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A NOTE ON CRITICAL TIMES OF 2×2 QUASILINEAR HYPERBOLIC SYSTEMS

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1. INTRODUCTION

In applications we often meet problems which lead to quasilinear hyperbolic systems.

In this note we consider a 2×2 quasilinear hyperbolic system of equations of the type

$$
\begin{aligned}
u_t + au_x + bv_x &= 0\,, \\
v_t + cu_x + dv_x &= 0\,,
\end{aligned}
\tag{1.1}
$$

where a, b, c, d are functions of u and v, u and v are functions of a space variable x ($x \geqq 0$) and a time variable t ($t \geqq 0$). The term "hyperbolic" means, as usual, that the matrix

$$
\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}
\tag{1.2}
$$

has real and distinct eigenvalues λ and μ for all relevant values of u and v. It is well known that solutions of the system (1.1) with correctly posed initial and boundary conditions develop singularities in a finite time unless the data are very special [2–4]. In our note we show a situation in which the critical time, after which the smooth solution to (1.1) does not exist, can be computed analytically, and find the relevant expression. This is useful especially in engineering problems, where the general estimates as derived e.g. in [1] and [2] may appear to be insufficient. We thus show that it is sometimes possible to localize the first appearance of a shock wave [2] in time and space.

In Section 2 we present some general facts and a detailed analysis of our problem. In Section 3 we give two examples arising from the one-dimensional compressible isentropic fluid flow.

2. THE PROBLEM AND ITS ANALYSIS

Suppose that for all relevant values u, v (i.e. for all values of unknown functions that must be considered) we have $\lambda(u, v) < 0 < \mu(u, v)$ (if $0 \notin (\lambda, \mu)$ we can achieve this, at least locally for (u, v), by the transformation $x := x - \alpha t$ with a suitable constant α). We consider the system (1.1) in $\{(x, t);\ x \geqq 0,\ 0 \leqq t < t_c\}$ with the initial and boundary conditions

$$(2.1) \qquad \begin{gathered} u(x, 0) = u_0 = \text{const}\,, \quad v(x, 0) = v_0 = \text{const.}\,, \\ u(0, t) = u_1(t)\,, \quad 0 \leqq t < t_c\,, \quad u_1(0) = u_0\,, \end{gathered}$$

where u_0, v_0, $u_1(t)$ ($t \in [0, \tau_1]$) are given data and t_c is the time of the first appearance of a singularity which is indicated by the intersecting of some characteristics of the same family. The value of t_c depends on the data and is to be determined. The system (1.1), (2.1) can be transformed [2] to the system in the Riemann invariants r, s:

$$(2.2) \qquad \begin{aligned} & r_t + \lambda r_x = 0\,, \\ & s_t + \mu s_x = 0\,, \quad x > 0\,, \quad 0 < t < t_c\,, \\ & r(x, 0) = r_0 = \text{const.}\,, \quad s(x, 0) = s_0 = \text{const.}\,, \quad x \geqq 0\,, \\ & s(0, t) \ = \psi(u_0, v_0, u_1(t)) \equiv \sigma(t)\,, \quad 0 < t < t_c\,, \end{aligned}$$

where we consider λ and μ as functions of r, s. As the general theory ensures the existence of the transformation $(u, v) \leftrightarrow (r, s)$ only locally, we must assume that it exists either globally or at least for all values of u, v induced by the initial and boundary conditions (2.1). If a continuously differentiable solution (with a possible discontinuity of its derivatives on a line) of (2.2) exists on $[0, \infty) \times (0, t_c)$ with some $t_c > 0$ then by the usual considerations on characteristics we find that $r(x, t) = r_0$, $s(x, t) = s_0$ in $\Omega_0 = \{(x, t);\ x \geqq \mu(r_0, s_0)\, t\}$ and $r = r_0$ in $\Omega_1 = \{(x, t);\ 0 \leqq x < < \mu(r_0, s_0)\, t,\ 0 \leqq t < t_c\}$. As $s = \text{const.}$ along any μ-characteristic, we have $s(x, t) = = s(0, \tau) = \sigma(\tau)$ along the μ-characteristic in Ω_1 issuing from a point $(0, \tau)$. Since r is constant in Ω_1, both r and s are constant along the characteristic

