 0$

$$\omega_{\hat{h}} \to \omega \quad \textit{in} \quad C([0, 1])\,,$$

and

$$\mathbf{q}^{0\hat{h}}(\omega_{\hat{h}}) \to \mathbf{q}(\omega) \quad \text{in} \quad [L^2(\Omega_\delta)]^2\,, \tag{5.2}$$

where $\mathbf{q}^{0\hat{h}}$ are the solutions of (4.1), *extended by means of zero, $\mathbf{q}(\omega)$ is the solution of* (3.2), *extended by means of zero and ω is a solution of* (1.0), $i = 1$. *Any uniformly convergent subsequence of $\{\omega_h\}$ tends to a solution of* (1.0) *and* (5.2) *holds.*

Proof. Let us consider a $v \in \mathscr{U}_{ad}$. There exists a sequence $\{v_h\}$, $h \to 0$, such that $v_h \in \mathscr{U}^h_{ad}$, $v_h \to v$ in $C([0, 1])$, (see e.g. [1], Lemma 7.1).

Since $\mathscr{U}_{ad}$ is compact in $C([0, 1])$, a subsequence $\{\omega_{\bar{h}}\}$ and $\omega \in \mathscr{U}_{ad}$ exist such that $\omega_{\bar{h}} \to \omega$ in $C([0, 1])$ for $\bar{h} \to 0$. By definition, we have

$$\mathscr{J}_1^*(\mathbf{q}^{\bar{h}}(\omega_{\bar{h}})) \leqq \mathscr{J}_1^*(\mathbf{q}^{\bar{h}}(v_{\bar{h}})) \quad \forall \bar{h}\,.$$

Applying Lemma 5.1 to both the sequences $\{\omega_{\bar{h}}\}$ and $\{v_{\bar{h}}\}$, we obtain

$$\mathscr{J}_1^*(\mathbf{q}(\omega)) \leqq \mathscr{J}_1^*(\mathbf{q}(v))\,.$$

Consequently, ω is a solution of the optimization problem (3.5), which is equivalent with (1.0). The assertion (5.2) follows from Lemma 4.1.

The rest of the Theorem is easy to prove by the argument used above.

Remark 5.1. The problem (5.1) has at least one solution for any h.

We have seen in Section 3, that the second cost functional can be written in terms of the solution u of an auxiliary problem (3.4). We shall solve instead an approximate problem, corresponding to (3.4), using the subspaces

$$V_h^c \subset C(\bar{\Omega}_h) \cap V^c(v_h)$$

of standard piecewise linear finite elements on the triangulations $\mathcal{T}_h(v_h)$.

We define the following problem: to find $u_h \in V_h^c$ such that

$$\int_{\Omega_h} (\nabla u_h \cdot \nabla w_h + u_h w_h)\,\mathrm{d}x = \int_{\Gamma_h} (\bar{\lambda} + \boldsymbol{q}^h) \cdot \nu w_h\,\mathrm{d}s \quad \forall w_h \in V_h^c . \tag{6.1}$$

Note that replacing $\Omega(v)$ by Ω_h, $\boldsymbol{q}(v)$ by $\boldsymbol{q}^h \in S_h$ and w by $w_h \in V_h^c$, the right-hand side of (3.4) can be transformed to that of (6.1) by means of the integration by parts. In fact, we have

$$\int_{\Omega_h} [(\bar{\lambda} + \boldsymbol{q}^h) \cdot \nabla w_h - f w_h]\,\mathrm{d}x = \int_{\Gamma_h} (\bar{\lambda} + \boldsymbol{q}^h) \cdot \nu w_h\,\mathrm{d}s . \tag{6.2}$$

The approximate second cost functional can be defined by means of (6.1) as follows

$$J_2^h(v_h) \equiv \mathcal{J}_{2h}^*(\boldsymbol{q}^h(v_h)) = \|u_h(\boldsymbol{q}^h(v_h))\|_{1,\Omega_h}^2 = \int_{\Gamma_h} (\bar{\lambda} + \boldsymbol{q}^h) \cdot \nu u_h\,\mathrm{d}s . \tag{6.3}$$

Then the optimization problem (1.0) for $i = 2$ will be replaced by the following approximate problem:

$$J_2^h(v_h) = \min , \quad v_h \in \mathcal{U}_{ad}^h . \tag{6.4}$$

To find the relation between the solutions of (6.4) and of the problem (1.0), we first have to analyze some properties of the solution of the problem (6.1), namely its dependence on the "control variable" v_h.

