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THE EXISTENCE AND UNIQUENESS THEOREM IN BIOT'S CONSOLIDATION THEORY

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The main aim of this paper is to prove the existence and uniqueness of the solution of a variational problem corresponding to an initial-boundary value problem the special case of which is Biot's model of consolidation of clay [1]. The method of proof is a modification of the compactness method used in [5] and [8]. The proof of existence has a constructive nature: First a completely discretized approximate solution is defined. The discretization in space is carried out by the finite element method using the simplest finite elements. For the discretization in time we use the Euler backward method. It is proved that an approximate solution defined in this way exists and is unique. Then it is shown that a sequence of approximate solutions (extended to the whole time interval $(0, T]$) has a weak limit if $h \to 0$, $\Delta t \to 0$ and that this weak limit is a solution of the problem.

Besides the existence and uniqueness theorem some error estimates are introduced. The considerations are restricted to the two-dimensional case. However, all results introduced can be proved similarly in the three-dimensional case.

1. FORMULATION OF THE PROBLEM

Let Ω be a bounded domain in the x_1, x_2-plane with a sufficiently smooth boundary Γ. We consider the following problem: Find a vector $\boldsymbol{u}(x_1, x_2, t)$ and a function $p(x_1, x_2, t)$ which satisfy the following equations and boundary and initial conditions:

$$kp_{,ii} + Q = \operatorname{div} \dot{\boldsymbol{u}} \quad \text{in} \quad \Omega \times (0, T] \tag{1}$$

$$\sigma_{ij,j} + X_i = 0 \qquad (i = 1, 2) \quad \text{in} \quad \Omega \times (0, T] \tag{2}$$

$$p = 0 \quad (t > 0) \quad \text{on} \quad \Gamma_{p1} \tag{3}$$

$$kp_{,i}\nu_i = b(x_1, x_2, t) \quad (t > 0) \quad \text{on} \quad \Gamma_{p2} \tag{4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad (t > 0) \quad \text{on} \quad \Gamma_{u1} \tag{5}$$

(6) $$\sigma_{ij}\nu_j = q_i(x_1, x_2, t) \quad (t > 0, i = 1, 2) \quad \text{on} \quad \Gamma_{u2}$$

(7) $$\operatorname{div} \boldsymbol{u}(x_1, x_2, 0) = \alpha(x_1, x_2), \quad (x_1, x_2) \in \Omega$$

where

(8) $$\Gamma = \Gamma_{p1} \cup \Gamma_{p2} = \Gamma_{u1} \cup \Gamma_{u2}$$

(9) $$\sigma_{ij} = D_{ijkm}\varepsilon_{km}(\boldsymbol{u}) - p\delta_{ij}$$

(10) $$D_{ijkm} = D_{jikm} = D_{kmij}$$

(11) $$\varepsilon_{ij}(\boldsymbol{v}) = (v_{i,j} + v_{j,i})/2$$

(12) $$D_{ijkm}\xi_{ij}\xi_{km} \geqq \mu_0\xi_{ij}\xi_{ij} \quad \forall\xi_{ij} = \xi_{ji} \in R$$

where $\mu_0 = \text{const} > 0$. The symbols Γ_{p1}, Γ_{p2} (and similarly Γ_{u1}, Γ_{u2}) denote two disjoint subsets of Γ each consisting of a finite number of parts. We assume

(13) $$\operatorname{mes} \Gamma_{p1} > 0, \quad \operatorname{mes} \Gamma_{u1} > 0.$$

A summation convention over a repeated subscript is adopted. A comma is employed to denote partial differentiation with respect to spatial coordinates and a dot denotes the derivative with respect to time t. The symbols Q, X_i, q_i, α denote prescribed sufficiently smooth functions, the symbol k in equation (1) is a positive constant. In relations (4) and (6), ν_1 and ν_2 denote the components of the outward unit normal to Γ. In relation (9) δ_{ij} is the Kronecker delta and D_{ijkm} are constants depending on the material only. We shall consider isotropic materials only.

