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# AN ENERGY ANALYSIS OF DEGENERATE HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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## 1. INTRODUCTION

Singular and/or degenerate partial differential equations arise in an extremely wide variety of physical situations. Problems with singularities or degeneracies in the steady state part of the operator describe transonic flow in aerodynamics. They can also arise in as simple a context as when the Poisson equation is written in polar coordinates:

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = r^2 f(r, \theta) .$$

Associated time dependent problems also occur. For example, the equation describing the vibrations of a homogeneous rod fixed at one end is

$$\frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}(x, t)\right), \quad 0 \leqq x \leqq 1 , \quad t \geqq 0 .$$

(see pp. 390–391 of Courant and Hilbert [7a]).

In this paper time dependent partial differential equations with degeneracies in the time variable are studied. Such equations frequently occur in fluid dynamics (see Weinstein [22a], Ames [1] for examples). The equation of this type about which the most is known is the Euler-Poisson-Darboux equation

$$u_{tt} + \frac{2p + 1}{t} - \Delta u = 0 , u(x, 0) = u_0(x), u_t(x, 0) = 0 ,$$

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which describes the isentropic flow of a perfect gas (see Ames [1] pp. 83–90). A similar degeneracy can occur if diffusion with a time delay effect occurs. The simplest effect of this type is perhaps averaging, giving

$$u_t(x, t) = \frac{1}{t}\int_0^t \Delta u(x, \tau)\, d\tau\,, \quad u(x, 0) = u_0(x)\,.$$

This averaging leads to the initial value problem

$$tu_{tt} + u_t = \Delta u\,, \quad u(x, 0) = u_0(x)\,,$$

which is of parabolic type at $t = 0$ and hyperbolic type for $t < 0$. In this paper a class of equations similar to the above is analyzed. Regularity results for the homogeneous equation are proved, then a method of approximating the solution to the inhomogeneous equation is analyzed.

The numerical method chosen in this paper for approximating the solutions to such equations is the usual, semidiscriete finite element method. Specifically, the semidiscrete finite element method is examined for the linear

$$(1.1) \qquad (tu_t)_t = -Lu + f(x, t)\,, \quad x \in \Omega\,, \quad 0 \leqq t \leqq T,$$

and semilinear

$$(1.2) \qquad (tu_t)_t = -Lu + f(x, t, u)\,, \quad x \in \Omega\,, \quad 0 \leqq t \leqq T,$$

degenerate hyperbolic equations. Here $L$ denotes a second order uniforly elliptic operator.

$$-Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a_0(x)\, u\,, \quad x \in \Omega\,,$$

where $a_0$, $a_{ij} = a_{ji}$ are smooth functions on $\bar{\Omega}$, $a_0(x) \geqq 0$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\, \zeta_i\zeta_j \geqq \alpha \sum_{i=1}^{N} \zeta_i^2$$

holds for some $\alpha > 0$ and all $(\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N$. Equations (1.1) and (1.2) are subject to the boundary and initial conditions

$$(1.3) \qquad u(x, 0) = u_0(x)\,, \quad x \in \Omega\,, \quad u(x, t) = 0\,, \quad x \in \partial\Omega\,, \quad t > 0\,,$$

where $\partial\Omega$ is assumed to be $C_\infty$.

Let $S^h$ be a finite element space of functions in $W^{2,1}(\Omega)$ vanishing on $\partial\Omega$. The Galerkin approximation to (1.1) and (1.2) is a differentiable map $U : [0, T) \to S^h$ satisfying

$$(1.4) \qquad ((tU_t)_t, v) + a(U, v) = (f, v) \quad \text{for all} \quad v \in S^h\,,$$

where $f = f(x, t)$ or $f(U)$, respectively. Here $(\cdot, \cdot)$ denotes the usual $L^2(\Omega)$ inner product and $a(\cdot, \cdot)$ is the bilinear form associated with $L$:

$$a(u, v) = \int_\Omega \left[ \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \right] \mathrm{d}x .$$

$U(0)$ will be taken as an approximation to $u_0$ in $S^h$. Also $f(x, t, w(x, t))$ will frequently be abbreviated as $f(w)$.

For the linear problem (1.1) the method (1.4) is shown to be stable in $W^{2,1}$ and convergent to the true solution. When $U(0)$ is picked as the "elliptic projection" of $u_0$, $U$ converges to $u$ in $L^2$ with optimal order.

The analogous convergence result is shown to hold for the semilinear equation (1.2) when $f$ is globally Lipschitz. It is then whown that this condition can be weakened considerably. That is, in one space dimension $U$ will converge to $u$ optimally when $f$ is only locally Lipschitz. It is shown that the same result holds in higher space dimensions when $S^h$ satisfies an inverse assumption that is typical of piecewise polynomial spaces under standard smoothness conditions.

Cahlon [6] has considered a finite difference method for approximating the Cauchy problem for the homogeneous equation of the form (1.1). He shows the method to be stable and convergent. Moreover, he uses an efficient method of discretizing the time variable by means of variable time steps. Genis [14] has considered finite element approximations for the related Euler-Poisson-Darboux equations and derived error estimates for the resulting methods.

Since the equation (1.1) is of hyperbolic-parabolic type, it is necessary to combine ideas used to obtain estimates for hyperbolic equations (as in Baker [3], Baker and Dougalis [4], and Dupont [10] with those used in parabolic equations, e.g. Douglas and Dupont [8], Fix and Nassif [11.a], Zlámal [25] and Thomée and Wahlbin [22]. In a previous paper [16] superconvergence estimates for solutions to the Cauchy problem were obtained. Here, energy type methods are used to derive error estimates for Galerkin approximations to solutions of initial boundary value problems for (1.1), (1.2), (1.3).

