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PROJECTIVE PLANE MOTIONS WITH INFINITELY MANY STRAIGHT TRAJECTORIES

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1. INTRODUCTION

The elliptical motion in the Euclidean plane is characterized as the only Euclidean plane motion with infinitely many straight trajectories (apart from a pure translation). Another characteristic property is that all trajectories of this motion are affinely equivalent (we again leave out translations). In the present paper we study the connection between these two properties in the (obvious) projective generalization. We define projective Darboux motions and show that they are motions with projectively equivalent trajectories. In the main part of the paper we classify all projective plane motions with the property that each point of the inflexion cubic has a straight trajectory. We also prove that they form a comparatively large class of motions.

Further on we show that the inflexion cubic of such a motion is in general irreducible and we give two examples. We also prove that these motions are in general Darboux motions.

Throughout the whole paper we suppose that the inflexion cubic is not trivial (not all trajectories are straight lines). The method used in the paper is that of [3], which modifies to the projective situation without any difficulties.

2. PRELIMINARIES

Let $V_{n+1}(\bar{V}_{n+1})$ be an $n + 1$ dimensional vector space over R, let $i : V_{n+1} - \{0\} \to P_n(\bar{\imath} : \bar{V}_{n+1} - \{0\} \to \bar{P}_n)$ be the natural projection onto the corresponding projective space P_n ($\bar{P}_n$, respectively). $\bar{P}_n$ is called the moving space, P_n is called the fixed space. Let us fix a base $\mathscr{R}_0 = \{f_0, \ldots, f_n\}$ ($\bar{\mathscr{R}}_0 = \{\bar{f}_0, \ldots, \bar{f}_n\}$) in $V_{n+1}(\bar{V}_{n+1})$, respectively. Further, let G denote the group $SL(n + 1, R)$, let $\mathfrak{G}$ be its Lie algebra. By a frame we shall mean any base $\mathscr{R}(\bar{\mathscr{R}})$ in $V_{n+1}(\bar{V}_{n+1})$ such that $\mathscr{R} = \mathscr{R}_0 \cdot \gamma_1$ ($\bar{\mathscr{R}} = \bar{\mathscr{R}}_0 \cdot \gamma_2$, respectively) for some $\gamma_1, \gamma_2 \in G$.

The group G acts as the group of projective transformations from $\bar{P}_n$ into P_n by the rule $g(\bar{\mathscr{R}}_0) = \mathscr{R}_0 \cdot g$ for any $g \in G$ through i and $\bar{i}$. A projective motion in P_n is by definition an immerson of an open interval I into G.

Let a projective motion $g(t)$: $I \to G$ be given. If we replace the basic frame in $V_{n+1}(\bar{V}_{n+1})$ by $\mathscr{R}_1 = \mathscr{R}_0 \cdot \gamma_1$ ($\bar{\mathscr{R}}_1 = \bar{\mathscr{R}}_0 \cdot \gamma_2$, respectively), $\gamma_1, \gamma_2 \in G$, the motion $g(t)$ changes to $\tilde{g}(t) = \gamma_1^{-1} g(t) \gamma_2$, as $g(\bar{\mathscr{R}}_1) = g(\bar{\mathscr{R}}_0 \gamma_2) = \mathscr{R}_0 g \gamma_2 = \mathscr{R}_1 \gamma_1^{-1} g \gamma_2$. This means that the motions $g(t)$ and $\tilde{g}(t)$ have to be considered as equivalent and we see that the group G has to be regarded as the homogeneous space $G \times G/\text{diag}\, G$ with the natural action $(g_1, g_2)(g) = g_1 g g_2^{-1}$, the isotropy group $\text{diag}\, G = \{(g, g) \mid | g \in G\}$ and the projection π: $\pi(g_1, g_2) = g_1 g_2^{-1}$.

By a moving frame $(\mathscr{R}(t), \bar{\mathscr{R}}(t))$ of a given motion $g(t)$ we mean any lift of $g(t)$ from G into $G \times G$. It is any pair of frames $(\mathscr{R}(t), \bar{\mathscr{R}}(t))$ such that $g(\bar{\mathscr{R}}(t)) = \mathscr{R}(t)$. (Let $\mathscr{R}(t) = \mathscr{R}_0\, g_1(t)$, $\bar{\mathscr{R}}(t) = \bar{\mathscr{R}}_0\, g_2(t)$. Then $g(t)(\bar{\mathscr{R}}(t)) = g(t)(\bar{\mathscr{R}}_0\, g_2(t)) = \mathscr{R}_0 \cdot \cdot g_2(t) = \mathscr{R}_0\, g_1(t)$ and so $g(t) = g_1(t)\, g_2^{-1}(t)$.)