Remark 1. If we set $Q \equiv 0$, $\alpha \equiv 0$, $b \equiv 0$, then relations (1)–(7) represent Biot's model of consolidation of clay ([1], [2]). The vector $\boldsymbol{u}$ has the meaning of the displacement vector of a compressible solid phase and the function p is the pore water pressure. Equations (2) are then Cauchy's equations of equilibrium, where σ_{ij} are the components of the stress tensor and X_i the components of the body forces per unit volume. As the vector $\boldsymbol{v}$ of the water velocity is given by the relation $\boldsymbol{v} = -k \operatorname{grad} p$, the physical meaning of the boundary condition

(4a) $$p_{,i}\nu_i = 0 \quad (t > 0) \quad \text{on} \quad \Gamma_{p2}$$

is that the part Γ_{p2} of the boundary is impermeable (the normal component of the water velocity is equal to zero). The physical meaning of the initial condition

(7a) $$\operatorname{div} \boldsymbol{u}(x_1, x_2, 0) = 0$$

is that at the time $t = 0$ the volume change $\varepsilon \equiv \operatorname{div} \boldsymbol{u}$ is equal to zero. (This is a consequence of the assumption that the pore water is incompressible: There cannot be any instantaneous volume change even though a load is applied suddenly.) Equation (1) can be written in the form

(1a) $$\operatorname{div} \boldsymbol{v} = -\dot{\varepsilon}$$

and has the following meaning: As the pore water is incompressible the rate of the volume decrease of an element is equal to the rate at which water is expelled.

Before presenting a variational formulation of problem (1)–(12) let us introduce some notation. The symbol $H^k(\Omega)$ denotes the usual Sobolev space,

$$H^k(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega)\ \forall |\alpha| \leqq k\},$$

equipped with the norm

$$\|v\|_k^2 = \sum_{|\alpha| \leqq k} \|D^\alpha v\|_{L^2(\Omega)}^2 .$$

$H_0^k(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_k$, $H^{-k}(\Omega)$ is the dual space to the space $H_0^k(\Omega)$. The symbols W and V denote the spaces

(14) $$W = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_{p1}\} .$$

(15) $$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{u1}\} .$$

Let X be a Banach space normed by $\|\cdot\|_X$ and let $0 < T < \infty$. For $p \geqq 1$ we denote by $L^p(0, T; X)$ the space of strongly measurable functions $f : (0, T) \to X$ such that

$$\|f\|_{L^p(0,T;X)} = \left[\int_0^T \|f(t)\|_X^p \,\mathrm{d}t\right]^{1/p} < \infty$$

with the usual $p = \infty$ modification. By $C([0, T]; X)$ we denote the space of continuous functions $f : [0, T] \to X$ normed by

$$\|f\|_{C([0,T];X)} = \max_{t \in [0,T]} \|f(t)\|_X .$$

The symbol $AC([0, T]; X)$ denotes the subspace of $C([0, T]; X)$ of all absolutely continuous functions $f : [0, T] \to X$.

The scalar product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$,

(16) $$(v, w) = \int_\Omega vw \,\mathrm{d}x , \quad v, w \in L^2(\Omega) .$$

In order to obtain a variational formulation of problem (1)–(12) let us multiply equation (1) by $w \in W$, integrate over Ω, and use (4) and Green's theorem. We easily find

(17) $$D(p, w) + (\operatorname{div} \dot{\boldsymbol{u}}, w) = (Q, w) + (b, w)_p \quad \forall w \in W, \quad \forall t \in (0, T] ,$$

where

(18) $$D(v, w) = k \int_\Omega v_{,i} w_{,i} \,\mathrm{d}x ,$$

(19) $$(b, w)_p = \int_{\Gamma_{p2}} bw \,\mathrm{d}s .$$

Multiplying equation (9) by $\varepsilon_{ij}(\boldsymbol{v})$, where $\boldsymbol{v} \in [V]^2 \equiv V \times V$, integrating over Ω and using relations (2), (10), (11) and Green's theorem we find

$$(20)\qquad a(\boldsymbol{u}, \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{X}, \boldsymbol{v}) + (\boldsymbol{q}, \boldsymbol{v})_u \quad \forall \boldsymbol{v} \in [V]^2, \quad \forall t \in (0, T],$$

where

$$(21)\qquad a(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega D_{ijkm} v_{i,j} w_{k,m} \, dx,$$

$$(22)\qquad (\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega v_i w_i \, dx, \quad (\boldsymbol{v}, \boldsymbol{w})_u = \int_{\Gamma_{u2}} v_i w_i \, ds.$$