For simplicity of analysis of the numerical method the inner products occuring in the Galerkin equations (1.4) will be assumed to be evaluated exactly and the finite element space $S^h$ will be assumed to satisfy the boundary conditions of the continuous problem exactly. In practice, neither of these conditions are fulfilled. However, once the basic algorithm is analyzed under these assumptions the "pollution effects" caused by these "variational crimes" can be analyzed as perturbations of the basic method. Based on the work in the present paper, this was carried out in Layton [16a] and rates of convergence were obtained when the finite element space consists of isoparametric elements (not satisfying the boundary conditions exactly) and the integrals in (1.4) are discretized by a suitable quadrature scheme.

For $s \geqq 0$, $W^{2,s}(\Omega)$ will denote the Sobolev space of real valued functions with $s$ weak derivatives in $L^2$. The norm on $W^{2,s}(\Omega)$ is defined in the usual manner and denoted by $\|\cdot\|_{2,s}$. The $L^\infty(\Omega)$ norm will denoted by $|\cdot|$, and $\|\cdot\|$ will denote the $L^2(\Omega)$ norm $|\cdot|_r$ will denote the norm on $W^{\infty,r}(\Omega)$. Also ,if $V$ is a normed space with a norm $\|\cdot\|_V$ and $f : [0, T] \to V$, then $f \in L^\infty(V)$ will mean that

$$\operatorname*{ess\,sup}_{0 \leqq t \leqq T} \|f(t)\|_V < \infty .$$

First, the continuous equation will be considered with a view to elucidating the special behavior caused by the degenerate term in (1.1), (1.2). Then, the Galerkin method for the linear and semilinear problem will be presented.

## 2. THE CONTINUOUS EQUATION

In this section the properties of the continuous equation that are relevant to its approximation are studied. In particular, the regularity and smoothing present in the homogeneous equation are analyzed. These properties are different globally ($0 \leqq t < \infty$) and locally ($0 < t_0 \leqq t \leqq T < \infty$) because of the hyperbolic-parabolic degeneracy at $t = 0$.

The nature of this dichotomy is most easily understood in the context of the Cauchy problem, as in Layton [16]. First, these results will be presented to motivate and provide insight into the later results for the boundary value problem. Consider first the pure initial value problem

$$(tu_t(x, t))_t = u_{xx}(x, t) , \quad u(x, 0) = u_0(x) , \quad x \in \mathbb{R} , \quad t \geqq 0 .$$

If $Fu = \hat{u}(\theta)$, $F^{-1}w = \check{w}(x)$ denote the Fourier and inverse Fourier transforms respectively, the solution to the above is given by

$$u(x, t) = F^{-1}(J_0(2\sqrt{\theta^2 t})) \, \hat{u}_0(\theta)) ,$$

where $J_0$ is the Bessel function of the first kind of order zero, (see formula (3.3) of [16] for details). Since $J_0(z) = O(z^{-1/2})$ as $z$ (real) $\to \infty$, the above formula implies that for $t > 0$, $\hat{u}(\theta, t)$ decays as $|\theta| \to \infty$ faster than $\hat{u}_0(\theta)$ and thus $u(x, t)$ is smoother than $u_0(x)$, (the precise amount of smoothness gained is given in Theorem 3.2 of [16] for the Cauchy problem and in Theorem 2.2 below for the initial boundary value problem).

The previous formula provides more insight if it is transformed back into the $x$ variables. Using the formula

$$\int_0^\infty J_0(\alpha x) \cos \beta x \, dx = \begin{cases} (\alpha^2 - \beta^2)^{-1} , & 0 < \beta < \alpha , \\ \infty & \alpha = \beta , \\ 0 , & \alpha < \beta , \end{cases}$$

it can be calculated exactly. The solution to the initial value problem is then given by

$$u(x, t) = K(x, t) * u_0(x) , \quad (\text{convolution over } \mathbb{R}) ,$$

where

$$K(x,t)=\begin{cases}2(2\pi)^{-1/2}(4t-x^2)^{-1/2}, & 0\leqq |x|<2t^{1/2},\\ \infty, & 2t^{1/2}=|x|,\\ 0, & 2t^{1/2}<|x|.\end{cases}$$

A graph of $K(x,t)$ illustrates the dual hyperbolic-parabolic nature of the equation. At $t=0$, $K$ is a $\delta$ function. For $t<0$, $K*u_0$ gives the solution to the initial value problem as a weighted average of $u_0(x)$ for $-2t^{1/2}<x<2t^{1/2}$. The weights converge to delta functions at $\pm 2t^{1/2}$ as $t\to\infty$. Thus, as $t$ grows, the domain of dependence of the continuous equation "converges" to two points.

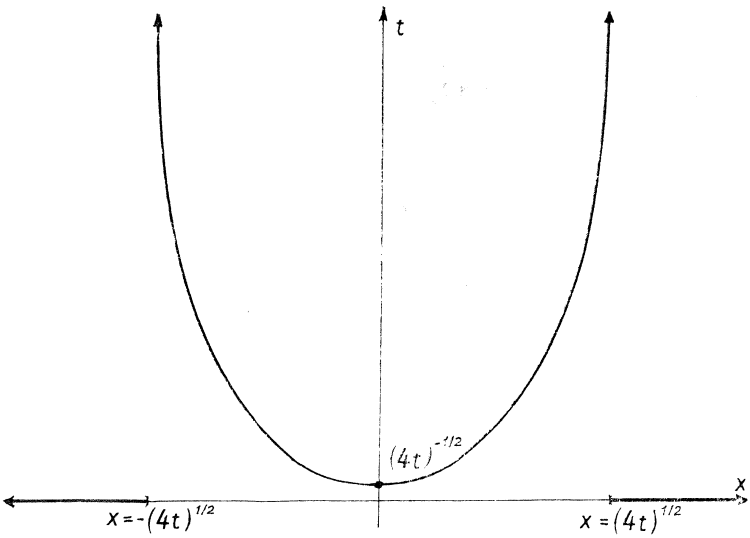


Figure 1. A sketch of $2^{-1}\sqrt{(2\pi)}\,K(x,t)$ for $t<0$.