Let $(\mathscr{R}(t), \bar{\mathscr{R}}(t))$ be a moving frame of a projective motion $g(t)$. Let us denote $\mathscr{R}' = \mathscr{R}\varphi$, $\bar{\mathscr{R}}' = \bar{\mathscr{R}}\psi$, $\varphi - \psi = \omega$, $\psi + \psi = \eta$. Then $\varphi, \psi, \omega, \eta \in \mathfrak{G}$ ($\text{Tr}\omega = \text{Tr}\eta = \text{Tr}\,\varphi = \text{Tr}\,\psi = 0$). Now we shall compute the operator of the k-th derivative of the trajectory $\mathscr{X}(t)$ of a point $\bar{\mathscr{X}} = \bar{\mathscr{R}}(t)\,\bar{X}(t)$ of the moving space. $\bar{X}(t)$ is fixed up to a factor and it is called the representative of the point $\bar{\mathscr{X}}$ in the frame $\bar{\mathscr{R}}(t)$. For the trajectory we have $\mathscr{X}(t) = \lambda(t)\,\mathscr{R}(t)\,\bar{X}(t)$. First we shall choose a fixed representative of $\bar{\mathscr{X}}$, $\bar{\mathscr{X}} = \bar{\mathscr{R}}_0 \bar{X}_0 = \bar{\mathscr{R}}(t)\,\bar{X}(t)$ and compute the derivatives with respect to the representative $\mathscr{X}(t) = \mathscr{R}(t)\,\bar{X}(t) = \mathscr{R}(t)\,X(t)$ of the trajectory of $\bar{\mathscr{X}}$. Then we have $\bar{\mathscr{X}}' = 0 = = \bar{\mathscr{R}}'\bar{X} + \bar{\mathscr{R}}\bar{X}' = \bar{\mathscr{R}}(\psi X + X')$ and so $\bar{X}' = -\psi\bar{X}$. Let Ω_k be the operator of the k-th derivative with respect to the chosen representatives. Then $\mathscr{X}^{(k)} = \mathscr{R}\Omega_k X$, $\Omega_0 = = E$, where E is the unit matrix and $\mathscr{X}^{(k+1)} = \mathscr{R}'\Omega_k X + \mathscr{R}\Omega_k' X + \mathscr{R}\Omega_k X' = = \mathscr{R}(\varphi\Omega_k - \Omega_k\psi + \Omega')\,X$. So we have

$$\Omega_{k+1} = \varphi\Omega_k - \Omega_k\psi + \Omega_k', \quad \Omega_0 = E. \tag{1}$$

Now let the representatives be arbitrary, so let us write $\bar{X}_1 = \lambda(t)\,\bar{X}$. Then we have $\bar{X}_1' = \lambda'\bar{X} + \lambda\bar{X}' = (\lambda' E - \lambda\psi)\,\bar{X} = (\nu E - \psi)\,\bar{X}_1$ where $\nu = \lambda'\lambda^{-1}$. Similarly, let $X_1(t) = \mu(t)\,X(t)$. Then $X_1' = \mu' X + \mu X' = [(\mu'\mu^{-1} + \nu)\,E - \psi]\,X_1$. If $\mathscr{X}^{(k)} = = \mathscr{R}\tilde{\Omega}_k X_1$, then $\mathscr{X}^{(k+1)} = \mathscr{R}'\tilde{\Omega}_k X_1 + \mathscr{R}\tilde{\Omega}_k' X_1 + \mathscr{R}\tilde{\Omega}_k X_1 = \mathscr{R}[\varphi\tilde{\Omega}_k - \tilde{\Omega}_n\psi + \tilde{\Omega}_k' + + (\mu'\mu^{-1} + \nu)\,\tilde{\Omega}_k]\,X_1$. So we get the following

Lemma 1. *The functions $\bar{X}(t)$ represent a fixed point of the moving space P_n iff $\bar{X}' = (\lambda E - \psi)\,X$ for some function λ; similarly, $X' = (\lambda E - \psi)\,X$ holds for the representatives of the fixed points of the fixed space P_n. The operator $\tilde{\Omega}_k$ of the k-th derivative of the trajectory of a point is given by the formula*

$$\tilde{\Omega}_k = \sum_{i=0}^{k} \beta_i \Omega_i \tag{2}$$

where $\beta_k = 1$, β_i are functions and Ω_i are given by (1).

Definition 1. *A projective motion $g(t)$ is called a D_r motion (a motion with the Darboux property of degree r) if there exist unique functions $\alpha_0(t), \ldots, \alpha_r(t)$, such that*

$$\tilde{\Omega}_{r+1} = \sum_{i=0}^{r} \alpha_i \tilde{\Omega}_i \,. \tag{3}$$

Remark. The D_r property is a geometrical property of a motion (it does not depend of the lift, representations and parameter) and according to Lemma 1 it can be defined by the operators Ω_i as well. The number r is the least number with the property (3) due to the unicity of α_i. Similarly as in [3] we have the following characterization of projective D_r motions:

Theorem 1. *A projective motion in P_n has the D_r property iff there exists a regular curve in P_r such that the trajectory of any point is a projective image of this curve. Then the trajectory of any point lies in a subspace of P_n of dimension at most r.*

Proof. Each trajectory is a solution of the differential equation

$$X^{(r+1)} - \sum_{i=0}^{r} \alpha_i X^{(i)} = 0 \,.$$

Remark. The projective map in Theorem 1 need not be regular. By a regular curve we mean such a curve that the r-th osculating space has dimension r at each t.

Let now $v_1, \ldots, v_r$ be vectors in V_{n+1}. By $|v_1, \ldots, v_r|^j$, $j = 1, \ldots, \binom{n+1}{r}$ we mean all subdeterminants of coordinates of the vectors $v_1, \ldots, v_r$ in some frame $\mathcal{R}$.

Theorem 2. *Let $g(t)$ be a projective D_r motion. Then the trajectory of any point X of $\bar{P}_n$ satisfying $|X, X', \ldots, X^{(r)}|^j = 0$ for $j = 1, \ldots \binom{n+1}{r+1}$ lies in a subspace of P_n of dimension at most $r - 1$.*

Proof. The proof is almost identical with the proof of Theorem 4 in [3].

3. PROJECTIVE PLANE MOTIONS

In what follows we shall prove the converse of Theorem 2 for projective plane motions. So from now on we shall consider only projective plane motions. The matrix ω can be given the real normal Jordan form in a suitable lift of the motion and in the following we shall restrict ourselves only to such lifts. Different Jordan normal forms of ω lead to 5 types of projective motions ($\mathrm{Tr}\,\omega = 0$):

Type 1. $\omega = \mathrm{Diag}\{\lambda_1, \lambda_2, \lambda_3\}$, λ_i mutually different.
Type 2. $\omega = \mathrm{Diag}\{1, 1, -2\}$.