Let us multiply relations (17) and (20) by an arbitrary function $f(t) \in C_0^\infty((0, T))$ and integrate the resulting relations in $(0, T)$. If we use integration by parts we get

$$(23)\qquad \int_0^T D(p, w) f \, dt - \int_0^T (\operatorname{div} \boldsymbol{u}, w) \dot{f} \, dt = \int_0^T (Q, w) f \, dt + \int_0^T (b, w)_p f \, dt$$

$$\forall w \in W, \quad \forall f \in C_0^\infty((0, T)),$$

$$(24)\qquad \int_0^T a(\boldsymbol{u}, \boldsymbol{v}) f \, dt - \int_0^T (p, \operatorname{div} \boldsymbol{v}) f \, dt = \int_0^T (\boldsymbol{X}, \boldsymbol{v}) f \, dt + \int_0^T (\boldsymbol{q}, \boldsymbol{v})_u f \, dt$$

$$\forall \boldsymbol{v} \in [V]^2, \quad \forall f \in C_0^\infty((0, T)).$$

Now we can present the variational formulation of the initial-boundary value problem (1)–(12) in the form of the following problem P:

Problem P: Find a vector $\boldsymbol{u} \in L^2(0, T; [V]^2)$ and a function $p \in L^2(0, T; W)$ with the following properties:

a) $\boldsymbol{u}$ and p satisfy relations (23) and (24);

b) for every $w \in W$ the expression $(\operatorname{div} \boldsymbol{u}(t), w)$ is equivalent to an absolutely continuous function on $[0, T]$ and the initial condition (7) is satisfied, i.e.

$$(25)\qquad (\operatorname{div} \boldsymbol{u}(0), w) = (\alpha, w) \quad \forall w \in W.$$

(As W is dense in $L^2(\Omega)$ relation (25) is equivalent to (7).)

Remark 2. In what follows we restrict our considerations to functions α with the following property: To a given α there exists such a vector $\boldsymbol{u}_0 \in [V]^2 \cap [H^2(\Omega)]^2$ that

$$(26)\qquad \operatorname{div} \boldsymbol{u}_0 = \alpha.$$

If $\alpha \equiv 0$ then we can set $\boldsymbol{u}_0 \equiv \boldsymbol{0}$.

2. EXISTENCE AND UNIQUENESS OF THE SOLUTION

In this section we give some sufficient conditions for the existence and uniqueness of the solution of problem P.

Theorem 1. *Let the boundary Γ of the domain Ω be piecewise of class C^2, let the function $\alpha \in L^2(\Omega)$ have the property introduced in Remark 2 and let the functions Q, b and the vectors $\boldsymbol{X}$, $\boldsymbol{q}$ have the properties:*

(27) $$Q \in L^2(0, T; L^2(\Omega)), \qquad b \in L^2(0, T; L^2(\Gamma_{p2})),$$

(28) $$\boldsymbol{X} \in AC([0, T]; [L^2(\Omega)]^2), \quad \dot{\boldsymbol{X}} \in L^2(0, T; [L^2(\Omega)]^2),$$

(29) $$\boldsymbol{q} \in AC([0, T]; [L^2(\Gamma_{u2})]^2), \quad \dot{\boldsymbol{q}} \in L^2(0, T; [L^2(\Gamma_{u2})]^2).$$

Then there exists just one pair p, $\boldsymbol{u}$ which is the solution of problem P.

Proof. The proof is divided into six parts A) – F). In the parts A) – E) the existence of the solution is proved; in the last part F) the uniqueness.

A) According to the assumption concerning Γ, the boundary can be divided into a finite number of arcs each of which has a parametric representation

(30) $$x_1 = \varphi(s), \quad x_2 = \psi(s), \quad a \leqq s \leqq b$$

with functions $\varphi(s)$, $\psi(s)$ belonging to $C^2([a, b])$ and such that at least one of the derivatives $\varphi'(s)$, $\psi'(s)$ is different from zero on $[a, b]$.

Let us triangulate the domain Ω, i.e. let us divide it into a finite number of triangles (the sides of which can be curved) in such a way that two arbitrary triangles are either disjoint, or have a common vertex, or a common side. Let every triangulation $\mathscr{T}$ have the property that each interior triangle (i.e. a triangle having at most one point common with the boundary Γ) has straight sides and each boundary triangle has at most one curved side. This side lies then on the boundary. Further we assume that the domain Ω is triangulated in such a way that the curved side of each boundary triangle lies on one arc of the type (30). The curved triangles of the triangulation $\mathscr{T}$ will be called ideal curved triangles.

