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ON THE TOPOLOGICAL CHARGE CONSERVATION IN THE THREE-DIMENSIONAL 0(3) σ-MODEL

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1. INTRODUCTION AND CONCLUSIONS

Field theories with nontrivial topological properties have been of permanent interest during the last decade. Convenient examples are σ-models with values of the field in a Riemannian manifold, a sphere in the simplest case [1]. The present paper deals with the topological-charge conservation for the 0(3) σ-model in 2 + 1 dimensional Minkowski space-time.

Let us consider the field φ of three-component unit vectors

$$\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3), \quad \boldsymbol{\varphi} . \boldsymbol{\varphi} = 1,$$

with the Lagrangian density

(1) $$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \boldsymbol{\varphi}) . (\partial^\mu \boldsymbol{\varphi})$$

($\mu = 0, 1, 2$). The energy of the field is

(2) $$E = \frac{1}{2} \int_{\mathbb{R}^2} \left[\left(\frac{\partial \boldsymbol{\varphi}}{\partial x^0} \right)^2 + (\nabla \boldsymbol{\varphi})^2 \right] \mathrm{d}^2 x .$$

The Euclidean action for the path of field configurations $\boldsymbol{\varphi}(t, x)$ is

(3) $$\mathscr{A} = \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \left[\left(\frac{\partial \boldsymbol{\varphi}}{\partial t} \right)^2 + (\nabla \boldsymbol{\varphi})^2 \right] \mathrm{d}^2 x \, \mathrm{d} t .$$

The Euclidean time $t \in \langle t_1, t_2 \rangle$ is a parameter along the path and should not be confused with the physical time x^0 in general.

We use polar coordinates r, ω in $\mathbb{R}^2$. The field $\boldsymbol{\varphi}$ is said to be continuous at infinity if there exists

(4) $$\lim_{r \to \infty} \boldsymbol{\varphi}(t, r, \omega) = \boldsymbol{a}(t)$$

uniform with respect to the polar angle ω, and if $\boldsymbol{a}(t)$ is independent of ω. This property allows us to extend the field φ to the $\mathbb{S}^2$-one-point compactification of $\mathbb{R}^2$. Then the topological charge

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \varphi \cdot (\partial_1 \varphi \times \partial_2 \varphi) \, d^2 x \tag{5}$$

is an integer number – the degree of the mapping from the sphere $\mathbb{S}^2$ (compactified plane $\mathbb{R}^2$) into $\mathbb{S}^2$. If the field is continuous at infinity at all times, and the limit (4) is uniform with respect to time and the polar angle, the topological charge is conserved as a homotopy invariant.

Our aim is to remove the assumption of continuity at infinity for finite action fields. We prove the following proposition.

Proposition. *Let $\varphi : \langle t_1, t_2 \rangle \times \mathbb{R}^2 \to \mathbb{S}^2$ be a field with continuous first derivatives such that the initial and final configurations $\varphi(t_1, \cdot)$ and $\varphi(t_2, \cdot)$ are continuous at infinity and have different topological charges $Q[\varphi(t_1, \cdot)] \neq Q[\varphi(t_2, \cdot)]$. Then the Euclidean action (3) is infinite.*

Proposition derived for the special model can be extended to some modified versions of the model [2] at least. The proof is performed without any assumption concerning the asymptotics of the field (and without reference to the topological charge) at intermediate times. Therefore the situations where compactification of $\mathbb{R}^2$ to $\mathbb{S}^2$ is impossible at $t \neq t_1, t_2$, and the usual homotopy arguments cannot be used, are included.

The continuity at infinity assumed at the initial and the final time can be replaced by the finiteness of energy. Finite-energy fields need not be continuous at infinity [3]. The two possible assumptions are independent in general. The more complicated proof for finite energy fields is not given here.

Proposition implies that the topological charge is conserved for a classical field with finite conserved energy. Such a field has a finite Euclidean action

$$\mathscr{A} = E(t_2 - t_1) \tag{6}$$

(for finite t_1 and t_2) if we put $t = x^0$.

It is well known by the scaling argument that no instantons exist for the considered model [4]. Proposition shows that there is no other smooth path of field configurations with different initial and final topological charges and with finite Euclidean action. Therefore there is no contribution of such paths to the Feyman path integral

$$\int \exp(-\mathscr{A}) \, \mathscr{D}\varphi$$

giving the quantum transition amplitude continued into imaginary time. Their existence cannot be excluded a priori – compare finite-action configurations in the massive $\mathbb{C}P^n$ model [5].