We note here in passing that if $K$ is extended to be zero for $t<0$ then $K$ satisfies

$$(tK_t)_t-\Delta K=\delta$$

in the distributional sense, and thus, a particular solution to $(tW_t)_t-\Delta w=f$ is given by $K*f$ (convolution over $(x,t)\in\mathbb{R}\times\mathbb{R}$).

Next, the initial boundary value problem will be considered. Regularity of solutions to the homogeneous equation of the form (1.1) is given for $t<0$ and for $t\geqq 0$. Abstract differential equations of the form (1.1) were studied by Bernardi [4.a] and by Povoas [19.a]. In these works, regularity results for the inhomogeneous equation are given in an abstract setting by (essentially) combining Duhamrl's principle with regularity results for the homogeneous equation.

Let $\{\lambda_j\}_{j\geqq 1}$ and $\{\varphi_j\}_{j\geqq 1}$ denote the eigenvalues (in nondecreasing order) and eigenfunctions of the elliptic operator $L$. The eigenfunctions are assumed to be orthonormal in $L^2(\Omega)$. For $-\infty < s < \infty$ define the space

$$\dot{H}^s(\Omega) = \{v \in L^2(\Omega) \mid \|v\|_s = (\sum_j \lambda_j^s |(v, \varphi_j)|^2)^{1/2} < \infty\}\,.$$

Following [25], [5] it is easily seen that for an integer $s \geqq 0$,

$$\dot{H}^s(\Omega) = \{v \in W^{2,s}(\Omega) \mid L^j v = 0 \text{ and } \Omega, j < s/2\}\,,$$

and that the norms $\|\cdot\|_s$ and $\|\cdot\|_{2,s}$ are equivalent on $\dot{H}^s$.

The smoothness conditions upon the coefficients of $L$ can be considerably weakened in the definition of $\dot{H}^s$. In fact, following Aubin [1] pp. 244–251, the interpolation spaces can be defined provided only that $L$ maps $\dot{H}^1$ in a continuous and $1-1$ manner onto $\dot{H}^{-1}$. For conditions ensuring this see Gilbarg and Trudinger [15].

Using these $\dot{H}^s$ spaces, the solution operator to the boundary value problem (1), (3) is shown to possess a smoothing property analogous to the one derived in [16] for the Cauchy problem for the constant coefficient equation $(tu_t)_t = \Delta u$. Regularity results, both global and asymptotic, for the time derivatives of $u$ are also given. Also, note that the dependence of the smoothness of $u(x, t)$ upon the boundary conditions of $u_0$ is also incorporated into this approach in the definition of the spaces $\dot{H}^s(\Omega)$.

**Theorem 2.1.** *Assume $f \equiv 0$ and $u_0 \in \dot{H}^s(\Omega)$. Then a unique solution to (1.1), (1.2) exists that is bounded as $t \to 0$. That solution $u(x, t)$ is in $\dot{H}^{s+1/2}(\Omega)$ for $t > 0$ and*

$$\|u(t)\|_{s+1/2} \leqq Ct^{-1/4}\|u_0\|_s\,.$$

Proof. Let $u$ be the solution to (1.1), (1.2) with $f \equiv 0$. Expanding

$$u = \sum_{j=1}^{\infty} u_j(t)\, \varphi_j(x)\,, \quad u_j(t) = (u(x, t)\,, \quad \varphi_j(x))\,,$$

and substituting $u$ into the differential equation gives

$$\sum_{j=1}^{\infty} [(tu_{j,t})_t + \lambda_j u_j]\, \varphi_j = 0\,.$$

Thus, $u_j(t)$ satisfies the differential equation

$$t\ddot{u}_j + \dot{u}_j + \lambda_j u_j = 0\,, \quad u_j(0) \text{ given}\,, \quad j = 1, 2, \ldots. \tag{2.1}$$

Problem (2.1) has a regular singular point at $t = 0$. The Frobenius-Fuchs Theorem implies that the differential equation (2.1) has a unique solution that is bounded as $t \to 0$ and that bounded solution is analytic. Thus, $u_j(t) = \omega_j(t)\, u_j(0)$ where

$$\omega_j(t) = 1 + \sum_{n=1}^{\infty} a_n t^n$$

satisfies (2.1) subject to $\omega_j(0) = 1$. Differentiating $\omega_j$ and plugging into (2.1) gives the recurrence formula for $a_n$

$$a_1 = -\lambda_j, \quad a_{n+1} = -\lambda_j(n+1)^{-2} a_n,$$

from which $a_n = (-1)_n \lambda_j^n (n!)^{-2}$ and

$$\omega_j(t) = \sum_{n=1}^{\infty} \frac{(-1)^n (\lambda_j t)^n}{(n!)^2}.$$

Since $\lambda_j > 0$, one can set $z = 2\sqrt{(\lambda_j t)}$ to obtain the Bessel function $J_0(z)$. Hence, $\omega_j(t) = J_0(2\sqrt{\lambda_j t})$ and

$$u(x, t) = \sum_{j=1}^{\infty} J_0(2\sqrt{\lambda_j t})\,(u_0(x), \ \varphi_j(x))\,\varphi_j(x).$$

It is known (see the Batemann manuscript Project [11]) that for $|\arg(z)| < \pi - \varepsilon$, $J_0(z)$ is asymptotic to $z^{-1/2}\cos(z - \Gamma/4)$. Thus,

$$|\omega_j(t)| \leqq Ct^{-1/3}\lambda_j^{-1/4}, \quad j = 1, 2, 3, \ldots,$$

where $C$ is a constant independen: of $j$.