Lemma 6.1. *Let $\{v_h\}$, $h \to 0$, be a sequence of $v_h \in \mathcal{U}_{ad}^h$, converging uniformly to a function v.*

Then a subsequence $\{u_{\hat{h}}\}$ of solutions $\{u_h\}$ of the problem (6.1) exists such that for $\hat{h} \to 0$

$$\tilde{u}_{\hat{h}} \rightharpoonup u \quad (\text{weakly}) \text{ in } \quad H^1(\Omega_\delta) , \tag{6.5}$$

where $\tilde{u}_{\hat{h}}$ is the extension of $u_{\hat{h}}$ according to (2.10), "symmetric" with respect to the curve $\Gamma_{\hat{h}}$, and the restriction

$$u|_{\Omega(v)} = u(y(v)) = u(\boldsymbol{q}(v)) \tag{6.6}$$

is the solution of (1.10) or (3.4), respectively.

Proof. Inserting $w_h = u_h$ into (6.1) and using (6.2), (4.2), we obtain

$$\|u_h\|^2_{1,\Omega_h} \leqq \|\bar{\lambda} + q^h\|_{0,\Omega_h} \|\nabla u_h\|_{0,\Omega_h} + \|f\|_{0,\Omega_h} \|u_h\|_{0,\Omega_h} \leqq C\|u_h\|_{1,\Omega_h},$$

so that

$$\|u_h\|_{1,\Omega_h} \leqq C. \tag{6.7}$$

For the extension $\tilde{u}_h$ we may write, using an analogue of (2.11), (2.12) and (6.7), the following estimate

$$\|\tilde{u}_h\|^2_{1,\Omega_\delta} \leqq (1 + C_0)\, C^2. \tag{6.8}$$

Consequently, a subsequence of $\{\tilde{u}_h\}$ exists (and we shall denote it by the same symbol) such that

$$\tilde{u}_h \rightharpoonup u \quad \text{(weakly) in} \quad H^1(\Omega_\delta), \quad u \in H^1(\Omega_\delta). \tag{6.9}$$

Let a $w \in V^c(v)$ be given. There exists a sequence $\{w_\varkappa\}$, $\varkappa \to 0$, $w_\varkappa \in C^\infty(\bar{\Omega}_\delta)$, $w_\varkappa|_{\Omega(v)} \in$ $\in V^c(v)$,

$$\operatorname{supp} w_\varkappa \cap \Gamma_1 = \emptyset, \quad \Gamma_1 = \partial\Omega(v) - \Gamma_0$$

$$w_\varkappa \to w \quad \text{in} \quad H^1(\Omega(v)) \quad \text{for} \quad \varkappa \to 0.$$

Let $\pi_h w_\varkappa$ denote the Lagrange linear interpolate of $w_\varkappa$ over the triangulation $\mathcal{T}_h$; consequently, $\pi_h w_\varkappa \in V^c(v_h) \cap C(\bar{\Omega}_h)\ \forall h$.

Let $\varkappa$ be fixed, for the time being. Obviously, we can insert $\pi_h w_\varkappa$ into (6.1) and use (6.2) to obtain

$$(u_h, \pi_h w_\varkappa)_{1,\Omega_h} = \int_{\Omega_h} [(\bar{\lambda} + \boldsymbol{q}^h) \,.\, \nabla\pi_h w_\varkappa - f\pi_h w_\varkappa]\, dx. \tag{6.10}$$

We shall pass to the limit with $h \to 0$ in (6.10). Denoting by m positive integers and

$$G_m = \left\{(x_1, x_2) : 0 < x_1 < v(x_2) - \frac{1}{m},\ 0 < x_2 < 1\right\},$$

$G_m \subset \Omega_h$ for $h < h_0(m)$. Then we may write

$$\begin{aligned} &|(u_h, \pi_h w_\varkappa)_{1,\Omega_h} - (u, w_\varkappa)_{1,G_m}| = \\ &= |(u_h, w_\varkappa)_{1,G_m} + (u_h, \pi_h w_\varkappa - w_\varkappa)_{1,G_m} + (u_h, \pi_h w_\varkappa)_{1,\Omega_h - G_m} - \\ &\quad - (u, w_\varkappa)_{1,G_m}| \leqq |(u_h - u, w_\varkappa)_{1,G_m}| + \\ &\quad + |(u_h, \pi_h w_\varkappa - w_\varkappa)_{1,G_m}| + |(u_h, \pi_h w_\varkappa)_{1,\Omega_h - G_m}|. \end{aligned} \tag{6.11}$$

Consider a positive ε. From (6.9) it follows that the first term on the right-hand side of (6.11) is not greater than $\varepsilon/6$ if $h < h_1(\varepsilon, m)$.