Type 3. $\omega = \begin{pmatrix} 1, & 1, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -2 \end{pmatrix}$,

Type 4. $\omega = \begin{pmatrix} 0, & \lambda, & 0 \\ 0, & 0, & \lambda \\ 0, & 0, & 0 \end{pmatrix}$, $\quad \lambda \neq 0$.

Type 5. $\quad = \begin{pmatrix} \lambda, & 1, & 0 \\ -1, & \lambda, & 0 \\ 0, & 0, & -2\lambda \end{pmatrix}$.

(In cases 2, 3, 5 we have chosen the canonical parameter).

First we shall prove a partial result showing that in the case of the general motion the converse of Theorem 2 is always true. To do it we shall need some definitions.

The set of all points of the moving plane which satisfy the equation

$$F(X) \equiv |X, X', X''| = |\Omega_0 X, \Omega_1 X, \Omega_2 X| = 0$$

is called the inflection cubic and will be denoted by F.

Definition 2. *The inflexion cubic F is called simple if it has the following property: Let $G(X)$ be a homogeneous polynomial of degree 3 such that $G(X) = 0$ for all points of F. Then $G(X) = \lambda\, F(X)$ for some $\lambda \in R$.*

Remark. The inflexion cubic is simple in the following three cases:

a) it is irreducible;
b) it splits into a straight line and a regular real conic section:
c) it splits into three district real straight lines.

In the other cases F is not simple (a straight line and an imaginary conic section, a straight line and a point, two lines and one line with the corresponding multiplicities).

Theorem 3. *Let a projective motion have the following properties:*

a) *The inflexion cubic is simple.* b) *All points of the inflexion cubic satisfy the equation $G(X) = |X, X'', X'''| = 0$. Then all points of the inflexion cubic have at most one-dimensional trajectories and the motion is a D_2 motion.*

Remark. The condition that $F(X) = 0$ implies $G(X) = 0$ is obviously necessary for a point X to have a straight trajectory.

In what follows, by a Darboux motion we understand a 2-Darboux motion.

Proof. Without any loss of generality, to any Ω_k we may add a linear combination of the preceding ones, which we shall do without mentioning it. To prove the theorem we shall consider each of the 5 types of the projective plane motions separately.

Type 1. $\Omega_0 = E$, $\Omega_1 = \text{Diag}\{0, 1, \mu\}$, $\mu \neq 0, 1$, $\Omega_2 = (c_j^i)$, where $c_1^1 = c_2^2 = 0$. Coordinates in the moving plane with respect to a suitable moving frame will be denoted by x, y, z. Then

$$F(X) = \begin{vmatrix} x, & 0, & c_2^1 y + c_3^1 z \\ y, & y, & c_1^2 x + c_3^2 z \\ z, & \mu z, & c_1^3 x + c_2^3 y + c_3^3 z \end{vmatrix} =$$

$$= c_1^3 x^2 y + c_2^3 x y^2 - c_1^2 \mu x^2 z - c_3^2 \mu x z^2 + (\mu - 1)\, c_2^1 y^2 z + c_3^1(\mu - 1)\, y z^2 + c_3^3 x y z.$$

A similar expression is obtained for $G(X)$ with $\Omega_3 = (d_j^i)$, $d_1^1 = d_2^2 = 0$. As F is simple, we have $G(X) = \lambda\, F(X)$, which immediately yields $\Omega_3 = \lambda \Omega_2$ and the statement follows.

Type 2. In this case F is never simple, as we shall see later on.

Type 3. $\Omega_0 = E, \Omega_1 = \begin{pmatrix} 0, & 1, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & -3 \end{pmatrix}$, $\Omega_2 = (c_j^i)$ with $c_1^1 = c_1^2 = 0$

and similarly for $\Omega_3 = (d_j^i)$. Computation yields $F(X) = -c_1^3 x y^2 - c_2^3 y^3 + (c_2^2 - c_3^3)\, z y^2 + (c_3^2 - 3c_3^1)\, y z^2 + 3c_1^2 x^2 z + 3c_3^2 x z^2 + (3c_2^2 + c_1^2)\, x y z$. From $G(X) = \lambda\, F(X)$ we get
$d_1^3 = \lambda c_1^3$, $d_2^3 = \lambda c_2^3$, $d_1^2 = \lambda c_1^2$, $d_3^2 = \lambda c_3^2$, $d_3^2 - 3c_3^1 = \lambda(c_3^2 - 3c_3^1)$ and so $d_3^1 = \lambda c_3^1$, $3d_2^2 + d_1^2 = \lambda(3c_2^2 + c_1^2)$ and $d_2^2 = \lambda c_2^2$. Finally, $d_2^2 - d_3^3 = \lambda(c_2^2 - c_3^3)$ and $d_3^3 = \lambda c_3^3$. This gives $\Omega_2 = \lambda \Omega_2$ and the statement follows. Types 4 and 5 are treated in quite a similar way.

Remark. If a projective plane motion satisfies the assumptions of Theorem 3, then any part of the moving centrode belongs to the preimage of the inflexion cubic. To see it, consider a pole X of the instantaneous motion. Then its preimage $\bar{X}$ belongs to the preimage of the inflexion cubic, which is fixed in the moving system. But $\bar{X}$ belongs to the moving centrode as well.

4. PROJECTIVE PLANE MOTIONS WITH F NOT SIMPLE

Now it remains to consider motions with inflexion cubic that is not simple. In the course of discussion of reducibility of F we shall also prove the existence of Darboux motions and give an example of a Darboux motion with irreducible inflexion cubic.