$$\frac{dx}{dt} = \mu(r_0, \sigma(\tau)) \equiv m(\tau)\,,$$

which is thus the straight line

$$(2.3) \qquad x = m(\tau)\,(t - \tau)\,.$$

We define $\Omega = \{(x, t);\ 0 \leqq x < m(0)\, t$, there exists a unique continuously differentiable solution $\tau = \tau(x, t) > 0$ of (2.3)$\}$ and

$$(2.4) \qquad r(x, t) = r_0\,, \quad s(x, t) = \sigma(\tau(x, t))\,.$$

Suppose that a, b, c, d and σ are continuously differentiable functions. Denote

$$(2.5) \qquad \varphi(\tau; x, t) = m(\tau)\,(t - \tau) - x\,, \quad \tau \geqq 0\,, \quad x \geqq 0\,, \quad t \geqq 0\,.$$

If $0 \leqq x \leqq m(0)\, t$ then we have $\varphi(0; x, t) = m(0)\, t - x \geqq 0$ and $\varphi(t; x, t) = -x \leqq 0$. Hence there exists at least one $\tau = \tau(x, t) \in [0, t]$ which solves (2.3). Choose a fixed (x, t), $0 \leqq x \leqq m(0)\, t$, and investigate the function φ for $\tau \in [0, t]$. If $m'(\tau) \leqq 0$ then

(2.6) $$\frac{\partial \varphi}{\partial \tau}(\tau; x, t) \equiv (t - \tau)\, m'(\tau) - m(\tau) < 0\,.$$

If

(2.7) $$m'(\tau) > 0\,, \quad \tau \in [0, \tau_1]$$

then (2.6) holds if and only if $t < \tau + m(\tau)\, m'(\tau)^{-1}$, $\tau \in [0, t]$. Thus (2.6) holds in $\{(x, t);\, 0 < x \leqq m(0)\, t, 0 < t < \min_{\tau \in [0, \tau_1]} [\tau + m(\tau)\, m'(\tau)^{-1}]\}$, which by the monotonicity of φ in τ and the Implicit Function Theorem is a part of Ω. To find the whole of Ω we investigate the cross intersections of the characteristics (2.3) issuing from two different points $(0, \sigma)$ and $(0, \tau)$, where $0 \leqq \sigma < \tau \leqq \tau_1$. In addition to (2.7) let us also have

(2.8) $$m''(\tau) < 0 \quad \text{for} \quad \tau \in [0, \tau_1]\,.$$

The characteristics

$$x = m(\tau)\,(t - \tau)\,,$$
$$m = m(\sigma)\,(t - \sigma)$$

intersect at the time

$$t = t(\sigma, \tau) = \tau + \frac{m(\sigma)\,(\tau - \sigma)}{m(\tau) - m(\sigma)}$$

where we set

$$t(\tau, \tau) = \tau + \frac{m(\tau)}{m'(\tau)}$$

by continuity. We have to find the minimal value of $t(\sigma, \tau)$ for $0 \leqq \sigma \leqq \tau \leqq \tau_1$. Our assumptions (2.7) and (2.8) imply that this value is $t(0, 0) = m(0)\, m'(0)^{-1}$. Indeed, we have

$$\frac{\partial t}{\partial \sigma}(\sigma, \tau) = \frac{m'(\sigma)\, m(\tau)\,(\tau - o) - m(\sigma)\,[m(\tau) - m(\sigma)]}{[m(\tau) - m(\sigma)]^2}$$

and there exists a $\vartheta\colon \sigma \leqq \vartheta \leqq \tau$ such that $m(\tau) - m(\sigma) = m'(\vartheta)\,(\tau - \sigma)$. Hence by virtue of $m(\tau) \geqq m(\sigma)$ we find