To estimate the second term, we employ the well-known inequality

$$\|w_\varkappa - \pi_h w_\varkappa\|_{1,\Omega_h} \leqq Ch\|w_\varkappa\|_{2,\Omega_h} \leqq Ch\|w_\varkappa\|_{2,\Omega_\delta}. \tag{6.12}$$

Combining (6.7) and (6.12), we obtain

$$|(u_h, \pi_h w_\varkappa - w_\varkappa)_{1,G_m}| \leqq \tilde{C}h\|w_\varkappa\|_{2,\Omega_\delta} < \varepsilon/6 \tag{6.13}$$

for $h < h_2$.

It remains to estimate the third term. For all triangles $K \in \mathcal{T}_h$ and h we have

$$\|\pi_h w_\varkappa\|_{1,K} \leqq C\|w_\varkappa\|_{2,K}\,.$$

Let G_m^h be the smallest union U of triangles $K \in \mathcal{T}_h$ such that $U \supset \Omega_h - G_m$. Obviously, we may write

$$\text{mes } G_m^h \leqq \frac{1}{m} + 2h + \|v_h - v\|_\infty\,, \tag{6.14}$$

where $\|\cdot\|_\infty$ denotes the norm in $C([0, 1])$.

Consequently,

$$\|\pi_h w_\varkappa\|_{1,\Omega_h - G_m}^2 \leqq \|\pi_h w_\varkappa\|_{1,G_m^h}^2 = \sum_{K \in G_m^h} \|\pi_h w_\varkappa\|_{1,K}^2 \leqq C^2 \sum_{K \in G_m^h} \|w_\varkappa\|_{2,K}^2 = C^2\|w_\varkappa\|_{2,G_m^h}^2\,.$$

Using again (6.7), we may write

$$|(u_h, \pi_h w_\varkappa)_{1,\Omega_h - G_m}| \leqq \|u_h\|_{1,\Omega_h} \|\pi_h w_\varkappa\|_{1,\Omega_h - G_m} \leqq C\|w_\varkappa\|_{2,G_m^h}\,. \tag{6.15}$$

Combining (6.11), (6.13) and (6.15), we derive for $h < h_3(\varepsilon, m)$

$$|(u_h, \pi_h w_\varkappa)_{1,\Omega_h} - (u, w_\varkappa)_{1,\Omega(v)}| \leqq |(u_h, \pi_h w_\varkappa)_{1,\Omega_h} - (u, w_\varkappa)_{1,Gm}| +$$
$$+ |(u, w_\varkappa)_{1,\Omega(v) - G_m}| \leqq \varepsilon/3 + C\|w_\varkappa\|_{2,G_m^h} + \|u\|_{1,\Omega(v)} \|w_\varkappa\|_{1,\Omega(v) - G_m}\,.$$

By virtue of (6.14), we conclude for $h \to 0$ that

$$(u_h, \pi_h w_\varkappa)_{1,\Omega_h} \to (u, w_\varkappa)_{1,\Omega(v)}\,. \tag{6.16}$$