Type 1. First we shall prove the existence of Darboux motions. Denote

$$\omega = 2\,\text{Diag}\{\lambda_1, \lambda_2, \lambda_3\}, \quad \eta = 2 \begin{pmatrix} m_1, & a_2, & b_2 \\ a_1, & m_2, & c_2 \\ b_1, & c_1 & m_3 \end{pmatrix},$$

$\mu = (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)^{-1}$, $\mu \neq 0, 1$, $\sigma = 2\mu(\lambda_3 - \lambda_2) + \mu'$, $m_1 - m_2 = k_3$, $m_2 - m_3 = k_1$, $m_3 - m_1 = k_2$.

We shall consider only the general case, when two of the products a_1a_2, b_1b_2, c_1c_2 are different from zero. Then we may choose the moving frame in such a way that $a_2 = \varepsilon a_1$, $b_2 = \nu b_1$, where $\varepsilon, \nu = \pm 1$. For simplicity let $\varepsilon = \nu = 1$. For Darboux motions we get the following system of equations:

$$
\begin{gathered}
2a_1k_3 + b_1(c_2 + c_1)(1 - 2\mu) = 0\,, \\
2a_1\lambda_3 + b_1(c_2 - c_1)(1 - 2\mu) - 2a_1' = -2\gamma a_1\,, \\
-2\mu b_1k_2 + a_1(c_1 + c_2)(\mu - 2) = 0\,, \\
2\mu b_1\lambda_2 + a_1(c_1 - c_2)(\mu - 2) - 2b_1\sigma - 2(\mu b_1)' = -2\mu\gamma b_1\,, \\
a_1\, b_1(1 + \mu) - c_1(1 - \mu)(k_1 + \lambda_1) + [c_1(1 - \mu)]' = \gamma\, c_1(1 - \mu) + c_1\sigma\,, \\
a_1\, b_1(1 + \mu) - c_2(1 - \mu)(k_1 - \lambda_1) - [c_2(1 - \mu)]' = -\gamma\, c_2(1 - \mu) - c_2\sigma\,, \\
2\, a_1^2(1 - 2\mu) + 2\, b_1^2(2 - \mu) + 2c_1\, c_2(\mu^2 - 1) + 2\sigma\lambda_3 + \sigma' = \gamma\sigma\,.
\end{gathered}
\tag{4}
$$

From the 1st row we compute k_3, from the 3rd row we get k_2. This gives also k_1, as $k_1 + k_2 + k_3 = 0$. We substitute the result in the other equations and choose the parameter in such a way that $\lambda_2 - \lambda_1 = 1$. Then $\lambda_1 = -(\mu + 1)/3$, $\lambda_2 = \lambda_1 + 1$, $\lambda_3 = -2\lambda_1 - 1$. We add the equation $\mu' = \sigma - 2\mu(\lambda_3 - \lambda_2)$, thus getting a system of 6 equations of the 1st order for unknown functions a_1, b_1, c_1, c_2, μ, σ, which are solved with respect to the first derivatives. This means that the solution exists and as the moving frame was uniquely fixed, the solution depends on 6 arbitrary constants and on an arbitrary function γ. This shows that in the case of Type 1 we have many Darboux motions. As there is apparently no sense in trying to find explicit solutions of equations (4) for Darboux motions in general, we shall present an example which proves the existence of Darboux motions with irreducible inflexion cubic.

Example 1. Put $\mu = -1$ in (4). Then $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$, $c_1 = c_2 = 0$, $k_i = m_i = 0$. Write $a_1 = a$, $b_1 = b$. Then we are left with the following equations:

$$a' = a(\gamma - 1)\,, \quad b' = b(\gamma + 5)\,, \quad \gamma = 3/2(a^2 - b^2) - 2\,.$$

One of the solutions is

$$a = \sqrt{(2)}\,(1 + \exp(12t))^{-1/2}\,, \quad b = \sqrt{(2)}\,(1 + \exp(-12t))^{-1/2}\,.$$

The equations $X' = -\psi X$ for fixed points of the moving plane are

$$x' = -ay - bz\,, \quad y' = -ax + y\,, \quad z' = -bx - z\,. \tag{5}$$

Further,

$$F(X) = 4xyz + bx^2y - ax^2z - 2ay^2z + 2byz^2\,.$$

The derivative of $F(X)$ with regard to (5) is

$$2xyz(a^2 - b^2) + (a'/3 + a)(bx^2y - ax^2z - 2ay^2z + 2byz^2),$$

which after substitution gives a multiple of F. This means that each point of F determines a fixed point of the moving plane, which remains on F during the motion, so that its trajectory must lie on a straight line. It remains to show that F is irreducible. This is left to the reader.

Now we shall consider the cases when F is not simple, and we shall find all motions with the property that all points of F have straight trajectories, The fact that F passes through all poles of the motion will simplify our considerations. Indeed, if F is not simple, it contains at least one straight line. Let its equation be $Ax + By + Cz = 0$. Then if $A \neq 0$, $B \neq 0$, $C \neq 0$, the remaining conic section contains at least three distinct points and F is simple. So let the equation of the line be $Ax + By = 0$ with $A \neq 0$, $B \neq 0$ (after a permutation of poles).

If the remaining conic section is regular, it has at least two different points and so it is real and F is simple. If the remaining conic section is singular, it must be the twice counted line $z = 0$. So we always have the line $z = 0$ as part of the inflexion cubic. Its equation is

$$-\mu b_1x^2y + c_1(1-\mu)xy^2 + a_1\mu x^2z + c_2\mu(1-\mu)xz^2 + (\mu-1)a_2zy^2 + \\ + (\mu-1)\mu b_2yz^2 = 0.$$

As $z = 0$ is part of F, we get $b_1 = c_1 = 0$. Then $z = 0$ is an integral of $X' = -\psi X$ as well as of $X' = -\varphi X$ since $z' = (\pm\lambda_3 - m_3)z$, and the line $z = 0$ is fixed during the motion. This means that our motion is projectively equivalent to affine motion.