Assuming $u_0 \in \dot{H}^s$:

$$u_0 = \sum_{j=1}^{\infty} u_{0,j}\,\varphi_j(x), \quad \|u_0\|_s^2 = \sum_{j=1}^{\infty} \lambda_j^s |u_{0,j}|^2 < \infty,$$

consider the $\dot{H}^{s+1/2}$ norm of $u(x, t)$:

$$\|u(x, t)\|_{s+1/2}^2 = \sum_{j=1}^{\infty} \lambda_j^{s+1/2} |u_j(t)|^2, \quad u_j(t) = (u(x, t), \varphi_j)$$
$$= \sum_{j=1}^{\infty} \lambda_j^{s+1/2} |\omega_j(t)|^2\, |u_{0,j}|^2 \leqq C \sum_{j=1}^{\infty} \lambda_j^{s+1/2} t^{-1/2} \lambda_j^{-1/2} |u_{0,j}|^2 \leqq$$
$$\leqq Ct^{-1/2} \|u_0\|_s^2 < \quad (\text{for } t > 0). \qquad \square$$

Note that the proof of Theorem 1 gives an explicit representation for the solution operator of (1.1), (1.2) when $f \equiv 0$:

$$u(x, t) = \sum_{j=1}^{\infty} J_0(2\sqrt{\lambda_j t})\,(u_0, \varphi_j)\,\varphi_j(x), \tag{2.2}$$

where $J_0$ is the Bessel function of order zero. Naturally, when $f(x, t) \not\equiv 0$ one also obtains and explicit representation for the solution operator of the inhomogeneous equation from (2.2) by using Duhamel's principle.

For numerical methods to converge rapidly it will generally be necessary for time derivatives of $u(x, t)$ to be smooth. Regularity results for $u_t$ and $u_{tt}$ (and higher time derivatives if necessary) follow from the representation (2.2). Noteworthy is the fact that $\|u_{tt}(t)\|$ must below up as $t \to 0$, but $\|tu_{tt}(t)\|$ will remain bounded. This fact is reflected in the estimates on the discrete equations: only smoothness of $tu_{tt}$ (rather than $u_{tt}$) can be assumed. The next theorem contains global (including $t = 0$) smoothness results, and smoothness results for $t > 0$.

**Theorem 2.2.** *Assume $f \equiv 0$, then for $-\infty < s < \infty$, the following hold:*

$$\begin{aligned}
&\|u_t(t)\|_s \leqq C\|u_0\|_{s+2}\,, \\
&\|u_t(t)\|_s \leqq Ct^{-3,4}\|u_0\|_{s+1/2}\,, \\
&\|u_{tt}(t)\| \to \infty\,, \quad as \quad t \to 0\,, \\
&\|tu_{tt}(t)\|_s \leqq C\|u_0\|_{s+2}\,, \\
&\|tu_{tt}(t)\|_s \leqq Ct^{-1/2}\|u_0\|_{s+1}\,,
\end{aligned}$$

*where $C$ is a constant independent of $u_0$ and $t$.*

The following lemma is useful for the proof of Theorem 2.2 and follows easily from the three term recurrence relation satisfied by Bessel functions and the known asymptotic expansions of $J_0$ and $J_1$, see Erdelyi, Magnus, Oberhettinger, and Tricomi [11].

**Lemma 2.1.** *For real $z$,*

$$|J_0(z)|\,, \quad |J_0'(z)|\,, \quad |J_0''(z)| \leqq C|z|^{-1/2}\,.$$

Proof. The known asymptotic expansions for $J_0(z)$ and $J_1(z)$ imply that the above holds for $J_0$ and $J_1$. $J_0$ satisfies

$$2J_0'(z) = J_{-1}(z) - J_{+1}(z) = -2J_1(z)\,,$$

so that $J_0'(z)$ is bounded by $C|z|^{-1/2}$. $J_0''$ is given by Bessel's equation

$$J_0''(x) = -x^{-1}J_0'(x) - J_0(x)\,,$$

so that $J_0''(z)$ will also be bounded by $C|z|^{-1/2}$ for $z$ real. □

Note also that $x^{-1}J_0'(x)$ is uniformly bounded for $x$ real.

Proof of Theorem 2.2. The representation (2.2) immediately gives that

$$u_t(x, t) = \sum_{j=1}^{\infty} J_0'(2\sqrt{\lambda_j t})\,\lambda_j^{1/2}t_j^{-1/2}u_{0,j}\,\varphi_j(x)\,,$$

$$u_{tt}(x, t) = \sum_{j=1}^{\infty}[J_0''(2\sqrt{\lambda_j t})\,\lambda_j t^{-1} - 1/2\lambda_j^{1/2}t^{-3/2}\,J_0'(2\sqrt{\lambda_j t})]\,u_{0j}\,\varphi_j(x)\,,$$

where $u_{0,j} = (u_0, \varphi_j)$.

A simple calculation with the series for $J_0(z)$ gives that $J_0'(z) = 0(z)$ as $z \to 0$ and $J_0''(z) = -1/2 + 0(z^2)$ as $z \to 0$. Hence it is evident that $\|u_{tt}(t)\|$ must blow up as $t \to 0$ only provided $u_0 \not\equiv 0$.

Next, consider $\|u_t(t)\|_s^2$;

$$\|u_t(t)\|_s^2 = \sum_{j=1}^{\infty} [J_0'(2\sqrt{\lambda_j t})(\lambda_j t)^{-1/2}]^2 \lambda_j^{2+s} u_{0,j}^2 .$$

Since $x^{-1} J_0'(x)$ is uniformly bounded for real $x$, it follows that

$$\|u_t(t)\|_s^2 \leqq C \sum_j \lambda_j^{2+s} u_{0,j} = C\|u_0\|_{2+s}^2 .$$

For the estimate with $t < 0$, Lemma 2.1 is used. Indeed,

$$\|u_t(t)\|_s^2 = \sum_{j=1}^{\infty} J_0'(2\sqrt{\lambda_j t})^2 \lambda_j^{1+s} t^{-1} u_{0,j}^2 \leqq$$

$$\leqq C \sum_{j=1}^{\infty} (\lambda_j t)^{-1/2} \lambda_j^{1+s} t^{-1} u_{0,j}^2 \leqq C t^{-3/2} \|u_0\|_{s+1/2}^2 .$$

