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ON HARDLY LINEARLY PROVABLE SYSTEMS

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A well-known theorem of Rabin yields a 'dimensional' lower bound on the width of complete polynomial proofs of a system of linear algebraic inequalities. In this note we investigate a practically motivated class of systems where the same lower bound can be obtained on the width of 'almost all' 'non-complete' linear proofs. The proof of our result is based on the Helly Theorem.

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let R^n denote the n-dimensional vector space over R, where $n \geqq 2$ is an integer; without loss of generality we shall assume that the elements of R^n are ordered n-tuples of real numbers (n-dimensional row vectors).

For two real matrices (in particular, row or column vectors) having the same size, $\boldsymbol{M}' = (m'_{i,j})$ and $\boldsymbol{M}'' = (m''_{i,j})$, we set

$$\boldsymbol{M}' \geqq \boldsymbol{M}'' \quad \text{if} \quad m'_{i,j} \geqq m''_{i,j}, \quad \text{and} \quad \boldsymbol{M}' > \boldsymbol{M}'' \quad \text{if} \quad m'_{i,j} > m''_{i,j}\,,$$

for each pair of subscripts i, j.

The matrix transposition will be denoted by the superscript $\dots^{\mathrm{T}}$. Symbol $\boldsymbol{0}$ will denote the zero matrix (particularly, the zero vector); the size of $\boldsymbol{0}$ will be always evident from the context. R^n_+ will denote the set $\{\boldsymbol{x} \in R^n \mid \boldsymbol{x} \geqq \boldsymbol{0}\}$ (the nonnegative cone in R^n).

Π_n (Λ_n) will denote the set of all polynomial functions $f : R^n \to R$ (respectively the set of all polynomial functions $f : R^n \to R$ having the degree at most 1).

The function sign : $R \to R$ is defined as usual:

$$\operatorname{sign} x = x \,.\, |x|^{-1} \quad \text{if} \quad x \neq 0\,; \quad \operatorname{sign} 0 = 0\,.$$

In [1] the following result is obtained (we present it in an equivalent formulation):

Theorem 1. *Let*

$$(f_{i,j}) \tag{1}$$

be a rectangular $p \times w$*-matrix of polynomial functions* $f_{i,j} \in \Pi_n$ $(i = 1, 2, \ldots, p;$ $j = 1, 2, \ldots, w)$, *where* p *and* w *are positive integers, let*

$$(2) \qquad l_k(\mathbf{x}) \geqq 0 \quad (k = 1, 2, \ldots, q)$$

be a system of linear algebraic inequalities, where $q \in \langle 2, n \rangle$ *is an integer and* $l_k \in \Lambda_n$ $(k = 1, 2, \ldots, q)$, *and let* $C \subseteq R^n$ *be a nonempty convex set.*

If

$$(3) \qquad \bigcup_{i=1}^{p} \bigcap_{j=1}^{w} \{\mathbf{x} \in C \mid f_{i,j}(\mathbf{x}) \geqq 0\} = \bigcap_{k=1}^{q} \{\mathbf{x} \in C \mid l_k(\mathbf{x}) \geqq 0\},$$

and if for each ordered q*-tuple*

$$(\sigma_1, \sigma_2, \ldots, \sigma_q) \in \{-1, 0, 1\}^q \text{ (cartesian power)}$$

there exists $\mathbf{x}_0 \in C$ *such that*

$$(4) \qquad \operatorname{sign}(l_k(\mathbf{x}_0)) = \sigma_k \quad (k = 1, 2, \ldots, q)$$

then

$$w \geqq q . \qquad \square$$

The matrix (1) satisfying condition (3) is called a *complete polynomial proof of (system)* (2) *in C*. The number w is called the *width* of the proof (1). Condition (4) in Theorem 1 is called the condition of *sign-independency* of (2). So, if the condition of sign-independency is fulfilled then the width of any complete polynomial proof of (2) equals at least the number of inequalities in (2).

Theorem 1 can be applied in particular to $f_{i,j} \in \Lambda_n$. In this case we speak about a *complete linear proof*.