With every triangulation $\mathscr{T}$ we associate two parameters h and ϑ defined by

(31) $$h = \max_{K \in \mathscr{T}} h_K, \quad \vartheta = \min_{K \in \mathscr{T}} \vartheta_K$$

where h_K and ϑ_K are the length of the greatest side and the magnitude of the smallest angle, respectively, of the triangle with straight sides which has the same vertices as the triangle K. We restrict ourselves to triangulations $\mathscr{T}$ satisfying

(32) $$\vartheta \geqq \vartheta_0, \quad \vartheta_0 = \text{const} > 0.$$

On every triangulation $\mathscr{T}$ we define the finite dimensional space Z_h with the following properties:

a) $Z_h \subset C(\bar{\Omega})$;

b) every function $v \in Z_h$ is uniquely determined by function values prescribed at the vertices of the triangles of $\mathscr{T}$;

c) the restriction of $v \in Z_h$ to an arbitrary interior triangle is a linear function.

In the case of a polygonal boundary Γ the space Z_h is formed by piecewise linear functions. The definition of the restriction of $v \in Z_h$ to an ideal curved triangle can be found in [7].

Let V_h and W_h be subspaces of Z_h defined by

(33) $$V_h = \{v \in Z_h : v = 0 \text{ on } \Gamma_{u1}\},$$

$$(34) \qquad W_h = \{w \in Z_h : w = 0 \text{ on } \Gamma_{p1}\} .$$

It follows from the properties of V_h and W_h that

$$(35) \qquad \lim_{h\to 0} \inf_{v \in V_h} \|u - v\|_1 = 0 \quad \forall u \in V ,$$

$$(36) \qquad \lim_{h\to 0} \inf_{w \in W_h} \|u - w\|_1 = 0 \quad \forall u \in W .$$

We introduce $\Delta t = T/r$, r being a natural number, and consider the partition of $[0, T]$ with the nodes

$$(37) \qquad t_i = i\, \Delta t \quad (i = 0, \ldots, r) .$$

We set

$$(38) \qquad Q^i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} Q(x, t)\, \mathrm{d}t , \quad b^i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} b(x, t)\, \mathrm{d}t ,$$

$$(39) \qquad \boldsymbol{X}^i = \boldsymbol{X}(x, t_i) , \quad \boldsymbol{q}^i = \boldsymbol{q}(x, t_i)$$

and define $P^i \in W_h$, $\boldsymbol{U}^i \in [V_h]^2$ $(i = 1, \ldots, r)$ by

$$(40) \qquad \Delta t\, D(P^i, w) + (\operatorname{div} \boldsymbol{U}^i - \operatorname{div} \boldsymbol{U}^{i-1}, w) = \Delta t(Q^i, w) + \Delta t(b^i, w)_p \quad \forall w \in W_h ,$$

$$(41) \qquad a(\boldsymbol{U}^i, \boldsymbol{v}) - (P^i, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{X}^i, \boldsymbol{v}) + (\boldsymbol{q}^i, \boldsymbol{v})_u \quad \forall \boldsymbol{v} \in [V_h]^2 ,$$

$$(42) \qquad \operatorname{div} \boldsymbol{U}^0 = \alpha_h \equiv \operatorname{div} \boldsymbol{u}_0^h ,$$

where $\boldsymbol{u}_0^h \in [V_h]^2$ is the interpolate of the vector $\boldsymbol{u}_0$ introduced in Remark 2. This interpolate has the following properties:

$$(43) \qquad \|\boldsymbol{u}_0^h - \boldsymbol{u}_0\|_1 \leqq Ch \|\boldsymbol{u}_0\|_2 ,$$

$$(44) \qquad \|\operatorname{div} \boldsymbol{u}_0^h - \alpha\|_0 = \|\operatorname{div} (\boldsymbol{u}_0^h - \boldsymbol{u}_0)\|_0 \leqq C \|\boldsymbol{u}_0^h - \boldsymbol{u}_0\|_1 \to 0 \quad \text{if} \quad h \to 0 ,$$

$$(45) \qquad \|\boldsymbol{u}_0^h\|_1 = \|\boldsymbol{u}_0^h - \boldsymbol{u}_0 + \boldsymbol{u}_0\|_1 \leqq C \|\boldsymbol{u}_0\|_2 .$$