Furthermore, we have

$$|(\bar{\lambda} + \boldsymbol{q}^h, \nabla\pi_h w_\varkappa)_{0,\Omega_h} - (\bar{\lambda} + \boldsymbol{q}(v), \nabla w_\varkappa)_{0,\Omega(v)}| \leqq \tag{6.17}$$
$$\leqq |(\bar{\lambda} + \boldsymbol{q}^h, \nabla\pi_h w - \nabla w_\varkappa)_{0,\Omega_h} + (\bar{\lambda} + \boldsymbol{q}^h, \nabla w_\varkappa)_{0,\Omega_h} -$$
$$- (\bar{\lambda} + \boldsymbol{q}^h, \nabla w_\varkappa)_{0,G_m} + (\bar{\lambda} + \boldsymbol{q}^h, \nabla w_\varkappa)_{0,G_m} - (\bar{\lambda} + \boldsymbol{q}(v), \nabla w_\varkappa)_{0,\Omega(v)}| \leqq$$
$$\leqq |\bar{\lambda} + \boldsymbol{q}^h, \nabla(\pi_h w_\varkappa - w_\varkappa))_{0,\Omega_h}| + |(\bar{\lambda} + \boldsymbol{q}^h, \nabla w_\varkappa)_{0,\Omega_h - G_m}| +$$
$$+ |(\bar{\lambda} + \boldsymbol{q}^h, \nabla w_\varkappa)_{0,G_m} - (\bar{\lambda} + \boldsymbol{q}(v), \nabla w_\varkappa)_{0,\Omega(v)}| = I_{1h} + I_{2h} + I_{3h}\,.$$

Using (4.2) and (6.12), we can write

$$I_{1h} \leqq (\|\bar{\lambda}\|_{0,\Omega_\beta} + \|\boldsymbol{q}^h\|_{0,\Omega_h}) \|\pi_h w_\varkappa - w_\varkappa\|_{1,\Omega_h} \to 0\,, \tag{6.18}$$

for $h \to 0$,

$$I_{2h} \leqq (\|\bar{\lambda}\|_{0,\Omega_\beta} + \|\boldsymbol{q}^h\|_{0,\Omega_h}) \|\nabla w_\varkappa\|_{0,\Omega_h - G_m} \to 0 \tag{6.19}$$

for $m \to \infty$, $h < h_0(m)$, $h \to 0$.

Finally, making use of (4.3), we obtain

$$(6.20)\qquad I_{3h} \leqq |(\bar{\lambda} + \mathbf{q}^h, \nabla w_\varkappa)_{0,G_m} - (\bar{\lambda} + \mathbf{q}(v), \nabla w_\varkappa)_{0,G_m}| + \\ + |(\bar{\lambda} + \mathbf{q}(v), \nabla w_\varkappa)_{0,\Omega(v)-G_m}| = \\ = |(\mathbf{q}^h - \mathbf{q}(v), \nabla w_\varkappa)_{0,G_m}| + |(\bar{\lambda} + \mathbf{q}(v), \nabla w_\varkappa)_{0,\Omega(v)-G_m}| \to 0$$

for $m \to \infty$, $h < h_1(m)$, $h \to 0$.

Combining (6.17)–(6.20), we deduce for $h \to 0$

$$(6.21)\qquad (\bar{\lambda} + \mathbf{q}^h, \nabla \pi_h w_\varkappa)_{0,\Omega_h} \to (\bar{\lambda} + \mathbf{q}(v), \nabla w_\varkappa)_{0,\Omega(v)}\,.$$

We also have

$$(6.22)\qquad |(f, \pi_h w_\varkappa)_{0,\Omega_h} - (f, w_\varkappa)_{0,\Omega(v)}| \leqq \\ \leqq |(f, \pi_h w_\varkappa - w_\varkappa)_{0,\Omega_h}| + |(f, w_\varkappa)_{0,\Omega_h} - (f, w_\varkappa)_{0,\Omega(v)}| \leqq \\ \leqq \|f\|_{0,\Omega_\beta} \| \pi_h w_\varkappa - w_\varkappa\|_{0,\Omega_h} + |(f, w_\varkappa)_{0,\,\Delta(\Omega_h,\Omega(v))}| \to 0$$

for $h \to 0$, where (6.12) has been used.

Passing to the limit with $h \to 0$ in (6.10) and using (6.16), (6.21) and (6.22), we derive the equation

$$(u, w_\varkappa)_{1,\Omega(v)} = (\bar{\lambda} + \mathbf{q}(v), \nabla w_\varkappa)_{0,\Omega(v)} - (f, w_\varkappa)_{0,\Omega(v)}\,.$$

Passing to the limit with $\varkappa \to 0$, we obtain

$$(u, w)_{1,\Omega(v)} = (\bar{\lambda} + \mathbf{q}(v), \nabla w)_{0,\Omega(v)} - (f, w)_{0,\Omega(v)}\,,$$

i.e., (3.4) is fulfilled by the restriction $u|_{\Omega(v)}$.