The remaining conic section is

$$H = a_1\mu x^2 + c_2\mu(1-\mu)xz + (\mu-1)a_2y^2 + (\mu-1)\mu b_2yz + \sigma xy = 0$$

We have the following possibilities:

a) H is just the point $(1, 0, 0)$. Then this point must be a fixed point of the moving plane and it must satisfy $X' = -\psi X$, so $b_2 = c_2 = 0$, $\sigma^2 - 4a_1a_2\mu(\mu-1) < 0$. The point $(1, 0, 0)$ is fixed in the fixed plane as well as we get a special case of the centroaffine motion – the inflexion set is trivial being only the center. This motion is in general not a Darboux motion.

b) $H = z(Ax + By)$. Then $a_1 = a_2 = \sigma = 0$, the remaining line is $c_2x - b_2y = 0$. The differentiation of the equation of this line yields the equation

$$b_2c_2' - c_2b_2' + b_2c_2(\lambda_1 - \lambda_2 - k_3) = 0,$$

which is exactly the equation for Darboux motions in this case. So in this case we get a Darboux motion.

c) $H = (Ax + By)^2$. Then $c_2 = b_2 = 0$, $\sigma^2 - 4\mu(\mu - 1)\, a_1 a_2 = 0$, the equation of the line is

$$a_1 \mu x^2 + \sigma xy + (\mu - 1)\, a_2 y^2 = (2a_1 \mu x + \sigma y)^2 = 0\,,$$

where $a_2 = \varepsilon a_1$, $\sigma = 2a_1[\mu(\mu - 1)\,\varepsilon]^{1/2}$, $\varepsilon = \pm 1$. The differentiation and substitution now leads to the equation $2k_3 = a_1'/a_1 - a_2'/a_2$, which is one of the equations for Darboux motions. Substitution in the second equation shows that it is satisfied as well. So this motion is a Darboux motion.

Theorem 4. *Let all inflexion points of a projective plane motion of Type 1 have straight trajectories. Then this motion either is a Darboux motion or is projectively equivalent to a centroaffine motion with the trivial inflexion set.*

Type 2. $\omega = 2\,\mathrm{Diag}\,\{1, 1, -2\}$, η is the same as for Type 1.

$$\Omega_0 = E\,, \quad \Omega_1 = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0, & 0, & b_2 \\ 0, & 0, & c_2 \\ -b_1, & -c_1, & 0 \end{pmatrix}.$$

The inflexion cubic is $F(X) = z^2(c_2 x - b_2 y) = 0$, so it is never simple. F is nontrivial only if $b_2^2 + c_2^2 \neq 0$, so we can choose a special moving frame in which $b_2 = 1$, $c_2 = 0$. Then $F(X) = yz^2$. The condition for a Darboux motion is $b_1 = c_1 = a_1 = 0$, which is at the same time the condition for the points of F to have straight trajectories. Here the condition that $F(X) = 0$ implies $G(X) = 0$ is not sufficient for a Darboux motion. The Darboux motion of this type is projectively equivalent to a centroaffine motion which preserves one direction.

Type 3.

$$\omega = 2\begin{pmatrix} 1, & 1, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -2 \end{pmatrix},$$

η is the same as in the case of Type 1.

$$\Omega_0 = E\,, \quad \Omega_1 = \begin{pmatrix} 0, & 1, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & -3 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0, & 0, & -3b_2 - c_2 \\ 0, & 2a_1, & -3c_2 \\ 3b_1, & 3c_1 + b_1, & 18 + a_1 + 3k_3 \end{pmatrix},$$

$$F(X) =$$

$$= -(3c_1 + b_1)\, y^3 - 3b_1 xy^2 - 9c_2 yz^2 + 9b_2 yz^2 + (a_1 - 18 - 3k_3)\, y^2 z + 6a_1 xyz\,.$$

First we shall prove the existence of Darboux motions of this type.

a) Let $b_1 \neq 0$. Then we can change the moving frame to get $b_1 = 1$, $c_1 = 0$. The equation of Darboux motions is $\Omega_3 = \gamma\Omega_2 + \beta\Omega_1 + \alpha\Omega_0$, where α, β, γ are functions of t. The explicit form is:

$$
\begin{aligned}
6b_2 + c_2 = \alpha\,, \qquad & 2a_1(1 + a_2) + b_2 = \beta\,, \\
-2a_1^2 + 6c_2 = 0\,, \qquad & 4a_1 + c_2 + 2a_1' = 2a_1\gamma + \alpha\,, \\
21 + 9m_1 + 2a_1 = -3\gamma\,, \quad & 2 - k_1 - 3a_2 = \gamma\,,
\end{aligned}
$$

$$b_2(21 + a_1 - 3k_1) + c_2(k_2 - 2 - 3a_2) - 3b_2' - c_2' = -\gamma(3b_2 + c_2)\,,$$

$$c_2(2a_1 - 21 + 9m_2) + 3a_1b_2 + 3c_2' = 3c_2\gamma\,,$$

$$-6b_2 - 2c_2 - 4(18 + a_1 + 3k_3) + a_1' + 3k_3' = (18 + a_1 + 3k_3)\gamma - 3\beta + \alpha\,.$$

aα) Let $a_1 \neq 0$. Then we have an explicit solution

$$c_2 = a_1^2/3,\ b_2 = -a_1^2/9,\ k_3 = -6 - a_1/3,\ a_2 = -1,\ k_2 = 3 + a_1'/a_1 + 2a_1/3\,,$$