Consider now $\|t u_{tt}(t)\|_s$:

$$\|t u_{tt}(t)\|_s^2 = \sum_{j=1}^{\infty} [J''(2\sqrt{\lambda_j t})\lambda_j - 1/2\lambda_j^{1/2} t^{-1/2} J_0'(2\sqrt{\lambda_j t})]^2 \lambda_j^s u_{0,j}^2 \leqq \tag{2.3}$$

$$\leqq C \sum_{j=1}^{\infty} J_0''(2\sqrt{\lambda_j t})^2 \lambda_j^{2+s} u_{0,j}^2 + C \sum_{j=1}^{\infty} \lambda_j^{1+s} t^{-1} J_0'(2\sqrt{\lambda_j t})^2 u_{0,j}^2 .$$

Since $J_0''(x)$ and $x^{-1} J_0'(x)$ are uniformly bounded, it follows that

$$\|t u_{tt}(t)\|_s^2 \leqq C \max |J_0''|^2 \sum_{j=1}^{\infty} \lambda_j^{2+s} u_{0,j}^2 + C \max |x^{-1} J_0'(x)| \sum_{j=1}^{\infty} \lambda_j^{2+s} u_{0,j}^2 \leqq C\|u_0\|_{2+s}^2 .$$

For $t > 0$, (2.3) and Lemma 2.1 give that

$$\|t u_{tt}(t)\|_s^2 \leqq C\Big[\sum_{j=1}^{\infty} [(\lambda_j t)^{-1/2}]^2 \lambda_j^{2+s} u_{0,j}^2 + \sum_{j=1}^{\infty} \lambda_j t^{-1} (\lambda_j t)^2 u_{0,j}^2\Big]$$

$$\leqq C t^{-1} \|u_0\|_{s+1}^2 . \qquad \square$$

Zlámal [23], [24] has shown that the time derivatives of the solution of the singular perturbation problem

$$\varepsilon w_{tt} + w_t + Lw = f(x, t), \quad w(0), \quad w_t(0) \quad \text{given},$$

have boundary layers at $t = 0$ as $\varepsilon \to 0$. It is interesting that $u$ and $u_0$ do not, but that $u_{tt}$ (and higher time derivatives) do share this feature of the above equation.

## 3. GALERKIN APPROXIMATIONS: STABILITY AND (LINEAR) CONVERGENCE

$S^h$ will denote a finite element space. That is, $S^h$ consists of functions in $W^{2,1}(\Omega)$ that vanish on $\partial\Omega$ that are typically piecewise polynomials on a tringulation of $\Omega$ satisfying a smoothness requirement across the edges. $S^h$ is assumed to have the following approximation property. Given $w \in \dot{H}^1 \cap H^s$,

$$\inf_{\chi \in S^h} (\|w - \chi\| + h\|w - \chi\|_1) \leqq ch^s\|w\|_s, \quad 1 \leqq s \leqq r.$$

First, the stability of the Galerkin method will be considered. There are two situations where a systematic treatment can be presented: the autonomous nonlinear equation and the forced linear equation.

First consider the autonomous nonlinear equation written in the convenient form

$$(3.1) \qquad (tu_t)_t = -Lu + F'(u).$$

The initial and boundary conditions (1.3) are also imposed. In this case the non-increasing energy of the continuous equation is easily found to be

$$(3.2) \qquad E(u) \equiv \int_\Omega [t(u_t)^2 - 2F(u)]\,\mathrm{d}x + a(u, u).$$

Thus, it is easy to see that when $F(u)$ is non-positive the continuous equation is stable:

$$(0 \leqq)\, E(u(t)) \leqq E(u_0), \quad 0 \leqq t < \infty.$$

The Galerkin approximation shares this feature of the continuous equation.

**Proposition 3.1.** *Let $U$ be the Galerkin approximation to* (3.1). *Then*

$$E(U(t)) \leqq E(U(0)), \quad 0 \leqq t < \infty.$$

*Thus, if $F \leqq 0$ the Galerkin approximation is stable in the same sense as the continuous equation.*

Proof. Set $v = U_t$ in the discrete equations for $U$. This gives

$$(tU_{tt}, U_t) + (U_t, U_t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,a(U, U) = (F'(U), U_t),$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega t(U_t)^2 - 2F(U)\,\mathrm{d}x + a(U, U)\right] = -\|U_t\|^2 \leqq 0. \qquad \square$$

For the error estimates the stability of the method in the case of a forced linear equation is particularly important. Since the stability result in this case follows in much the same way as the above, the proof of the next proposition will be omitted.

**Proposition 3.2.** *Let $U$ be Galerkin approximation to* (1.1), (1.3). *Then*

$$t\|U_t(t)\|^2 + a(U(t), U(t)) \leqq a(U(0), U(0)) + \int_0^t \|f(\cdot, s)\|^2\,\mathrm{d}s. \qquad \square$$

Since, in the linear case, the error in the method satisfies an inhomogeneous equation of the form (1.1), error estimates follow from the stability result in Proposition 3.2. The next theorem asserts that the Galerkin approximation is optimal in $\dot{H}^1$. When $U(0)$ is taken to be the elliptic projection of $u_0$, optimality in $L^2$ will also follow.

**Theorem 3.1.** *Assume $u \in L^\infty(\dot{H}^r)$, $u_t, tu_{tt} \in L^\infty(\dot{H}^{r-1})$. Then*

$$\max_{0 \leqq t \leqq T} \|u - U\|_1 \leqq Ch^{r-1}[\|u\|_r + \|u_t\|_{r-1} + \|tu_{tt}\|_{r-1}] + C\|U(0) - u_0\|_1$$

Proof. Let $w$ be the elliptic projection of $u$ into $S^h$, i.e. $w$ satisfies $a(u - w, v) = 0$, for all $v \in S^h$.