Now we prove that the solution P^i, $\boldsymbol{U}^i$ $(i = 1, \ldots, r)$ of problem (40)–(42) exists and is unique. As (40) and (41) represent a system of linear algebraic equation it is sufficient to prove the uniqueness. Let $Q^i = b^i = 0$, $\boldsymbol{X}^i = \boldsymbol{q}^i = \boldsymbol{0}$ $(i = 1, \ldots, r)$ and $\alpha_h = 0$. In the case $i = 1$ let us set $w = P^1$ in (40) and $\boldsymbol{v} = \boldsymbol{U}^1$ in (41) and sum up the obtained relations. We get

$$(46) \qquad \Delta t\, D(P^1, P^1) + a(\boldsymbol{U}^1, \boldsymbol{U}^1) = 0 .$$

Friedrichs' inequality and Korn's inequality together with (46) imply $P^1 \equiv 0$, $\boldsymbol{U}^1 \equiv \boldsymbol{0}$. In the case $i > 1$ let as assume that we have proved $\boldsymbol{U}^{i-1} \equiv \boldsymbol{0}$. Setting $w = P^i$ in (40) and $\boldsymbol{v} = \boldsymbol{U}^i$ in (41) we obtain

$$\Delta t\, D(P^i, P^i) + a(\boldsymbol{U}^i, \boldsymbol{U}^i) = 0$$

which implies $\boldsymbol{U}^i \equiv \boldsymbol{0}$, $P^i \equiv 0$.

B) Now we prove estimates (59)–(61). Let us set $w = P^i$, $\boldsymbol{v} = \Delta \boldsymbol{U}^i \equiv \boldsymbol{U}^i - \boldsymbol{U}^{i-1}$, where $\boldsymbol{U}^0 = \boldsymbol{u}_0^h \in [V_h]^2$, add up (40) and (41) and sum the result from $i = 1$ to $i = j$. We obtain

$$\Delta t \sum_{i=1}^{j} D(P^i, P^i) + \sum_{i=1}^{j} a(\boldsymbol{U}^i, \Delta \boldsymbol{U}^i) = \Delta t \sum_{i=1}^{j} (Q^i, P^i) + \Delta t \sum_{i=1}^{j} (b^i, P^i)_p + \sum_{i=1}^{j} (\boldsymbol{X}^i, \Delta \boldsymbol{U}^i) + \sum_{i=1}^{j} (\boldsymbol{q}^i, \Delta \boldsymbol{U}^i)_u . \tag{47}$$

Using the relations

$$2a(\boldsymbol{U}^i, \Delta \boldsymbol{U}^i) = a(\boldsymbol{U}^i, \boldsymbol{U}^i) - a(\boldsymbol{U}^{i-1}, \boldsymbol{U}^{i-1}) + a(\Delta \boldsymbol{U}^i, \Delta \boldsymbol{U}^i), \tag{48}$$

$$\sum_{i=1}^{j} (\boldsymbol{X}^i, \Delta \boldsymbol{U}^i) = (\boldsymbol{X}^j, \boldsymbol{U}^j) - (\boldsymbol{X}^1, \boldsymbol{u}_0^h) - \sum_{i=1}^{j-1} (\Delta \boldsymbol{X}^{i+1}, \boldsymbol{U}^i) \tag{49}$$

and Friedrichs' and Korn's inequalities we obtain from (47):

$$\Delta t \sum_{i=1}^{j} \|P^i\|_1^2 + \|\boldsymbol{U}^j\|_1^2 \leqq C\{\|\boldsymbol{u}_0^h\|_1^2 + (\boldsymbol{X}^j, \boldsymbol{U}^j) - (\boldsymbol{X}^1, \boldsymbol{u}_0^h) - \sum_{i=1}^{j-1} (\Delta \boldsymbol{X}^{i+1}, \boldsymbol{U}^i) + (\boldsymbol{q}^j, \boldsymbol{U}^j)_u - (\boldsymbol{q}^1, \boldsymbol{u}_0^h)_u - \sum_{i=1}^{j-1} (\Delta \boldsymbol{q}^{i+1}, \boldsymbol{U}^i)_u + \Delta t \sum_{i=1}^{j} (Q^i, P^i) + \Delta t \sum_{i=1}^{j} (b^i, P^i)_p\} . \tag{50}$$