2. PROOF OF PROPOSITION

We summarize some properties of the degree of a mapping needed in the proof. Let $\psi : \mathbb{R}^2 \to \mathbb{S}^2$ be a map with continuous first derivatives, which is continuous at infinity. For almost all points $\mathbf{y} \in \mathbb{S}^2$, the set of inverse images $\psi^{-1}(\mathbf{y}) = \{x_1, \ldots, x_n\}$ is finite. There exist neighbourhoods U_i of points x_i in $\mathbb{R}^2$ such that $\psi_{|U_i}$ (restrictions of the map ψ to the neighbourhoods U_i) are diffeomorphisms of the neighbourhoods U_i onto a neighbourhood of the point $\mathbf{y}$ in $\mathbb{S}^2$. We put $\varepsilon_i = 1$ if the diffeomorphism $\psi_{|U_i}$ conserves the orientation and $\varepsilon_i = -1$ if $\psi_{|U_i}$ reverses the orientation. The degree of the mapping ψ at the point $\mathbf{y}$ is then defined as

$$\deg_{\mathbf{y}} \psi = \sum_{i=1}^{n} \varepsilon_i \,. \tag{7}$$

The globally defined topological charge $Q[\psi]$ (5) equals the local degree of the mapping (7) at almost all points $\mathbf{y} \in \mathbb{S}^2$:

$$Q[\psi] = \deg_{\mathbf{y}} \psi \,. \tag{8}$$

This can be seen, for example, with the help of Theorem 61 and Corollary 1 to Theorem 63 of [6] and from the methods of their proofs (see also Sect. 4.6.5 in [7]).

The definition (7) is used also for the restrictions $\psi_r = \psi_{|B(r)} : B(r) \to \mathbb{S}^2$ of the map ψ to the disc $B(r) = \{x \in \mathbb{R}^2 \mid |x| \leqq r\}$ with a radius $r > 0$. Let us denote $C(r) = \{x \in \mathbb{R}^2 \mid |x| = r\}$, a circle which is the boundary of the disc $B(r)$.

The following lemma is an analogue of Theorem 66 of [6] or Lemma 4.6.5.2 of [7]. We give a simple proof of its required variant.

Lemma. *Let* $\varphi_r : \langle t_1, t_2 \rangle \times B(r) \to \mathbb{S}^2$ *be a homotopy of smooth maps. Let* $\mathbf{y} \in \mathbb{S}^2$ *be a point such that* $\deg_{\mathbf{y}} \varphi_r(t_1, \cdot)$ *and* $\deg_{\mathbf{y}} \varphi_r(t_2, \cdot)$ *are defined according to Eq.* (7). *If* $\mathbf{y} \notin \varphi_r(\langle t_1, t_2 \rangle \times C(r))$, *then* $\deg_{\mathbf{y}} \varphi_r(t_1, \cdot) = \deg_{\mathbf{y}} \varphi_r(t_2, \cdot)$.

Proof. Since $\mathbf{y}$ is not an element of the compact set $\varphi_r(\langle t_1, t_2 \rangle \times C(r))$, there exists a number $0 < \varepsilon < 2$ such that $V \cap \varphi_r(\langle t_1, t_2 \rangle \times C(r)) = \emptyset$, where $V = \{\mathbf{z} \in \mathbb{S}^2 \mid |\mathbf{z} - \mathbf{y}| < \varepsilon\}$. It is easy to construct a map $\mathbf{F} : \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$\mathbf{F}_{|V} : V \to \mathbb{S}^2 \setminus \{-\mathbf{y}\}$$

is an orientation preserving diffeomorphism, $\mathbf{F}(\mathbf{y}) = \mathbf{y}$, and

$$\mathbf{F}(\mathbb{S}^2 \setminus V) = \{-\mathbf{y}\} \,.$$

Then $\Phi = \mathbf{F} \circ \varphi_r$ is a homotopy such that

$$\Phi(\langle t_1, t_2 \rangle \times C(r)) = \{-\mathbf{y}\} \,. \tag{9}$$

We identify all points from $C(r)$. The disc $B(r)$ is homeomorphic to the sphere $\mathbb{S}^2$ after the identification. Eq. (9) allows us to consider Φ as a homotopy $\Phi : \langle t_1, t_2 \rangle \times$

$\times\ \mathbb{S}^2 \to \mathbb{S}^2$. Since the degree of a mapping is a homotopy invariant for maps $\mathbb{S}^2 \to \mathbb{S}^2$, we have

$$\deg_{\mathbf{y}} \varphi_r(t_1, \cdot) = \deg_{\mathbf{y}} \Phi(t_1, \cdot) = \deg_{\mathbf{y}} \Phi(t_2, \cdot) = \deg_{\mathbf{y}} \varphi_r(t_2, \cdot)\,.$$

Lemma is proved.