Define $\varphi = U - w$, $\eta = u - w$. It is known that $\eta$ satisfies the estimate (see Babuska and Aziz [2])

$$\|D_t^j\eta\| + h\|D_t^j\eta\|_1 \leqq Ch^s\|D_t^j u\|_s\,, \quad j = 0, 1, 2, , \quad 1 \leqq s \leqq r\,. \tag{3.3}$$

$\varphi = U - w$ satisfies the equation

$$((t\varphi)_t, v) + a(\varphi, v) = ((t\eta_t)_t, v) + a(\eta, v)$$

for all $v$ in $S^h$. Since $a(\eta, v) = 0$ this becomes

$$((t\varphi_t)_t, v) + a(\varphi, v) = ((t\eta_t)_t, v)\,, \quad \text{for all} \quad v \in S^h\,. \tag{3.4}$$

Applying the stability result (Proposition 3.2), with identification $\varphi \equiv U$, $(t\eta_t)_t \equiv f$, we obtain

$$t\|\varphi_t\|^2 + \|\varphi\|_1^2 \leqq C\|\varphi(0)\|_1^2 + C\int_0^t \|(t\eta_t)_t\|^2\, ds\,. \tag{3.5}$$

This inequality, (3.3) with $s = r$ and the triangle inequality, $\|U - u\| \leqq \|\varphi\| + \|\eta\|$, yield the theorem. □

One choice of the initial data $U(0)$ is the elliptic projection of $u_0$:

$$a(U(0) - u_0, v) = 0 \quad \text{for all} \quad v \in S^h\,. \tag{3.6}$$

Computationally, this represents no additional work over choosing $U(0)$ to be the $L^2$ projection of $u_0$. Both choices involve solving a linear system for $c(0)$ with matrices $S$ and $M$, respectively.

With this choice of $U(0)$, the Galerkin approximation is optimal in $L^2$.

**Theorem 3.2.** *Assume that $U(0)$ is chosen as (3.6) and that, $u$, $u_t$, $tu_{tt} \in L^\infty(\dot{H}^r)$. Then*

$$\max_{0 \leqq t \leqq T} \{t^{1/2}\|U_t - u_t\| + \|U - u\|\} \leqq Ch^r \max_{0 \leqq t \leqq T} \{\|u\|_r + \|u_t\|_r + \|tu_{tt}\|_r\}\,.$$

Proof. Define $w$, $\varphi$, $\eta$ as in the proof of the previous theorem. $\varphi$ satisfies the equation (3.4):

$$((t\varphi_t)_t, v) + a(\varphi, v) = ((t\eta_t)_t, v)\,.$$

This is an equation of the form (3.1) where $\varphi \equiv U$ and $(t\eta_t)_t \equiv f$. Proposition 3.2 gives

$$t\|\varphi_t(t)\|^2 + \|\varphi(t)\|_1^2 \leqq C \int_0^t \|(t\eta_t)_t\|^2 \, \mathrm{d}s \,, \tag{3.7}$$

where the equivalence of $a(\cdot, \cdot)$ with $\|\cdot\|_1$ and the choice (3.6) of $U(0)$ has been used.

Since $\|\varphi\|_1 \geqq \|\varphi\|$, the result follows. □

Remark. A close look at (3.7) shows that when $U(0)$ is chosen by (3.7), then $\|U - w\|_1 = 0(h^r)$. In one space dimension $(n = 1)$ $L^\infty$ estimates follow from the fact that $\|\varphi\|_1 = 0(h^r)$, the known $L^\infty$ estimates for the steady state problem, and the Sobolev theorem.

In particular, in Douglas, Dupont and Wahlbin [9] it was shown that if $S^h$ consists of $C^k(0 \leqq k \leqq r - 1)$ piecewise polynomials of degree $r - 1$ on a quasiuniform mesh then

$$|u - w| \leqq Ch^r|u|_r \,.$$

Thus, in this case,

$$\max_{0 \leqq t \leqq T} |U(t) - u(t)| \leqq Ch^r \max_{0 \leqq t \leqq T} [|u|_r + \|u\|_r + \|u_t\|_r + \|tu_{tt}\|_r]$$

holds and the Galerkin method is also optimal in $L^\infty$.

## 4. THE SEMILINEAR EQUATION

In this section the Galerkin method for the semilinear equation (1.2), (1.3) is considered; $U : [0, T] \to S^h$ satisfies

$$((tU_t)_t, v) + a(U, v) = (f(U), v) \quad \text{for all} \quad v \in S^h \,.$$

It is shown that when $f$ is globally Lipschitz the Galerkin approximation $U$ converges to $u$ optimally in $\dot{H}^1$ and in $L^2$ when $U(0)$ is chosen to be the elliptic projection of the initial data.

The situation when $f(u)$ is only locally Lipschitz is more delicate. In one space dimension, $\dot{H}^1$ estimates on the approximate solution are obtained. These imply that the $L^\infty$ norm of $U$ must be bounded so that the argument of the globally Lipschitz case gives that $U$ converges to $u$ with optimal rates. In higher space dimensions this device is not available and $L^\infty$ estimates require the use of inverse assumptions on $S^h$. When these hold, optimal rates of convergence can be shown.

**Theorem 4.1.** *Assume $f$ is globally Lipschitz and $u, u_t, tu_{tt} \in \dot{H}^r$. Then*

$$\max_{0 \leqq t \leqq T} \|u(t) - U(t)\|_1 \leqq Ch^{r-1} \max_{0 \leqq t \leqq T} [\|u(t)\|_r + \|u_t(t)\|_r + \|tu_{tt}\|_r] + C\|u_0 - U(0)\|_1 \,.$$

*When $U(0)$ is chosen by (3.6), it follows that*

$$\max_{0 \leqq t \leqq T} [t^{1/2}\|u_t(t) - U_t(t)\| + \|u(t) - U(t)\|] \leqq$$
$$\leqq Ch^r \max_{0 \leqq t \leqq T} [\|u(t)\|_r + \|u_t(t)\|_t + \|tu_{tt}(t)\|_r] .$$