In (50) and in what follows the symbol C denotes a positive constant not depending on h and Δt and not necessarily the same at any two places. We have

$$C \,\Delta t \sum_{i=1}^{j} (b^i, P^i)_p = C \sum_{i=1}^{j} \left(\int_{t_{i-1}}^{t_i} b \,\mathrm{d}t, P^i \right)_p \leqq C \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \|b\|_{L^2(\Gamma_{p2})} \,\mathrm{d}t \|P^i\|_{L^2(\Gamma_{p2})} \leqq$$
$$\leqq C_1 C \,\Delta t^{1/2} \sum_{i=1}^{j} \left\{ \int_{t_{i-1}}^{t_i} \|b\|_{L^2(\Gamma_{p2})}^2 \,\mathrm{d}t \right)^{1/2} \|P^i\|_1 \leqq (C_1 C)^2 \|b\|_{L^2(0,T;L^2(\Gamma_{p2}))}^2 + \tfrac{1}{4} \Delta t \sum_{i=1}^{j} \|P^i\|_1^2$$

where we used Cauchy's inequality, the trace theorem with the constant C_1 and the inequality

$$|ab| \leqq \tfrac{1}{2}\gamma a^2 + \frac{1}{2\gamma} b^2 \tag{51}$$

with $\gamma = 2$. Further,

$$- C \sum_{i=1}^{j-1} (\Delta \boldsymbol{X}^{i+1}, \boldsymbol{U}^i) \leqq C \sum_{i=1}^{j-1} \|\Delta \boldsymbol{X}^{i+1}\|_0 \|\boldsymbol{U}^i\|_0 \leqq$$
$$\leqq C \sum_{i=1}^{j-1} \left\{ \int_{t_i}^{t_{i+1}} \|\dot{\boldsymbol{X}}\|_0^2 \,\mathrm{d}t \right\}^{1/2} \|\boldsymbol{U}^i\|_1 \leqq (C^2/2) \|\dot{\boldsymbol{X}}\|_{L^2(0,T;[L^2(\Omega)]^2)}^2 + \tfrac{1}{2} \Delta t \sum_{i=1}^{j-1} \|\boldsymbol{U}^i\|_1^2 .$$

The remaining terms on the right-hand side of (50) can be estimated similarly. We obtain from (50):

$$\Delta t \sum_{i=1}^{j} \|P^i\|_1^2 + \|U^j\|_1^2 \leqq C + \Delta t \sum_{i=1}^{j-1} \|U^i\|_1^2 , \tag{52}$$

where the constant C depends on Q, b, $\boldsymbol{X}$, $\dot{\boldsymbol{X}}$, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and $\boldsymbol{u}_0$. Using the discrete Gronwall inequality we obtain from (52)

$$\Delta t \sum_{i=1}^{j} \|P^i\|_1^2 + \|U^j\|_1^2 \leqq C . \tag{53}$$

Inequality (53) implies

$$\Delta t \sum_{i=1}^{r} \|P^i\|_1^2 \leqq C , \tag{54}$$

$$\|U^i\|_1 \leqq C \quad (i = 1, \dots, r) . \tag{55}$$

It follows from (55) that

$$\Delta t \sum_{i=1}^{r} \|U^i\|_1^2 \leqq C . \tag{56}$$

Let us define extensions of the approximate solutions P^i, U^i on the whole interval $(0, T]$ by

$$P^\delta = P^i \quad \text{in} \quad (t_{i-1}, t_i] , \quad i = 1, \dots, r ; \quad \delta = (h, \Delta t) , \tag{57}$$

$$U^\delta = U^i \quad \text{in} \quad (t_{i-1}, t_i] , \quad i = 1, \dots, r ; \quad \delta = (h, \Delta t) . \tag{58}$$

As $P^i \in W$, $U^i \in [V]^2$ relations (54)–(58) imply

$$\|P^\delta\|_{L^2(0,T;W)} \leqq C , \tag{59}$$

$$\|U^\delta\|_{L^2(0,T;[V]^2)} \leqq C , \tag{60}$$

$$\|U^\delta(T)\|_{[V]^2} \leqq C \tag{61}$$

where the norms of W and $[V]^2$ are induced by the norms of $H^1(\Omega)$ and $[H^1(\Omega)]^2$, respectively.