$$m'(\sigma)\, m(\tau)\,(\tau - \sigma) - m(\sigma)\,[m(\tau) - m(\sigma)] \geqq m(\sigma)\,(\tau - \sigma)\,[m'(\sigma) - m'(\vartheta)] \geqq 0\,,$$

wherefrom

$$\frac{\partial t}{\partial \sigma}(\sigma, \tau) \geqq 0\,.$$

Thus we have

$$\min_{0 \leqq \sigma \leqq \tau} t(\sigma, \tau) = t(0, \tau).$$

As for any $\tau \in (0, \tau_1]$ there exists a $\vartheta \in [0, \tau]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} t(0, \tau) \equiv 1 + m(0) \frac{m(\tau) - m(0) - \tau\, m'(\tau)}{[m(\tau) - m(0)]^2} =$$

$$= 1 + m(0)\, \tau \frac{m'(\vartheta) - m'(\tau)}{[m(\tau) - m(0)]^2} \geqq 1,$$

our assertion immediately follows. We have proved the following theorem.

Theorem 2.1. *Let a, b, c, d and $u_1(t)$ be continuously differentiable functions and let the eigenvalues λ, μ of the matrix (1.2) satisfy $\lambda < 0 < \mu$, all in a sufficiently large neighbourhood of (u_0, v_0) and for $t \in [0, \tau_1], (\tau_1 > 0)$. Besides, suppose that $m(\tau) \equiv \mu(r_0, \sigma(\tau))$ satisfies (2.7), (2.8) for $\tau \in [0, \tau_1]$. Then the time of breakdown of the solution to (2.2) (equivalently to (1.1), (2.1)) which is given by (2.4) is*

$$t_c = t(0) = \frac{m(0)}{m'(0)} = \frac{\mu(r_0, s_0)}{\mu_s(r_0, s_0)\, \sigma'(0)}. \tag{2.9}$$

Remark. If we proceeded as Lax in [1] we should obtain the time of the blowing up of the function s_x (or equivalently s_t) on the μ-characteristic issuing from $(0, \tau)$ as $t = t(\tau, \tau)$. We should minimize this function with respect to $\tau \in [0, \tau_1]$ under a milder assumption than (2.8), namely $2m'^2 - mm'' \geqq 0$ (or $\leqq 0$) on $[0, \tau_1]$. But such a reasoning would give only an upper bound for $t_c : t_c \leqq \min_{\tau \in [0, \tau_1]} t(\tau, \tau)$.

The region Ω is now described by

$$\Omega = \Big\{(x, t);\ \max\,[0, m(\tau_1)(t - \tau_1)] \leqq x < m(0)\, t,$$

$$0 \leqq t < \tau_1 + \frac{m(0)\, \tau_1}{m(\tau_1) - m(0)}\Big\}.$$

However, this region must be reduced by the λ-characteristic $x = x(t)$ issuing from $(m(0)\, t_c, t_c)$ to the region illustrated in Fig. 1, since on the right from this characteristic but still in $\{x \leqq m(0)\, t\}$ the solution is influenced by its singularity at $(m(0)\, t_c, t_c)$ and the μ-characteristics are no longer straight lines. Thus the region

$$\Omega_c = \{(x, t);\ \max\,[0, m(\tau_1)(t - \tau_1)] \leqq x < x(t), 0 \leqq t < \tau_1\} \cup \{(x, t);$$

$$m(0)\, t \leqq x,\ 0 \leqq t < t_c\}$$

is the largest one where the classical solution to (2.2) (with a discontinuity of its derivatives on $x = m(0)\, t$) exists. As the explicit description of Ω_c is desirable, we find the formula for $x = x(t)$ as follows.