The space $V^c(\Omega_\delta)$ is weakly closed in $H^1(\Omega_\delta)$. Since $\tilde{u}_h \in V^c(\Omega_\delta)$, $u \in V^c(\Omega_\delta)$ follows and $u|_{\Omega(v)} \in V^c(v)$. The uniqueness of solution of (3.4) implies the assertion (6.6).

Lemma 6.2. *Let the assumption of Lemma* 6.1 *be satisfied. Then a subsequence* $\{\tilde{h}\}$, $\tilde{h} \to 0$, *exists such that*

$$\mathscr{J}^*_{2\tilde{h}}(\mathbf{q}^{\tilde{h}}(v_{\tilde{h}})) \to \mathscr{J}^*_2(\mathbf{q}(v)) \quad \textit{for} \quad \tilde{h} \to 0\,,$$

where $\mathbf{q}(v)$ *is the solution of* (3.2), $\mathscr{J}^*_{2h}$ *is defined by means of* (6.3).

Proof. On the basis of (6.3), (6.2) we may write

$$(6.23)\qquad \mathscr{J}^*_{2h}(\mathbf{q}^h(v_h)) = \int_{\Omega_h} [(\bar{\lambda} + \mathbf{q}^h) \cdot \nabla u_h - f u_h]\, dx\,,$$

where u_h is the solution of (6.1).

Henceforth, we shall denote all subsequences of $\{h\}$ by the same unchanged symbol.

First we have

$$\left|(\bar{\lambda}, \nabla u_h)_{0,\Omega_h} - (f, u_h)_{0,\Omega_h} - (\bar{\lambda}, \nabla u)_{0,\Omega(v)} + (f, u)_{0,\Omega(v)}\right| \leqq$$
$$\leqq \left|(\bar{\lambda}, \nabla u_h)_{0,\Omega_h} - (\bar{\lambda}, \nabla \tilde{u}_h)_{0,\Omega(v)}\right| + \left|(\bar{\lambda}, \nabla \tilde{u}_h - \nabla u)_{0,\Omega(v)}\right| +$$
$$+ \left|(f, u_h)_{0,\Omega h} - (f, \tilde{u}_h)_{0,\Omega(v)}\right| + \left|(f, \tilde{u}_h - u)_{0,\Omega(v)}\right| =$$
$$= I_{1h} + I_{2h} + I_{3h} + I_{4h} .$$

Using (6.8), we obtain

$$I_{1h} \leqq \left|(\bar{\lambda}, \nabla \tilde{u}_h)_{0,\, \Delta(\Omega_h,\Omega(v))}\right| \leqq \left\|\tilde{u}_h\right\|_{1,\Omega_\delta} \left\|\bar{\lambda}\right\|_{0,\, \Delta(\Omega_h,\Omega(v))} \to 0 .$$

From the weak convergence (6.9) it follows that

$$I_{2h} \to 0 , \quad I_{4h} \to 0 .$$

Furthermore, we have

$$I_{3h} \leqq \left\|\tilde{u}_h\right\|_{0,\Omega_\delta} \left\|f\right\|_{0,\, \Delta(\Omega_h,\Omega(v))} \to 0 .$$

Altogether, we can write

$$(6.24) \qquad (\bar{\lambda}, \nabla u_h)_{0,\Omega_h} - (f, u_h)_{0,\Omega_h} \to (\bar{\lambda}, \nabla u)_{0,\Omega(v)} - (f, u)_{0,\Omega(v)} .$$

Next we shall estimate (for $\mathbf{q}(v) \equiv \mathbf{q}$)

$$\left|(\mathbf{q}^h, \nabla u_h)_{0,\Omega_h} - (\mathbf{q}, \nabla u)_{0,\Omega(v)}\right| \leqq \left|(\mathbf{q}^h, \nabla u_h)_{0,\Omega_h} - (\mathbf{q}, \nabla u_h)_{0,\Omega_h}\right| +$$
$$+ \left|(\mathbf{q}, \nabla u_h)_{0,\Omega_h} - (\mathbf{q}, \nabla \tilde{u}_h)_{0,\Omega(v)}\right| + \left|(\mathbf{q}, \nabla \tilde{u}_h)_{0,\Omega(v)} - (\mathbf{q}, \nabla u)_{0,\Omega(v)}\right| =$$
$$= I_{5h} + I_{6h} + I_{7h} .$$