In accordance with [1] we introduce now the concept of the ('non-complete') proof. A w-tuple

$$(5) \qquad (f_1, f_2, \ldots, f_w)$$

of polynomials of Π_n will be called a *polynomial proof of* (2) *in C if*

$$\emptyset \neq \bigcap_{v=1}^{w} \{\mathbf{x} \in C \mid f_v(\mathbf{x}) \geqq 0\} \subseteq \bigcap_{k=1}^{q} \{\mathbf{x} \in C \mid l_k(\mathbf{x}) \geqq 0\} .$$

The number w is called the *width* of (5). The polynomial proof (5) is called *linear* if

$$f_v \in \Lambda_n \quad (v = 1, 2, \ldots, w) .$$

Remark. While the concept of the complete proof corresponds to nondeterministic checking whether a given $\mathbf{x} \in R^n$ is a solution of (2) the concept of the ('non-complete') proof corresponds merely to proving for each element $\mathbf{x}$ of some subset of R^n that $\mathbf{x}$ is a solution of (2).//

For ('non-complete') polynomial (even linear) proofs we have no such lower bound as in Theorem 1. Indeed, (g_1) with $g_1(x_1, x_2, \ldots, x_n) = -x_1 - x_2 - \ldots - x_n$

is obviously a linear proof of an arbitrary sign-independent system (2) in $C = R^n_+$ provided $l_k(\mathbf{0}) \geqq 0$ $(k = 1, 2, \ldots, q)$.

In this note we construct a practically motivated class of linear-inequalities systems (2) for which one has the same lower bound as in Theorem 1 on the width of 'almost all' linear proofs in $C = R^n_+$.

To this aim we shall write the system (2) in the matrix form

$$(2') \qquad x_1 \boldsymbol{a}_1^{\mathrm{T}} + x_2 \boldsymbol{a}_2^{\mathrm{T}} + \ldots + x_n \boldsymbol{a}_n^{\mathrm{T}} + \boldsymbol{a}_{n+1}^{\mathrm{T}} \geqq \mathbf{0}$$

or equivalently

$$\boldsymbol{A} \, . \, (x_1, x_2, \ldots, x_n, 1)^{\mathrm{T}} \geqq \mathbf{0} \, ,$$

where $\boldsymbol{A} = (\boldsymbol{a}_1^{\mathrm{T}}, \boldsymbol{a}_2^{\mathrm{T}}, \ldots, \boldsymbol{a}_n^{\mathrm{T}}, \boldsymbol{a}_{n+1}^{\mathrm{T}})$ is a $q \times (n + 1)$-rectangular matrix with real entries.

A linear proof (5) of (2) (equivalently, of (2')) in C will be called *full-dimensional* if

$$\dim \left(\bigcap_{\upsilon=1}^{w} \{ \boldsymbol{x} \in C \mid f_\upsilon(\boldsymbol{x}) \geqq 0 \} \right) = n \, .$$

The system (2') will be called *hardly linearly provable in C* if each full-dimensional linear proof of (2') in C has a width at least q.

Now we propose the following open problem: Find a "good" characterization of the set of all real $q \times (n + 1)$ — matrices $\boldsymbol{A}$ such that (2') is hardly linearly provable in $C = R^n_+$.

The main result of this paper is to give a sufficient condition for $\boldsymbol{A}$ under which (2') is hardly linearly provable in $C = R^n_+$ (Theorem 2).

A trivial sufficient condition for it is

$$\dim \{ \boldsymbol{x} \in R^n_+ \mid \boldsymbol{A} \, . \, (x_1, x_2, \ldots, x_n, 1)^{\mathrm{T}} \geqq \mathbf{0} \} < n \, .$$

(If this condition is fulfilled then there is no full-dimensional linear proof of (2') in R^n_+ at all.) Thus, in the rest of this paper we shall usually assume that the following condition is fulfilled:

$$(6) \qquad \dim \{ \boldsymbol{x} \in R^n_+ \mid \boldsymbol{A} \, . \, (x_1, x_2, \ldots, x_n, 1)^{\mathrm{T}} \geqq \mathbf{0} \} = n \, .$$

Theorem 2. *Let a real $q \times (n + 1)$ — matrix $\boldsymbol{A} = (a_{i,j})$ satisfy the conditions:*

(7) *Each row of $\boldsymbol{A}$ contains at least one negative element.*

(8) *Each column of $\boldsymbol{A}$ contains at most one positive element.*

Then (2') is hardly linearly provable in R^n_+. □

Corollary. *The system of $n - 1$ linear inequalities*

$$x_i - x_n \geqq 0 \quad (i = 1, 2, \ldots, n - 1)$$

(the proof of which is equivalent to the verification of $x_n = \min(x_1, x_2, \ldots, x_n)$) is hardly linearly provable in R^n_+. □

Theorem 2 will be proved by contradiction: Let us assume that conditions (7) and (8) are fulfilled and there exists a full-dimensional linear proof $(f_1, f_2, \ldots, f_w)$ of (2′) in R^n_+ with $1 \leqq w \leqq q - 1$.