Let $y(t) = \tau(x(t), t)$. Then clearly

(2.10) $$\frac{dx}{dt}(t) = \lambda(r_0, \sigma(y(t))) .$$

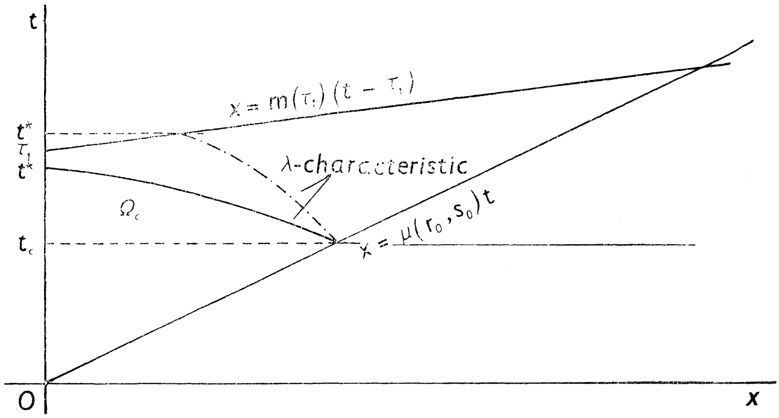

Fig. 1.

On the other hand,

$$x = m(\tau(x, t)) \left[t - \tau(x, t)\right] \quad \text{for any} \quad (x, t) \in \Omega .$$

Hence in particular

(2.11) $$x(t) = m(y(t))(t - y(t)) .$$

Differentiating (2.11) by t we get

$$\frac{dx}{dt} = m'(y)(t - y) - m(y)\frac{dy}{dt} + m(y) .$$

Comparing this with (2.10) we find

$$\frac{dy}{dt} = \left[\lambda(r_0, \sigma(y)) - m(y)\right]\left[m'(y)(t - y) - m(y)\right]^{-1} > 0 \quad \text{for} \quad t > t_c .$$

(As all cross intersections of μ-characteristics lie in Ω_0 we have $t < y + m(y)\, m'(y)^{-1}$ for $t > t_c$.) Interchanging the role of the dependent and independent variables we get

$$\frac{dt}{dy} = \frac{m'(y)}{\lambda(r_0, \sigma(y)) - m(y)}\, t - \frac{m(y) + m'(y)\, y}{\lambda(r_0, \sigma(y)) - m(y)} ,$$

which yields

$$t(y) = \exp\left(\int_0^y \frac{m'(\eta)}{\lambda(r_0, \sigma(\eta)) - m(\eta)}\,\mathrm{d}\eta\right) t_c -$$

$$- \int_0^y \exp\left(\int_\eta^y \frac{m'(\vartheta)}{\lambda(r_0, \sigma(\vartheta)) - m(\vartheta)}\,\mathrm{d}\vartheta\right) \frac{m(\eta) + m'(\eta)\,.\,\eta}{\lambda(r_0, \sigma(\eta)) - m(\eta)}\,\mathrm{d}\eta\,.$$

Taking the inverse function to $t(y)$ we obtain the function $y(t)$ which set into (2.10) provides an analytic description of the λ-characteristic issuing from $(\mu(r_0, s_0)\,t_c, t_c)$.

We have, of course, no formula for t_c if (2.7), (2.8) fail to be satisfied, and consequently the minimum of $t(\sigma, \tau)$ may not be achieved at $(0, 0)$. When solving concrete problems it might be convenient to search for the minimum of $t(\sigma, \tau)$ numerically.

3. EXAMPLES

The isentropic fluid flow in a right semi-infinite tube with the velocity disturbance on the left boundary $x = 0$ is described by the Euler equations, i.e. by (1.1) with $u := u$, $v := \varrho$, $a := u$, $b := c^2/\varrho$, $c := \varrho$, $d := u$, where $c = c(\varrho) = p'(\varrho)^{1/2}$, $p'(\varrho) \geqq 0$, $p = p(\varrho)$, $u_0 = 0$, $v_0 = \varrho_0$, $u(0, t) = u_1(t)$ ($u(x, t)$ is the velocity of the fluid at a position x and a time t, ϱ-density, p – pressure, c – sound speed). Then

$$r = u - \int_{\varrho_0}^{\varrho} \frac{c(\sigma)}{\sigma}\,\mathrm{d}\sigma\,,$$

$$s = u + \int_{\varrho_0}^{\varrho} \frac{c(\sigma)}{\sigma}\,\mathrm{d}\sigma\,,$$

and conversely

$$u = \frac{r + s}{2}\,, \quad \varrho = h\left(\frac{s - r}{2}\right),$$

where h is defined by

$$u - \int_{\varrho_0}^{h(u)} \frac{c(\sigma)}{\sigma}\,\mathrm{d}\sigma = 0\,.$$