C) Let $h_n, \Delta t^n > 0$ for $n = 1, 2, \dots$ and

$$h_n \to 0 , \quad \Delta t^n \to 0 \quad \text{if} \quad n \to \infty$$

and consider the sequences $\{U^{\delta_n}\}_{n=1}^{\infty}$, $\{P^{\delta_n}\}_{n=1}^{\infty}$ with $\delta_n = (h_n, \Delta t^n)$. For simplicity we leave out the subscript n and write δ, h and Δt instead of δ_n, h_n and Δt^n. Then (59)–(61) and the compactness theorem (see, e.g., [3, p. 24]) imply: There exist subsequences, denoted here again by U^δ, P^δ, such that

$$U^\delta \to \boldsymbol{u} \quad \text{weakly in} \quad L^2(0, T; [V]^2) , \tag{62}$$

$$P^\delta \to p \quad \text{weakly in} \quad L^2(0, T; W) , \tag{63}$$

$$U^\delta(T) \to \boldsymbol{g} \quad \text{weakly in} \quad [V]^2 . \tag{64}$$

It can be arranged that the set of indices $\{\delta\}$ is the same in all the three cases (62)–(64).

D) Consider a function $f(t) \in C^\infty([0, T])$. Let

$$f^i = f(t_i), \quad i = 1, \ldots, r, \quad f^{r+1} = f^r = f(T)$$

and let us define two functions $f_{\Delta t}, \tilde{f}_{\Delta t}$ by

$$(65) \qquad f_{\Delta t} = f^{i+1} \quad \text{in} \quad (t_i, t_{i+1}], \quad i = 0, \ldots, r-1,$$

$$(66) \qquad \tilde{f}_{\Delta t} = (f^{i+2} - f^{i+1})/\Delta t \quad \text{in} \quad (t_i, t_{i+1}], \quad i = 0, \ldots, r-1.$$

Using Taylor's theorem we can prove

$$(67) \qquad \|f_{\Delta t} - f\|_{L^2(0,T)} \leqq C\,\Delta t, \quad \|\tilde{f}_{\Delta t} - \dot{f}\|_{L^2(0,T)} \leqq C\,\Delta t.$$

We also define

$$(68) \qquad Q_{\Delta t} = Q^{i+1}, \quad b_{\Delta t} = b^{i+1}, \quad \boldsymbol{X}_{\Delta t} = \boldsymbol{X}^{i+1}, \quad \boldsymbol{q}_{\Delta t} = \boldsymbol{q}^{i+1} \quad \text{in} \quad (t_i, t_{i+1}],$$
$$i = 0, \ldots, r-1.$$

We set $w = f^i z^h$, $z^h \in W_h$ and $\boldsymbol{v} = f^i \boldsymbol{y}^h$, $\boldsymbol{y}^h \in [V_h]^2$ in (40) and (41), respectively, and sum each relation from $i = 1$ to $i = r$. As

$$\sum_{i=1}^{r} (\operatorname{div} \boldsymbol{U}^i - \operatorname{div} \boldsymbol{U}^{i-1}, z^h) f^i = (\operatorname{div} \boldsymbol{U}^r, z^h) f^r - (\operatorname{div} \boldsymbol{u}_0^h, z^h) f^1 - \\ - \sum_{i=1}^{r-1} (\operatorname{div} \boldsymbol{U}^i, z^h)(f^{i+1} - f^i),$$

we obtain (due to (57) and (58))

$$(69) \qquad \int_0^T D(P^\delta, z^h) f_{\Delta t}\,dt - \int_0^T (\operatorname{div} \boldsymbol{U}^\delta, z^h) \tilde{f}_{\Delta t}\,dt = \int_0^T (Q_{\Delta t}, z^h) f_{\Delta t}\,dt + \\ + \int_0^T (b_{\Delta t}, z^h)_p f_{\Delta t}\,dt + (\operatorname{div} \boldsymbol{u}_0^h, z^h) f(\Delta t) - (\operatorname{div} \boldsymbol{U}^\delta(T), z^h) f(T),$$

$$(70) \qquad \int_0^T a(\boldsymbol{U}^\delta, \boldsymbol{y}^h) f_{\Delta t}\,dt - \int_0^T (P^\delta, \operatorname{div} \boldsymbol{y}^h) f_{\Delta t}\,dt = \\ = \int_0^T (\boldsymbol{X}_{\Delta t}, \boldsymbol{y}^h) f_{\Delta t}\,dt + \int_0^T (\boldsymbol{q}_{\Delta t}, \boldsymbol{y}^h)_u f_{\Delta t}\,dt.$$