We may assume without loss of generality that each f_υ ($\upsilon = 1, 2, \ldots, w$) is non-constant. (Observe that for each constant function $f : R^n \to R$ we have either $\{\mathbf{x} \in R^n \mid f(\mathbf{x}) \geqq 0\} = R^n$ or $\{\mathbf{x} \in R^n \mid f(\mathbf{x}) \geqq 0\} = \emptyset$.)

Thus there exists $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*) \in R^n$ such that

$$\mathbf{x}^* > \mathbf{0}\,, \tag{9}$$

and

$$f_\upsilon(\mathbf{x}^*) > 0 \quad (\upsilon = 1, 2, \ldots, w)\,. \tag{10}$$

Let us write

$$f_\upsilon(\mathbf{x}) = c_{\upsilon,1}x_1 + c_{\upsilon,2}x_2 + \ldots + c_{\upsilon,n}x_n + c_{\upsilon,n+1} \tag{11}$$
$$(\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n;\ \upsilon = 1.\,2, \ldots, w)\,.$$

Now, it follows from condition (8) that there exists a $(q + 1)$ -tuple $(L_1, L_2\,.. \ldots, L_q, L_{q+1})$ of pair-wise disjoint subsets of $\{1, 2, \ldots, n + 1\}$ such that

$$L_1 \cup L_2 \cup \ldots \cup L_q \cup L_{q+1} = \{1, 2, \ldots, n, n + 1\}\,, \tag{12}$$

and

$$a_{i,j} > 0 \Leftrightarrow j \in L_i \quad (1 \leqq i \leqq q;\ 1 \leqq j \leqq n + 1)\,. \tag{13}$$

(Some of the sets $L_1, L_2, \ldots, L_q, L_{q+1}$ may be empty.) In particular, it follows from the definition of $L_1, L_2, \ldots, L_q$ and L_{q+1} that

$$a_{i,j} \leqq 0 \quad \text{if} \quad j \in L_{q+1}\,, \quad l \leqq i \leqq q\,. \tag{14}$$

Using the coefficients $c_{\upsilon,j}$ from (11) and the sets $L_1, L_2, \ldots, L_q, L_{q+1}$ we introduce q convex polyhedra $P_1, P_2, \ldots, P_q$ in R^w as follows: P_k ($k = 1, 2, \ldots, q$) is the set of all $\mathbf{y} = (y_1, y_2, \ldots, y_w) \in R^w_+$ such that

$$\sum_{\upsilon=1}^{w} c_{\upsilon,j}y_\upsilon \leqq 0 \quad \text{if} \quad j \in L_k \cup L_{q+1}\,, \tag{15}$$

and

$$y_1 + y_2 + \ldots + y_w = 1\,. \tag{16}$$

Let us observe that $P_1, P_2, \ldots, P_q$ are subsets of the hyperplane

$$H = \{\mathbf{y} \in R^w \mid y_1 + y_2 + \ldots + y_w = 1\}$$

in R^w. Thus, we have $q \geqq w + 1$ convex sets $P_1, P_2, \ldots, P_q$ lying in the $(w - 1)$-dimensional affine set H.

We shall apply the Helly Theorem (see [2], p. 117) to the q-tuple $(P_1, P_2, \ldots, P_q)$. To this aim we shall verify that

$$P_1 \cap P_2 \cap \ldots \cap P_{k-1} \cap P_{k+1} \cap \ldots \cap P_q \neq \emptyset \tag{17}$$

holds for each $k \in \{1, 2, \ldots, q\}$.

Indeed, the k-th inequality

$$a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,n}x_n + a_{k,n+1} \geqq 0$$

from (2′) is a consequence of the consistent system of $w + n$ linear inequalities

$$c_{\upsilon,1}x_1 + c_{\upsilon,2}x_2 + \ldots + c_{\upsilon,n}x_n + c_{\upsilon,n+1} \geqq 0 \quad (1 \leqq \upsilon \leqq w),$$

and

$$x_1 \geqq 0, \quad x_2 \geqq 0, \ldots, x_n \geqq 0.$$

Hence, using a well-known theorem due to Farkas (see e.g. [3], p. 108) we conclude: There exists $\mathbf{z} = (z_1, z_2, \ldots, z_w) \in R^w_+$ such that

$$\sum_{\upsilon=1}^{w} c_{\upsilon,j}z_\upsilon \leqq a_{k,j} \quad (1 \leqq j \leqq n + 1). \tag{18}$$

It follows from (7) that $a_{k,j} < 0$ for some $j \in \{1, 2, \ldots, n + 1\}$. Thus we have from (18)

$$z_1 + z_2 + \ldots + z_w > 0. \tag{19}$$

Further, it follows from (12), (13), (14) and (18) that

$$\sum_{\upsilon=1}^{w} c_{\upsilon,j}z_\upsilon \leqq 0 \tag{20}$$

for each $j \in L_1 \cup L_2 \cup \ldots \cup L_{k-1} \cup L_{k+1} \cup \ldots \cup L_q \cup L_{q+1} = (\{1, 2, \ldots, n, n + 1\} \setminus \setminus L_k)$.