This implies $r_0 = s_0 = 0$, $s(0, t) = 2u_1(t)$ and

$$\lambda = \lambda(r, s) = \frac{r + s}{2} - c\left(h\left(\frac{r + s}{2}\right)\right),$$

$$\mu = \mu(r, s) = \frac{r + s}{2} + c\left(h\left(\frac{r + s}{2}\right)\right),$$

$$m(\tau) = u_1(\tau) + c(h(u_1(\tau)))\,.$$

We suppose the flow to be subsonic, i.e. $|u_1(\tau)| < c(h(u_1(\tau)))$, $\tau \in [0, \tau_1]$. The assumptions of Theorem 2.1, namely (2.7) and (2.8) read, respectively,

$$[2p'(h) + p''(h)\,h]\,u_1'(\tau) \geqq 0 \tag{3.1}$$

and

$$\frac{p'''p'h + p''p' - p''^2h}{2p'^2}\frac{h}{c}u_1'(\tau)^2 + \left(1 + \frac{p''h}{2p'}\right)u_1''(\tau) \leqq 0 \tag{3.2}$$

$$\text{for } \tau \in [0, \tau_1],$$

where $p' = p'(h) = p'(h(u_1(\tau)))$ etc. If $u_1' > 0$ (compression wave) and $u_1'' \leqq 0$ then (3.1) is implied by

$$\frac{d^2}{d\varrho^2}(\varrho \cdot p(\varrho)) \geqq 0 \tag{3.3}$$

and (3.2) by (3.3) and $\varrho \cdot (p'''p' - p''^2) + p''p' \leqq 0$. These conditions have been derived in [5]. They are satisfied e.g. for liquids with the equation of state $p(\varrho) = p_0 + E \cdot \ln(\varrho/\varrho_0)$ ($E = \text{const.}$ is the elasticity modulus of the liquid) and also for the polytropic ideal gas with $p = p(\varrho) = A \cdot \varrho^\gamma + \text{const.}$ ($A > 0, \gamma > 1$ constants). The formula (2.9) reads

$$t_c = \frac{2c_0^3}{(2c_0^2 + \varrho_0 p''(\varrho_0))\, u_1'(0)}, \quad (c_0 = \sqrt{p'(\varrho_0)}).$$

If we consider the problem with the left moving boundary formed e.g. by a piston, and employ Lagrange's coordinates, we have the system

$$u_t + p_x = 0,$$
$$V_t - u_x = 0,$$
$$u(x, 0) = 0, \quad V(x, 0) = \frac{1}{\varrho_0} = V_0,$$
$$u(0, t) = u_1(t)$$

with $p = p(V)$, $p' < 0$, $c = \sqrt{-p'(V)}$, V being the specific volume. Here we have $\lambda = -c$, $\mu = c$,

$$r = u + \int_{V_0}^{V} c(\sigma)\, d\sigma, \quad s = u - \int_{V_0}^{V} c(\sigma)\, d\sigma.$$

The equations for Riemann invariants are

$$r_t - cr_x = 0,$$
$$s_t + cs_x = 0,$$
$$r(x, 0) = s(x, 0) = 0,$$
$$s(0, t) = 2u_1(t) = \sigma(t).$$

In this case $m(\tau) = c(h(\sigma(\tau)))$, where the function h is defined by the relation $u + \int_{V_0}^{h(u)} c(\eta)\, d\eta = 0$. As $h'(u) = -c(h(u))^{-1}$, we find $m' = -p''\sigma'/2p'$ and

$$m'' = \frac{(p'p''' - p''^2)(\sigma')^2 - p''p'\sigma''}{2p'^2c}$$

(here again $p' = p'(h(\sigma(\tau)))$, $c = c(h(\sigma(\tau)))$ etc.). The assumptions (2.7) and (2.8) can be written as