The solution depends on one arbitrary function $a_1(t) \neq 0$.

aβ) Let $a_1 = 0$. Then $a_2 = 3 + 2k_3/3$, $k_3' = -3(6 + k)(1 + m_1)$, $c_2 = b_2 = 0$, m_1 arbitrary.

b) Let $b_1 = 0$. Then $a_1 = 0$.

bα) $c_2 \neq 0$. We specialize to $b_2 = 0$, $c_2 = 1$ obtaining $a_2 = -3 - 2k_3/3$, $k_3' = 3(6 + k_3)(m_2 - 1)$, $a_1, m_1. n_2$ are arbitrary.

bβ) $c_2 = 0$. We get the following equations:

$$b_2' = b_2(\gamma + 7 - k_1)\,, \quad c_1' = c_1(\gamma + 7 - k_2)\,, \quad k_3' = (\gamma + 4)(k_3 + 6) - 6c_1b_2\,.$$

These equations always have a solution, a_2 is arbitrary. This proves the existence of Darboux motions in all subcases.

The inflexion cubic for motions of this type always splits. It is given by the following equations:

aα): $F(X) = (3x + y)(y - a_1z)^2$; aβ): $F(X) = y^2[y + 3x + (18 + 3k_3)z]$;

bα): $F(X) = z[3xz + (6 + k_3)y^2]$; b$\beta$): $F(X)\,y[c_1y^2 + (6 + k_3)yz - 3b_2z^2]$

Now we are going to prove the converse of Theorem 2. To this end let us suppose that the motion is of Type 3 and that all points of F have trajectories on straight lines. We may suppose that F is not simple, because in the opposite case the statement follows from Theorem 3. We shall again consider separately several cases.

a) $b_1 = 1$, $c_1 = 0$. $F(X) = y^3 + 3xy^2 + 9c_2xz^2 - 9b_2yz^2 + My^2z - 6a_1xyz$, with $M = 18 + 3k_3 - a_1$. F splits, so let us write

$$F(X) = (Ax + y + Cz)(\alpha x^2 + y^2 + \gamma z^2 + \delta xy + \varphi xz + \psi y\delta)\,.$$

Comparison gives

$$A\alpha = 0,\ C\gamma = 0,\ A + \delta = 3,\ A\delta + \alpha = 0,\ A\gamma + C\varphi = 9c_2,\ A\varphi + C\alpha = 0\,,$$

$$\gamma + C\psi = -9b_2,\ \psi + C = M,\ A\psi + \varphi + C\delta = -6a_1\,.$$

Let $\alpha \neq 0$. Then $A = 0$ and $\alpha = 0$ is a contradiction. Hence always $\alpha = 0$.

I) $A \neq 0$. Then $\delta = 0$, $\varphi = 0$, $C\gamma = 0$, $A = 3$, $\gamma = 3c_2$, $\psi = -2a$, $C = M + 2a_1$.

Iα) $C \neq 0$. Then $\gamma = c_2 = 0$, $F(X) = (3x + y + Cz)\, y(y - 2a_1 z)$, which is not simple only for $a_1 = 0$, $\psi = 0$ and $b_2 = 0$. The equation $X' = -\psi X$ is

$$\begin{aligned} x' &= (1 - m_1)\, x + (1 - a_2)\, y - b_2 z\,, \\ y' &= -a_1 x + (1 - m_2)\, y - c_2 z\,, \\ z' &= -x - (2 + m_3)\, z\,. \end{aligned}$$

As a consequence of $y = 0$ we have $c_2 = 0$. Further, $C = 18 + 3k_3$ and the differentiation of $3x + y + Cz = 0$ gives the condition for a Darboux motion.

Iβ) $C = 0$. Then $F = (3x + y)(y^2 + \psi yz + \gamma z^2)$, $\gamma = 3c_2$, $\psi = -2a_1$, $M = -2a_1$, $c_2 = -3b_2$. The differentiation of $3x + y = 0$ gives $a_1 = -18 - 3k_3$, $a_2 = -1$. F is not simple iff $\psi^2 - 4\gamma \leqq 0$.

Let first $\psi^2 = 4\gamma$, so $a_1^2 = 3c_2$. The conic section is $(y - a_1 z)^2 = 0$, the differentiation gives a Darboux motion of the case aα) for $a_1 \neq 0$ and of the special case $k_3 = -6$ of aβ) for $a_1 = 0$.

Further let $\psi^2 - 4\gamma < 0$. Then the conic section $y^2 + \psi yz + \gamma z^2 = 0$ has the only real point $y = z = 0$, which is an isolated point of F. Hence it must satisfy $X' = -\psi X$, which is impossible. So such a motion does not exist.

II) Let $A = 0$. Then the case $C \neq 0$ gives a motion from Iβ), the case $C = 0$ gives a Darboux motion from aβ).

The case b) is treated in a similar way; we also get only Darboux motions. The detailed computation is omitted.