By virtue of Lemma 4.1 and (6.7) we may write

$$I_{5h} \leqq \left\|\tilde{\mathbf{q}}^h - \mathbf{q}\right\|_{0,\Omega_\delta} \left\|u_h\right\|_{1,\Omega_h} \to 0 ;$$

using also (6.8), we obtain

$$I_{6h} \leqq \left\|\mathbf{q}\right\|_{0,\Delta(\Omega_h,\Omega(v))} \left\|\tilde{u}_h\right\|_{1,\Omega_\delta} \to 0 .$$

Finally,

$$I_{7h} \to 0$$

follows from the weak convergence (6.9).

We combine the latter three results to obtain

$$(6.25) \qquad (\mathbf{q}^h, \nabla u_h)_{0,\Omega_h} \to (\mathbf{q}, \nabla u)_{0,\Omega(v)} .$$

Making use of (6.23), (6.24) and (6.25), we arrive at

$$\mathscr{I}^*_{2h}(\mathbf{q}^h(v_h)) \to (\bar{\lambda} + \mathbf{q}(v), \nabla u)_{0,\Omega(v)} - (f, u)_{0,\Omega(v)} .$$

By comparison of the limit with the right-hand side of (3.4), one finds the assertion of the Lemma 6.2.

Remark 6.3. The problem (6.4) has at least one solution for any h.

Theorem 6.1. *Let* $\{v_h\}$, $h \to 0$, *be a sequence of solutions of the approximate problem* (6.4). *Then a subsequence* $\{v_{\bar{h}}\}$, $\bar{h} \to 0$, *exists such that*

$$v_{\bar{h}} \to v \quad in \quad C([0, 1]),$$

where v *is a solution of the problem* (1.0), $i = 2$.

The corresponding solutions $\mathbf{q}^{\bar{h}}(v_{\bar{h}})$ *of the approximate state problem* (4.1) *and the solutions* $u_{\bar{h}}(\mathbf{q}^{\bar{h}}(v_{\bar{h}}))$ *of the problem* (6.1) *converge in accordance with Lemma* 4.1. *and Lemma* 6.1, *respectively.*

Any uniformly convergent subsequence of $\{v_h\}$ *has the properties mentioned above (the limit is a solution of* (1.0) *a.s.o.).*

Proof. Consider an arbitrary $\eta \in \mathscr{U}_{ad}$ and a sequence $\{\eta_h\}$, $h \to 0$, such that $\eta_h \in \mathscr{U}^h_{ad}$, $\eta_h \to \eta$ in $C([0, 1])$ (see [1] – Lemma 7.1).

Since $\mathscr{U}_{ad}$ is compact in $C([0, 1])$, a subsequence $\{v_{\bar{h}}\}$ and $v \in \mathscr{U}_{ad}$ exist such that $v^h \to v$ in $C([0, 1])$.

By definition, we have (see (6.3), (6.4))

$$J_2^{\bar{h}}(v_{\bar{h}}) \leqq J_2^{\bar{h}}(\eta_{\bar{h}}) \quad \forall \bar{h} . \tag{6.26}$$

Applying the Lemma 6.2 to both sequences in (6.26), we arrive at

$$\mathscr{J}_2^*(\mathbf{q}(v)) \leqq \mathscr{J}_2^*(\mathbf{q}(\eta)) .$$

Consequently, v is a solution of (3.5), which is equivalent with (1.0). The rest of the Theorem is obvious.

7. SEVERAL REMARKS ON THE NUMERICAL SOLUTION

To solve the approximate optimization problems (5.1) and (6.4), respectively, one has to apply some algorithm of nonlinear programming (see [10], [11] et al.). It is well-known, that an efficient algorithm requires the knowledge of the gradient of the cost functional.

To calculate the gradient, one can use the method of an adjoint problem. We shall sketch the latter approach briefly on the example of the problem (5.1).