$$p''\sigma' > 0 \tag{3.4}$$

and

$$(p''^2 - p'p''')\sigma'^2 + cp'p''\sigma'' > 0\,, \tag{3.5}$$

respectively. If $\sigma' > 0$ (compression wave) then (3.4) is implied by $p'' > 0$. The state equations

$$p(V) = p_0 + E \ln \frac{V_0}{V} \tag{3.6}$$

and

$$p(V) = AV^{-\gamma} \tag{3.7}$$

satisfy (3.4) if $\sigma' > 0$. The condition (3.5) for $p(V)$ given by (3.6) and (3.7) leads us respectively to the inequalities

$$\sigma'^2 + \left(V_0 - \frac{\sigma}{2}\right)\sigma'' < 0 \tag{3.8}$$

and

$$\sigma'^2 + \left((A\gamma)^{1/2} V_0^{(1-\gamma)/2} + \frac{\gamma - 1}{2}\right)\sigma'' < 0\,. \tag{3.9}$$

The critical time is then given by (see (2.9))

$$t_c = \frac{2c_0^3}{p''(V_0)\,\sigma'(0)}\,.$$

As a concrete example investigate

$$\sigma(t) = a(t + b)^\alpha - ab^\alpha \tag{3.10}$$

with constants $a > 0$, $b > 0$, $0 < \alpha < 1$, $(t \in [0, \tau_1])$. If the state equation (3.6) takes place then σ given by (3.10) should satisfy (3.8), which after elementary calculations results in

$$(t + b)^\alpha < \frac{1 - \alpha}{a(1 + \alpha)}(2V_0 - ab^\alpha)\,, \quad t \in [0, \tau_1]\,. \tag{3.11}$$

This is certainly satisfied if (3.11) holds for $t = \tau_1$. If (3.7) takes place then (3.9) has to be investigated. We find it equivalent to

$$a[(\gamma - 1)(1 - \alpha) - 2\alpha](t + b)^\alpha > (1 - \alpha)\left[ab^\alpha(\gamma - 1) - 2(A\gamma)^{1/2} V_0^{(1-\gamma)/2}\right], \quad t \in [0, \tau_1]\,.$$

After distinguishing the cases

$$\alpha < \tfrac{1}{2}(\gamma - 1)(1 - \alpha) \quad \text{and} \quad \alpha \geqq \tfrac{1}{2}(\gamma - 1)(1 - \alpha)$$

we arrive at either

$$\alpha < \tfrac{1}{2}(\gamma - 1)(1 - \alpha) \wedge \alpha a b^{\alpha} < (1 - \alpha)(A\gamma)^{1/2} V_0^{(1-\gamma)/2}$$

or

$$\alpha \geqq \tfrac{1}{2}(\gamma - 1)(1 - \alpha) \wedge (\tau_1 + b)^{\alpha} < (1 - \alpha) \frac{2(A\gamma)^{1/2} V_0^{(1-\gamma)/2} - (\gamma - 1) a b^{\alpha}}{a[\alpha(\gamma + 1) - \gamma + 1]} .$$

Thus for $p(V)$ given by (3.6) and (3.7) with σ given by (3.10) we get (under the respective conditions on a, b, α and τ_1 given above)

$$t_c = \frac{2b^{1-\alpha} V_0}{\alpha a} \quad \text{and} \quad t_c = \frac{2b^{1-\alpha}(A\gamma)^{1/2} V_0^{(1-\gamma)/2}}{\alpha a(\gamma + 1)} ,$$

respectively.

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Souhrn

POZNÁMKA O KRITICKÝCH ČASECH HYPERBOLICKÝCH KVAZILINEÁRNÍCH SOUSTAV DVOU ROVNIC

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V práci je za jistých předpokladů odvozen přesný vzorec pro výpočet kritického času vzniku nespojitosti (rázové vlny) u řešení kvazilineární hyperbolické soustavy dvou rovnic. Použitelnost tohoto vzorce v inženýrské praxi je ukázána na rovnicích jednorozměrného isentropického neviskózního proudění stlačitelné tekutiny.

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