Thus, for

$$(y_1, y_2, \ldots, y_w) = (\sum_{\upsilon=1}^{w} z_\upsilon)^{-1} \cdot (z_1, z_2, \ldots, z_w)$$

we have

$$\sum_{\upsilon=1}^{w} c_{\upsilon,j}y_\upsilon \leqq 0 \quad (j \in (\{1, 2, \ldots, n, n + 1\} \setminus L_k)),$$

and

$$y_1 + y_2 + \ldots + y_w = 1,$$

i.e.

$$(y_1, y_2, \ldots, y_w) \in P_1 \cap P_2 \cap \ldots \cap P_{k-1} \cap P_{k+1} \cap \ldots \cap P_q.$$

This verifies (17).

Now, using the Helly Theorem we have

$$P_1 \cap P_2 \cap \ldots \cap P_q \neq \emptyset,$$

hence, the exists $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_w^*) \in R_+^w$ such that

$$(21) \qquad \sum_{\upsilon=1}^{w} c_{\upsilon,j} y_\upsilon^* \leqq 0 \quad (1 \leqq j \leqq n+1),$$

and

$$(22) \qquad y_1^* + y_*^2 + \ldots + y_w^* = 1 .$$

If follows immediately from (22) that $\mathbf{y}^* \neq \mathbf{0}$. Furthermore, since $\mathbf{x}^* > \mathbf{0}$ (see (9)), we have from (21)

$$\sum_{j=1}^{n} x_j^* \sum_{\upsilon=1}^{w} c_{\upsilon,j} y_\upsilon^* + \sum_{\upsilon=1}^{w} c_{\upsilon,n+1} y_\upsilon^* \leqq 0$$

i.e.

$$\sum_{\upsilon=1}^{w} y_\upsilon^* f_\upsilon(\mathbf{x}^*) \leqq \mathbf{0} .$$

The above inequality contradicts the set of relations $\mathbf{y}^* \geqq \mathbf{0}$, $\mathbf{y}^* \neq \mathbf{0}$ and (10), which concludes the proof.

III. CONCLUDING REMARKS

(A) If a matrix $\mathbf{A}$ satisfies condition (6) then condition (7) is necessary for (2′) to be hardly linearly provable in R_+^n. Indeed, each inequality $a_{k,1}x_1 + a_{k,2}x_2 + \ldots \ldots + a_{k,n}x_n + a_{k,n+1} \geqq 0$ with $a_{k,1} \geqq 0$, $a_{k,2} \geqq 0, \ldots, a_{k,n+1} \geqq 0$ is a consequence of the system $x_1 \geqq 0$, $x_2 \geqq 0, \ldots, x_n \geqq 0$.

(B) Condition (8) is not necessary for (2′) to be hardly linearly provable in R_+^n, even if condition (6) is fulfilled. Indeed, let us consider the example

$$q = n = 3 , \quad \mathbf{A}_0 = \begin{pmatrix} -1, & 1, & 1, & 0 \\ 1, & -1, & 1, & 0 \\ 1, & 1, & -1, & 0 \end{pmatrix} .$$

The matrix $\mathbf{A}_0$ satisfies condition (6) since $\mathbf{A}_0 \,.\, (1, 1, 1, 1)^{\mathsf{T}} > \mathbf{0}$. We shall prove, however, that the system (2′) corresponding to $\mathbf{A}_0$ is hardly linearly provable in R_+^3. Let us assume by contradiction that there exists a full-dimensional linear proof of (2′) corresponding to $\mathbf{A}_0$ in R_+^3, the proof having the width at most 2.