Proof. As in the proof of Theorem 3.2, let $w \in S^h$ satisfy $a(u - w, v) = 0$ for all $v \in S^h$, and define $\varphi = U - w$, $\eta = u - w$. Since $a(\eta, v) = 0$, $\varphi$ satisfies the equation

$$((t\varphi_t)_t, v) + a(\varphi, v) = (f(U) - f(w), v) +$$
$$(f(w) - f(u), v) + ((t\eta_t)_t, v) , \quad \text{for all} \quad v \in S^h .$$

Setting $v = \varphi_t \in S^h$ and using the Cauchy-Schwarz inequality we obtain

$$\frac{1}{2} t \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi_t\|^2 + \|\varphi_t\|^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} a(\varphi, \varphi) \leqq \frac{1}{2\varepsilon} [\|(t\eta_t)_t\|^2 + \|f(U) - f(w)\|^2 +$$

$$+ \|f(u) - f(w)\|^2] + \frac{3}{2} \varepsilon \|\varphi_t\|^2 \leqq \frac{1}{2\varepsilon} [\|(t\eta_t)_t^2\| + L^2\|\eta\|^2] + \frac{1}{2\varepsilon} L^2 \|\varphi\|^2 + \frac{3}{2} \varepsilon \|\varphi_t\|^2 ,$$

where $L$ is the Lipschitz constant of $f$. Picking $\varepsilon = 1/3$ and rearranging the left hand side of the inequality gives

$$\frac{\mathrm{d}}{\mathrm{d}t} [t\|\varphi_t\|^2 + a(\varphi, \varphi)] \leqq 3[\|(t\eta_t)_t\|^2 + L^2\|\eta\|^2] + 3L\|\varphi\|^2 .$$

Since $a(\varphi, \varphi) \geqq \|\varphi\|^2$, Gronwall's inequality can be applied to the above to yield

$$t\|\varphi_t(t)\|^2 + a(\varphi(t), \varphi(t)) \leqq$$
$$\leqq a(\varphi(0), \varphi(0)) + C \max_{0 \leqq t \leqq T} [\|(t\eta_t)_t\|^2 + \|\eta(t)\|^2] .$$

The $\dot{H}^1$ estimate now follows from the triangle inequality and the equality of $a(\cdot, \cdot)$ with the $\dot{H}^1$ norm.

When $U(0)$ is chosen as the elliptic projection of $u_0$, then $\varphi(0) = 0$ and $L^a$ estimate follows from the fact that $a(\cdot, \cdot) \geqq c\|\cdot\|^2$. □

The next theorem cosniders the case of one space dimension and locally Lipschitz $f$. The main problem is that of showing that $|U|$ is bounded uniformly in $h$. Once this is accomplished the proof of convergence in the globally Lipschits case goes over to the present context.

**Theorem 4.2.** *Assume $u$, $u_t$, $tu_{tt} \in \dot{H}^r$ for $0 \leqq t \leqq T$, $U(0)$ is chosen as the elliptic projection of $u_0$ and $f$ is locally Lipschitz. Then for $0 \leqq h \leqq h_0$,*

$$\max_{0 \leqq t \leqq T} \|u(t) - U(t)\| \leqq Ch^r$$

*holds, where C depends upon T and the $\dot{H}^1$ norms of $u$, $u_t$ and $tu_{tt}$.*

Proof. Suppose that $|u(x, t)| \leqq K$ for $0 \leqq t \leqq T$. Let $\delta < 0$ be a fixed positive number. It will be shown that for $h$ sufficiently small, $|U(x, t)| \leqq K + \delta$ for $0 \leqq t \leqq T$ Let $A^h$ denote $\{t : 0 \leqq t \leqq T \text{ and } |U(x, t)| < K + \delta\}$.

Since $U(0)$ is chosen to be the elliptic projection of $u_0$, $U(0)$ satisfies

$$\|U(0) - u_0\|_1 \leqq Ch^{r-1}.$$

Hence $|U(0) - u_0| \leqq Ch^{r-1} \leqq \delta/2$ for $h \leqq h_1$. Thus $0 \in A^h$, and $A^h$ is nonempty. Next suppose that $t^*$ is the largest $t$ such that $[0, t^*] \subset A^h$. Thus, for $0 \leqq t \leqq t^*$, Theorem 4.1 shows that

$$|U - u| \leqq \|U - u\|_1 \leqq C(T)\, h^{r-1}, \quad 0 \leqq t \leqq t^*.$$

Thus, for $h \leqq h_2$ it follows that $|U| \leqq K + (\delta/2) < K + \delta$. By continuity, $t^*$ cannot be the largest such $t$ unless $t^* = T$.

The proof now follows the proof of Theorem 4.1, with all results holding for $h \leqq \min(h_1, h_2)$. □

Remark 2. In one dimension, the above estimates imply that the Galerkin approximation is optimal in $L^\infty$ when $U(0)$ is chosen as $w(0)$. The proof of this fact is identical to the linear case, see Remark 1.

In higher space dimensions, following Thomée and Wahlbin [22], the following two assumptions upon the space $S^h$ will be made. The first is a standard inverse assumption and the second an approximation theoretic assumption concerning the space $S^h$ and the smoothness of $u(x, t)$.

A 1 For all $\chi \in S^h$, $|x| \leqq Ch^{-v} \|\chi\|$ holds for $h \leqq h_1$ and some $v < r$.

A 2 $\lim\limits_{h \to 0} [\sup\limits_{0 \leqq t \leqq T} [\inf\limits_{\chi \in S^h} \{|u(t) - \chi| + h^{-v}\|u(t) - \chi\|\}]] = 0.$

When $S^h$ consists of piecewise polynomials of degree $r - 1$ satisfying certain regularity assumptions on the triangulation the elliptic projection of the steady state problem has been shown to converge in $L^\infty$ to $u$. Nitsche [18] has shown that if $L = -\Delta$ and $r < 2$ then $|u - w| \leqq Ch^r|u|_2$. Scott [21] has shown in the case of Neumann boundary conditions (not considered here) that for $r = 2$, $\||u - w| \leqq Ch^2 \ln(h)\, |u|_2$. Analogous results have been shown to hold in the variable coefficient and nonlinear cases, cf. among others the papers of Nitsche [19] and Freshe and Rannacher [12]. When any of these estimates apply, so that $|u - w| \to 0$ as $h \to 0$, A 2 can be dispensed with.