Theorem 5. *Let all points of the inflexion cubic of a projective plane motion with a double characteristic root have straight trajectories. Then this motion is a Darboux motion.*

Type 4.

$$\omega = 2\begin{pmatrix} 0, & \lambda, & 0 \\ 0, & 0, & \lambda \\ 0, & 0, & 0 \end{pmatrix}, \quad \eta \text{ is the same as for Type 1.}$$

$$\Omega_0 = E\,, \quad \Omega_1 = \begin{pmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ 0, & 0, & 0 \end{pmatrix}. \quad \Omega_2 = \begin{pmatrix} -a_1, & k_3 & 2 + a_2 - c_2 \\ -b_1, & a_1 - c_1, & k_1 \\ 0, & b_1 & c_1 \end{pmatrix}.$$

The formula for Ω_3 is too complicated to be presented here. Let us write only the terms necessary for the specialization:

$$(\Omega_3)_{21} = b_1(k_3 - k_1) - a_1(2a_1 - c_1) - b_1'\,, \quad (\Omega_3)_{31} = -2b_1(a_1 + c_1)\,,$$
$$(\Omega_3)_{32} = b_1(k_3 - k_1) + c_1(a_1 - 2c_1) + b_1'\,.$$

We shall first prove the existence of Darboux motions. The specialisation of the frame is more complicated for this type and therefore we shall present more detailed computations. The isotropy group preserving ω is

$$H = \begin{pmatrix} r, & rs, & u \\ 0, & 1, & s \\ 0, & 0, & r^{-1} \end{pmatrix}, \quad r \neq 0,\, s,\, u \in R\,.$$

The formulas for coefficients of η in the new frame are:

$$\tilde{\lambda} = \lambda r^{-1}\,, \quad \tilde{a}_1 = a_1 r - b_1 s r^2\,, \quad \tilde{b}_1 = b_1 r^2\,, \quad \tilde{c}_1 = b_1 s r^2 + c_1 r\,.$$

a) Let $b_1 \neq 0$. Then we may specialize the frame to $b_1 = 1$, $c_1 = 0$. Then $a_1 = 0$. The equations for Darboux motions now simplify considerably and the solution is $k_1 = k_3$, $b_1 = 1$, $a_1 = c_1 = 0$, $a_2 = -1$, $c_2 = 1$. So $m_2 = 0$ and we may further specialize to $m_1 = m_3 = 0$; b_2 is arbitrary, the frame is uniquely fixed.

b) $b_1 = 0$. If $c_1 \neq 0$, we immediately see that such a motion does not exist. So $c_1 = 0$ and this implies $a_1 = 0$. We get equations

$$-k_2(3 + a_2 - c_2) + k_1 a_2 - k_3 c_2 + a_2' - c_2' = \gamma(2 + a_2 - c_2)\,,$$
$$k_3^2 - k_1^2 + k_3' - k_1' = \gamma(k_3 - k_1)\,,$$

which always have a solution.

Now we are going to prove the converse of Theorem 2. We have

$$F(X) = b_1 y^3 + (2 + a_2 - c_2)\, z^3 - (a_1 + c_1)\, xz^2 + (2c_1 - a_1)\, y^2 z + (k_3 - k_1)\, yz^2\,.$$

a) $b_1 = 1$, $c_1 = 0$. The inflexion cubic is reducible only if $a_1 = 0$. Then $F(X) = y^3 + (2 + a_2 - c_2)\, z^3 + (k_3 - k_1)\, yz^2$. It contains a real line of the form $y + Cz = 0$. From the equation $X' = -\psi X$ we get $C = 0$, $c_2 = 1$ and $a_2 = -1$. Now $F(X) = y(y^2 + \gamma z^2)$. Let $\gamma < 0$. Then we have two real lines $y \pm \sqrt{(-\gamma)}\, z = 0$ and differentiation gives $\gamma = 0$ which is a contradiction. So $\gamma \geqq 0$. Then the real points of F form only the line $y = 0$ and the motion need not be a Darboux motion. Specialization to $m_1 = 0$ finally yields

$$b_1 = c_2 = 1\,, \quad c_1 = a_1 = m_1 = 0\,, \quad a_2 = -1\,, \quad m_2 \leqq 0\,. \tag{6}$$

b) Let $b_1 = 0$. Then F contains $z = 0$ and this implies $c_1 = 0$. The conic section from F is $F_1(X) = (2 + a_2 - c_2)\, z^2 - a_1 xz - a_1 y^2 + (k_3 - k_1)\, yz$, which contains the point $(1, 0, 0)$. So if F is not simple, it is singular. But F_1 is singular only if $a_1 = 0$. The derivative now gives the condition for a Darboux motion.

Theorem 6. *Let all points of inflexion cubic of a projective plane motion with a triple characteristic root have straight trajectories. Then it is either a Darboux motion or is given by* (6).

Type 5.

$$\omega = 2\begin{pmatrix} \lambda, & 1, & 0 \\ -1, & \lambda, & 0 \\ 0, & 0, & -2 \end{pmatrix}, \quad \eta \text{ is as in Type 1}.$$

Here we may always specialize the moving frame to have $m_1 = m_2 = m$. The equations for Darboux motions will be

$$(2\lambda - \gamma)(a_1 + a_2) + 3\lambda(c_1c_2 - b_1b_2) + (a_1 + a_2)' = 0,$$

$$a_2^2 - a_1^2 + 3\lambda(c_1b_2 + b_1c_2) = 0,$$

$$-6\lambda(b_1b_2 + c_1c_2) + 3(1 + 3\lambda^2)(b_2c_1 - b_1c_2) - (4\lambda + \gamma)(\mu - 1) + \mu' =$$
$$= -3\lambda(a_1 + a_2)^2,$$

$$(3\lambda b_1 - c_1)(\lambda + 3m) + b_1(\mu + 2a_1 + a_2) + 3\lambda c_1(1 + a_1) - (3\lambda b_1 - c_1)' =$$
$$= -\gamma(3\lambda b_1 - c_1),$$

$$-(3\lambda c_1 + b_1)(\lambda + 3m) + c_1(-\mu + a_1 + 2a_2) + 3\lambda b_1(1 - a_2) + (3\lambda c_1 + b_1)' =$$
$$= \gamma(3\lambda c_1 + b_1),$$

$$(3\lambda b_2 + c_2)(\lambda - 3m) + b_2(\mu + a_1 + 2a_2) - 3\lambda c_2(1 + a_2) - (3\lambda b_2 + c_2)' =$$
$$= -\gamma(3\lambda b_2 + c_2),$$

$$(-3\lambda c_2 + b_2)(-\lambda + 3m) + c_2(\mu - 2a_1 - a_2) + 3\lambda b_2(1 - a_1) + (-3\lambda c_2 + b_2)' =$$
$$= \gamma(-3\lambda c_2 + b_2),$$

where $\mu = 18\lambda^2 - 3\lambda' + 3$.
From these equations we see similarly as in Type 1 that in the general case of $a_1 + a_2 \neq 0$, $c_1 = c_2 \neq 0$ Darboux motions exist. The solution depends on one arbitrary function m and six constants of integration.