Then (using the theorem of Farkas) there exist real matrices

$$\mathbf{U} = \begin{pmatrix} u_{1,1}, & u_{1,2} \\ u_{2,1}, & u_{2,2} \\ u_{3,1}, & u_{3,2} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_{1,1}, & v_{1,2}, & v_{1,3}, & v_{1,4} \\ v_{2,1}, & v_{2,2}, & v_{2,3}, & v_{2,4} \end{pmatrix}$$

and a vector $\mathbf{y} = (y_1, y_2, y_3, 1) \in R^4$ such that the following conditions are fulfilled:

$$(23) \qquad \mathbf{U} \geqq \mathbf{0} ,$$

$$\mathbf{A}_0 \geqq \mathbf{U} \, . \, \mathbf{V} \, , \tag{24}$$

$$\mathbf{V} \, . \, \mathbf{y}^{\mathrm{T}} > \mathbf{0} \quad \text{and} \quad \mathbf{y} > \mathbf{0} \, . \tag{25}$$

Now, it follows from (25) and from the well-known fact concerning the existence of basic feasible solutions of linear optimization problems (see e.g. [4], p. 18) that there exists a vector $\mathbf{z} = (z_1, z_2, z_3, 1) \in R_+^4$ such that

$$\mathbf{V} \, . \, \mathbf{z}^{\mathrm{T}} > \mathbf{0} \, , \tag{26}$$

and exactly one of the components z_1, z_2, z_3 is zero.

Since the matrix $\mathbf{A}_0$ is invariant with respect to the simultaneous and equal permutations of rows and the first three columns, we may assume without loss of generality that

$$z_1 > 0 \, , \quad z_2 > 0 \, , \quad z_3 = 0 \, . \tag{27}$$

On the other hand, each row of $\mathbf{A}_0$ contains a negative element. Hence, it follows from (23) and (24) that each row of $\mathbf{U}$ contains a positive element. By combining this fact with (23), (24), (26) and (27) we obtain

$$-z_1 + z_2 \geqq u_{1,1}(v_{1,1}z_1 + v_{1,2}z_2 + v_{1,4}) + u_{1,2}(v_{2,1}z_1 + v_{2,2}z_2 + v_{2,4}) > 0$$
$$z_1 - z_2 \geqq u_{2,1}(v_{1,1}z_1 + v_{1,2}z_2 + v_{1,4}) + u_{2,2}(v_{2,1}z_1 + v_{2,2}z_2 + v_{2,4}) > 0 \, .$$

This contradiction completes the proof.

(C) On the other hand, condition (8) is necessary for (2′) to be hardly linearly provable in R_+^n if $q = 2$. (Thus, if condition (6) is fulfilled then (7) and (8) is the set of necessary and sufficient conditions in this case.) Indeed, it is easy to see that if condition (8) is not fulfilled then the system consisting of the single inequality

$$\min(a_{1,1}, a_{2,1})\, x_1 + \min(a_{1,2}, a_{2,2})\, x_2 + \ldots + \min(a_{1,n}, a_{2,n})\, x_n + \\ + \min(a_{1,n+1}, a_{2,n+1}) \geqq 0$$

is a full-dimensional linear proof of the system

$$a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n + a_{i,n+1} \geqq 0 \quad (i = 1, 2)$$

in R_+^n.

(D) The concept of "full-dimensionality" can be in an obvious way generalized to polynomial proofs. It is, however, easy to show that for each system (2′) satisfying (6) there exists a "full-dimensional" polynomial proof having the width 1.

(E) A nontrivial generalization of Theorem 1 was obtained in [5].

References

[1] *M. O. Rabin:* Proving simultaneous positivity of linear forms. JCSS 6 : 639—650 (1972).

[2] *Joseph Stoer, Christoph Witzgall:* Convexity and Optimization in Finite Dimensions I. Springer-Verlag, Berlin—Heidelberg—New York, 1970.

[3] *Ky Fan:* 'On systems of linear inequalities' in 'Linear Inequalities and Related Systems' (H. W. Kuhn and A. W. Tucker, eds.). Princeton Univ. Press, 1956.
[4] *David G. Luenberger:* Introduction to Linear and Nonlinear Programming. Addison-Wesley Publishing Company, 1973.
[5] *J. W. Jaromczyk:* 'An extension of Rabin's complete proof concept'. MFCS 1981, Lecture Notes in Computer Science 118, Springer-Verlag 1981, 321—326.

Souhrn

O TĚŽCE LINEÁRNĚ ODVODITELNYCH SOUSTAVÁCH

JAROSLAV MORÁVEK

Rabinův výsledek dává ‚dimensionální' dolní odhad pro šířku úplných polynomiálních dedukcí dané soustavy lineárních algebraických nerovnic. V poznámce se vyšetřuje prakticky motivovaná třída soustav, pro které lze stejný dolní odhad získat i pro šířku „skoro všech" „neúplných" lineárních dedukcí. Důkaz výsledku je založen na Hellyově větě.

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