We are prepared to prove the Proposition now. Let $\varphi : \langle t_1, t_2 \rangle \times \mathbb{R}^2 \to \mathbb{S}^2$ be a field with continuous first derivatives such that $\varphi(t_1, \cdot)$, $\varphi(t_2, \cdot)$ are continuous at infinity, and

$$Q[\varphi(t_1, \cdot)] \neq Q[\varphi(t_2, \cdot)]\,. \tag{10}$$

Let U_i be neighbourhoods of the points $\mathbf{a}_i = \lim_{r\to\infty} \varphi(t_i, r, \omega)$ $(i = 1, 2)$ such that the set $\mathbb{S}^2 \setminus (\bar{U}_1 \cup \bar{U}_2)$ is a nonempty open set with a measure (area) $s > 0$. There exists $R > 0$ such that

$$\varphi(t_i, r, \omega) \in U_i \tag{11}$$

for all $r > R$, $\omega \in \langle 0, 2\pi \rangle$, $i = 1, 2$. For almost all points $\mathbf{y} \in \mathbb{S}^2 \setminus (\bar{U}_1 \cup \bar{U}_2)$ and all $r > R$, we have

$$\deg_{\mathbf{y}} \varphi_r(t_1, \cdot) = Q[\varphi(t_1, \cdot)] \neq Q[\varphi(t_2, \cdot)] = \deg_{\mathbf{y}} \varphi_r(t_2, \cdot)$$

according to Eqs. (8), (10) and (11). Then

$$\mathbf{y} \in \varphi(\langle t_1, t_2 \rangle \times C(r))$$

according to Lemma. Therefore the area covered by $\varphi(\langle t_1, t_2 \rangle \times C(r))$ is larger than the area of $\mathbb{S}^2 \setminus (\bar{U}_1 \cup \bar{U}_2)$:

$$\int_{t_1}^{t_2} \int_0^{2\pi} \left| \boldsymbol{\varphi} \cdot \left(\frac{\partial \boldsymbol{\varphi}}{\partial t} \times \frac{\partial \boldsymbol{\varphi}}{\partial \omega} \right) \right| \mathrm{d}t\, \mathrm{d}\omega \geqq s \tag{12}$$

for $r > R$. Using the inequalities

$$(\nabla \boldsymbol{\varphi})^2 \geqq r^{-2} \left(\frac{\partial \boldsymbol{\varphi}}{\partial \omega} \right)^2$$

and

$$\left(\frac{\partial \boldsymbol{\varphi}}{\partial t} \right)^2 + r^{-2} \left(\frac{\partial \boldsymbol{\varphi}}{\partial \omega} \right)^2 \geqq 2r^{-1} \left| \boldsymbol{\varphi} \cdot \left(\frac{\partial \boldsymbol{\varphi}}{\partial t} \times \frac{\partial \boldsymbol{\varphi}}{\partial \omega} \right) \right|,$$

we obtain

$$\mathscr{A} \geqq \int_R^\infty \frac{s}{r}\, r\, \mathrm{d}r = \infty$$

according to Eqs. (3) and (12). Proposition is proved.

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Souhrn

K ZACHOVÁNÍ TOPOLOGICKÉHO NÁBOJE V TROJROZMĚRNÉM 0(3) σ-MODELU

JAROSLAV DITTRICH

Je uvažováno pole tříkomponentních jednotkových vektorů na 2 + 1 rozměrném prostoročasu. Dvě konfigurace pole s různými topologickými náboji nemohou být spojeny dráhou polních konfigurací s konečnou euklidovskou akcí. Proto mezi nimi nedochází k přechodům. Předpokládáme, že počáteční a konečná konfigurace jsou spojité v nekonečnu. Asymptotické chování intermediálních konfigurací může být libovolné. Důkaz je založen na vlastnostech stupně zobrazení.

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