**Theorem 4.3.** *Assume $f$ is locally Lipschitz, $u$, $u_t$, $tu_{tt} \in \dot{H}^r$,* A 1 *holds and either $|u - w| \to 0$ as $h \to 0$ or* A 2*holds. Then, if $U(0)$ is chosen by (3.6),*

$$\max_{0 \leqq t \leqq T} \|u(t) - U(t)\| \leqq Ch^r \max_{0 \leqq t \leqq T} [\|u\|_r + \|u_t\|_r + \|tu_{tt}\|_r + |u|_r].$$

Proof. As before, the result will follow provided an $L^\infty$ estimate can be obtained for $w$ and $U$. Suppose $|u| \leqq K$ and let $\delta < 0$ be given. If $|u - w| \to 0$ as $h \to 0$ then clearly $|w| \leqq K + \delta$ for $h$ sufficiently small. On the other hand, suppose A 2 holds. Let $t^*$ be the largest $t < 0$ such that $|w(t)| \leqq K + \delta$ for $0 \leqq t \leqq t^*$. In this case, for $\chi \in S^h$,

$$|u - w| \leqq |\eta| \leqq |u - \chi| + |\chi - w| \leqq |u - \chi| + Ch^{-v}\|\chi - w\| .$$

Thus, since $\|u - w\| = 0(h^r)$.

$$|u - w| \leqq |u - \chi| + Ch^{-v}[\|u - \chi\| + \|\eta\|] \leqq$$
$$\leqq C \inf_{\chi \in S^h} [|u - \chi| + h^{-v}\|u - \chi\|] + h^{-v} Ch^r .$$

Since $r < \mu$, A 2 implies that $|u - w| \to 0$ as $h \to 0$ and, for $h$ sufficiently small, $|w| < K + (\delta/2)$. Thus, $t^* = T$.

Next, it will be shown that $|U(t)| < K + \delta$ for $0 \leqq t \leqq T$ and $h$ sufficiently small. Indeed, $|U(0) - u_0| \to 0$ as $h \to 0$ so that $|U(0)| < K + (\delta/2)$ for $h$ sufficiently small. Let $t^*$ be the largest $t$ such that $|U(t)| \leqq K + \delta$ for $0 \leqq t \leqq t^*$. A 1 implies

$$|U - w| \leqq Ch^{-v}\|U - w\| \leqq Ch^{r-v} \quad (r < v) .$$

Thus, $|U - w| \to 0$ as $h \to 0$ and $t^* = T$. □

Thus, even when $f$ is locally Lipschitz the Galerkin approximation will converge to $u$ optimally, locally in time. The situation for large time, $0 \leqq t < \infty$, is much more complicated. For example, when $f(u)$ has the form $f(u) = \lambda u^p$ with $p < 1$, and $\lambda < 0$, then the true solution to

$$(tu_t)_t = \Delta u + \lambda u^p , \quad p < 1 , \quad \lambda < 0$$

can blow up in finite time in $L^\infty$. (Levine [17] has shown this for the Euler-Poisson-Darboux Equation, an analogous result can be expected here, see also Reed [20]). For example, when $p = 3$ the nonincreasing energy of the continuous equation and the Galerkin approximation can easily shown to be (see (3.1), (3.2))

$$E(u(t)) = \int_\Omega \left[ t(u_t)^2 + |\nabla u|^2 - \frac{\lambda}{2} u^4 \right] \mathrm{d}x .$$

When $\lambda < 0$ the energy $E(t)$ can become negative and thus the blow-up can be expected. However, when $\lambda < 0$, both the continuous and discrete equation will not blow up since $E$ positive define (Proposition 3.1).

Thus, the good and bad points of the estimates in this section are apparent: they hold for very general $f(u)$ terms but are only local in $t$ (i.e. valid cnly for $0 \leqq t \leqq \leqq T^* < \infty$). This reflects the state of the continuous equation. Estimates on semi-linear evolution equations that are valid for all $t$, $0 \leqq t < \infty$, are much more difficult to obtain than estimates valid for small time. As the above example indicates, a unified treatment of such problems for $0 \leqq t < \infty$ is probably not possible, since the specific form of the nonlinearity is critical to the behavior of the continuous equation for large $t$.

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## Souhrn

# ENERGETICKÁ ANALÝZA DEGENEROVANÝCH HYPERBOLICKÝCH PARCIÁLNÍCH DIFERENCIÁLNÍCH ROVNIC

WILLIAM J. LAYTON

Je provedena energetická analýza obvyklé semidiskrétní Galerkinovy metody pro semilineární rovnici v oblasti $\Omega$

$$(tu_t)_t = \sum_{i,j=1}^{N} (a_{ij}(x)\, u_{x_i})_{x_j} - a_0(x) + f(u)\,, \tag{E}$$

s okrajovou a počáteční podmínkou $u = 0$ na $\partial\Omega$ a $u(x, 0) = u_0$. Uvažovaná rovnice je degenerovaná pro $t = 0$ a proto i v případě $f \equiv 0$ mohou časové derivace $u$ být neomezené při $t \to 0$. V případě lokálně Lipschitzovské funkce $f$ mohou řešení divergovat při $t > 0$ v nekonečném čase.

V lineárním případě je dokázána stabilita a konvergence ve $W^{2,1}$ bez předpokladu hladkosti $u_{tt}$ (která může být při $t \to 0$). Konvergence aproximací k $u$ je dokázána v případě nelineární lokálně lipschitzovské funkce $f$. Konvergence nastává v oblasti, kde $u(x, t)$ existuje a je hladká. Je udána rychlost konvergence.

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