The next example shows that also in this case we have Darboux motions with an irreducible inflexion cubic.

Example 2. Put $a_1 = a_2 = m = \lambda = 0$. Then $\mu = 3$ and we get

$$b_2c_1 - b_1c_2 = 2\gamma/3,\ 3b_1 + c_1' = c_1\gamma, \quad -3c_1 + b_1' = b_1\gamma,$$
$$3b_2 - c_2' = -c_2\gamma, \quad 3c_2 + b_2' = b_2\gamma.$$

Let us denote $w = (C - A\sin(6t + \vartheta))^{-1/2}$, where ϑ, A, C are constants. Then the solution is

$$c_1 = \sqrt{(2)}\, w\cos 3t,\ b_1 = \sqrt{(2)}\, w\sin 3t,\ b_2 = \sqrt{(2)}\, w\cos(3t + \vartheta),$$
$$c_2 = \sqrt{(2)}\, w\sin(3t + \vartheta) = 3Aw^2\cos(6t + \vartheta).$$

The proof of irreducibility of F can be done by direct computation and is omitted. Easy computation also shows that the derivative of $F(X)$ is

$$\gamma(-c_1x^3 + b_1y^3 - c_1xy^2 + b_1x^2y + c_2xz^2 - b_2yz^2) + \\ + 3(y^2z + x^2z)(b_2c_1 - b_1c_2).$$

Substitution shows that this is a multiple of F and this completes the example.

Now we would like to prove the converse of Theorem 2. To this end we have to discuss the cases when F is not simple. A rather long computation shows that in the case when F is not simple we do not get a Darboux motion in general. (Here we have the case when F consists of a straight line and an imaginary conic section or of a straight line and a point. This means that the differentiation of the real part of F gives only few conditions, which are too weak to guarantee the motion to be a Darboux motion.)

In the end we present a projective characterization of the elliptical motion from the Euclidean plane kinematics to show that this motion is distinguishable even among projective motions.

Theorem 7. *Let $g(t)$ be a projective Darboux motion with one real and two imaginary poles for each t. Let the inflexion cubic be a regular real conic section passing through the imaginary poles, and a straight line connecting them. Then if the instantaneous motion has conic sections as trajectories, the motion is projectively equivalent to the Euclidean elliptic motion.*

Proof. The conditions for the instantaneous motion give $\lambda = b_1 = c_1 = a = 0$. Then $b_2^2 + c_2^2 \neq 0$, as otherwise the inflexion conic section would be singular. So we may specialize to $b_2 = 1$, $c_2 = 0$. Then we get $a_2 = -3$, $a_1 = 3$, $m = 0$ and this motion is the elliptic motion in a suitable Euclidean structure on P_2.

Example 3. For illustration we shall present a typical example of a projective Darboux motion. Let us consider the projective plane motion given (up to a multiple) by the matrix $g(t) = E + At + Bt^2$, where E is the unit matrix, A and B are arbitrary 3×3 matrices. Then $g(t)$ is a regular matrix in some neighborhood of 0 and so it determines a projective motion. This motion is a 2-Darboux motion, as $g'''(t) = 0$. Further, all trajectories of this motion are conic sections, as every trajectory $X(t)$ of X_0 is given by $X(t) = g(t)X_0 = X_0 + AX_0t + BX_0t^2$, which is a parametric form of a conic section. The inflexion cubic is given by the equation $|X, X', X''| = 0$, which is

$$|(E + At + Bt^2)X_0, (A + 2Bt)X_0, 2BX_0| = |X_0, AX_0, BX_0| = 0.$$

This means that the preimage of the inflexion cubic in the moving plane is a fixed cubic which contains the preimages of all poles of the motion. Hence we see that all moving poloids of the motion are parts of the cubic $|X_0, AX_0, BX_0| = 0$. It is easy to see that any irreducible cubic can be written in the form $|X, AX, BX| = 0$ for suitable matrices A and B (for instance, the choice $A = \text{Diag}\{0, 1, 2\}$, B arbitrary will do). This means that the inflexion cubic of the motion considered is in general irreducible, as A and B can be chosen quite arbitrarily. We also see that the inflexion cubic of a 2-Darboux projective plane motion has no special properties, as any irreducible cubic is the inflexion cubic of a certain motion of this kind.

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Souhrn

PROJEKTIVNÍ POHYBY V ROVINĚ S NEKONEČNĚ MNOHA PŘÍMKOVYMI TRAJEKTORIEMI

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Článek je věnován jednoparametrickým pohybům v projektivní rovině majícím tu vlastnost, že všechny body inflexní kubiky mají přímkové trajektorie. Je ukázáno, že tyto pohyby mají v obecném případě irreducibilní inflexní kubiku a projektivně ekvivalentní trajektorie. Dále jsou podrobně diskutovány případy těchto pohybů s rozpadlou inflexní kubikou. Diskutuje se též souvislost s Darbouxovskými pohyby.

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