We may write

$$\mathbf{q}^h = \sum_{j=1}^{n} x_j \psi^j ,$$

where $\{\psi^j\}_1^n$ are the basis functions of the space S_h, x_j are real coefficients and denote

$$v_h(ih) = v_i\,, \quad i = 0, 1, \ldots, N\,.$$

Then

$$\mathscr{J}_1^*(\mathbf{q}^h(v_h)) = j_h(\mathbf{v}, x(\mathbf{v})) \equiv J(\mathbf{v})\,, \quad \mathbf{v} \in \mathbb{R}^{N+1}\,,$$

and the approximate state problem (4.1) takes the form of the following system of n linear equations

$$A(\mathbf{v})\,\mathbf{x} = \mathscr{F}(\mathbf{v}) \tag{7.1}$$

(see [7] – Lemma 2.2).

The problem

$$A(\mathbf{v})\,\xi = \frac{\partial j_h}{\partial \mathbf{x}}(\mathbf{v}, \mathbf{x}(\mathbf{v})) \tag{7.2}$$

is called adjoint to (7.1). It is not difficult to derive that

$$\nabla J(\mathbf{v}) = \frac{\partial j_h}{\partial \mathbf{v}}(\mathbf{v}, \mathbf{x}(\mathbf{v})) + \left[\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\mathbf{v}}\right]^{\mathrm{T}} \xi - \left[\frac{\mathrm{d}A(\mathbf{v})}{\mathrm{d}\mathbf{v}}\,\mathbf{x}(\mathbf{v})\right]^{\mathrm{T}} \xi\,,$$

where ξ is the solution of (7.2).

The matrices

$$\frac{\partial j_h}{\partial \mathbf{v}}\,, \quad \frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\mathbf{v}}\,, \quad \frac{\mathrm{d}A(\mathbf{v})}{\mathrm{d}\mathbf{v}}$$

can be assembled from "local" increments, i.e. from those parts connected with single triangular elements. For details, see e.g. [9], where an analogous optimization problem has been solved.

References

[1] *D. Begis, R. Glowinski:* Application de la méthode des éléments finis à l'approximation d'un problème de domaine optimal. Appl. Math. & Optim. 2 (1975), 130–169.

[2] *J. Haslinger, I. Hlaváček:* Convergence of a finite element method based on the dual variational formulation. Apl. Mat. 21 (1976), 43–65.

[3] *I. Hlaváček:* The density of solenoidal functions and the convergence of a dual finite element method. Apl. Mat. 25 (1980), 39–55.

[4] *I. Hlaváček:* Dual finite element analysis for some elliptic variational equations and inequalities. Acta Applic. Math. 1, (1983), 121–20.

[5] *J. Haslinger, J. Lovíšek:* The approximation of the optimal shape problem governed by a variational inequality with flux cost functional. To appear in Proc. Roy. Soc. Edinburgh.

[6] *I. Hlaváček, J. Nečas:* Optimization of the domain in elliptic unilateral boundary value problems by finite element method. R.A.I.R.O. Anal. numér, 16, (1982), 351–373.

[7] *M. Křížek:* Conforming equilibrium finite element methods for some elliptic plane problems R.A.I.R.O. Anal. numér. 17, (1983), 35–65.

[8] *J. Haslinger, P. Neittaanmäki:* On optimal shape design of systems governed by mixed Dirichlet-Signorini boundary value problems. To appear in Math. Meth. Appl. Sci.

[9] *P. Neittaanmäki, T. Tiihonen:* Optimal shape design of systems governed by a unilateral boundary value problem. Lappeenranta Univ. of Tech., Dept. of Physics and Math., Res. Rept. 4/1982.
[10] *B. A. Murtagh:* Large-scale linearly constrained optimization. Math. Programming, 14 (1978), 41–72.
[11] *R. Fletcher:* Practical methods of optimization, vol. 2, constrained optimization. J. Wiley, Chichester, 1981.

Souhrn

OPTIMALIZACE OBLASTI V ELIPTICKÝCH ÚLOHÁCH DUÁLNÍ METODOU KONEČNÝCH PRVKŮ

IVAN HLAVÁČEK

Vyšetřuje se úloha najít optimální část hranice oblasti pro kombinovanou okrajovou úlohu s Poissonovou rovnicí. Účelový funkcionál je buď (a) vnitřní energie, tj. Dirichletův integrál nebo (b) norma vnějšího toku hranicí.

K numerickému řešení stavové úlohy se užívá duální variační formulace – prostřednictvím gradientu řešení, a prostory solenoidálních po částech lineárních funkcí.

Dokazuje se existence optimální oblasti a některé konvergenční výsledky.

Author's address: Ing. *Ivan Hlaváček*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.