Let $w \in W$ and $\boldsymbol{v} \in [V]^2$ be given. We choose $z^h \in W_h$ and $\boldsymbol{y}^h \in [V_h]^2$ such that

$$(71) \qquad \|z^h - w\|_1 \to 0, \quad \|\boldsymbol{y}^h - \boldsymbol{v}\|_1 \to 0.$$

According to (37), (38), such a choice is always possible. We want to prove that if we pass to the limit for $\delta \to 0$ in (69), (70) then we obtain

$$(72) \qquad \int_0^T D(p, w) f\,dt - \int_0^T (\operatorname{div} \boldsymbol{u}, w) \dot{f}\,dt = (\alpha, w) f(0) - (\operatorname{div} \boldsymbol{g}, w) f(T) + \\ + \int_0^T (Q, w) f\,dt + \int_0^T (b, w)_p f\,dt \quad \forall f \in C^\infty([0, T]), \quad \forall w \in W,$$

$$(73) \qquad \int_0^T a(\boldsymbol{u}, \boldsymbol{v}) f \, dt - \int_0^T (p, \operatorname{div} \boldsymbol{v}) f \, dt = \int_0^T (\boldsymbol{X}, \boldsymbol{v}) f \, dt + \int_0^T (\boldsymbol{q}, \boldsymbol{v})_u f \, dt$$
$$\forall f \in C^\infty([0, T]), \quad \forall \boldsymbol{v} \in [V]^2,$$

where $\boldsymbol{u}$, p, $\boldsymbol{g}$ are the weak limits from (62)–(64).

First we prove that the left-hand sides of (69) and (70) converge to the left-hand sides of (72) and (73), respectively. To this end let us note that for given functions $f \in C^\infty([0, T])$, $w \in W$ and a given vector $\boldsymbol{v} \in [V]^2$ the functionals

$$F_1(z) = \int_0^T D(z, w) f \, dt, \quad F_2(z) = \int_0^T (z, \operatorname{div} \boldsymbol{v}) f \, dt$$

are linear functionals on $L^2(0, T; W)$ and the functionals

$$G_1(\boldsymbol{z}) = \int_0^T a(\boldsymbol{z}, \boldsymbol{v}) f \, dt, \quad G_2(\boldsymbol{z}) = \int_0^T (\operatorname{div} \boldsymbol{z}, w) \dot{f} \, dt$$

are linear functionals on $L^2(0, T; [V]^2)$. For example,

$$|G_1(\boldsymbol{z})| = \left| \int_0^T a(\boldsymbol{z}, \boldsymbol{v}f) \, dt \right| \leqq C \int_0^T \|\boldsymbol{z}\|_1 \, \|\boldsymbol{v}f\|_1 \, dt \leqq C \|\boldsymbol{z}\|_{L^2(0,T;[V]^2)} \, \|\boldsymbol{v}f\|_{L^2(0,T;[V]^2)} \, .$$

Thus, according to (62), (63),

$$(74) \qquad F_i(P^\delta) \to F_i(p), \quad G_i(\boldsymbol{U}^\delta) \to G_i(\boldsymbol{u}) \quad (i = 1, 2).$$

We have

$$\int_0^T D(P^\delta, z^h) f_{\Delta t} \, dt - \int_0^T (\operatorname{div} \boldsymbol{U}^\delta, z^h) \tilde{f}_{\Delta t} \, dt = \int_0^T D(P^\delta, w) f \, dt +$$
$$+ \left\{ \int_0^T D(P^\delta, z^h - w) f \, dt + \int_0^T D(P^\delta, z^h)(f_{\Delta t} - f) \, dt \right\} - \int_0^T (\operatorname{div} \boldsymbol{U}^\delta, w) \dot{f} \, dt -$$
$$- \left\{ \int_0^T (\operatorname{div} \boldsymbol{U}^\delta, z^h - w) \dot{f} \, dt + \int_0^T (\operatorname{div} \boldsymbol{U}^\delta, z^h) (\tilde{f}_{\Delta t} - \dot{f}) \, dt \right\}.$$

It follows from (59), (60), (67) and (71) that the integrals in the brackets tend to zero. For example,

$$\left| \int_0^T D(P^\delta, z^h - w) f \, dt \right| \leqq k \|z^h - w\|_1 \int_0^T \|P^\delta\|_1 \, |f| \, dt \leqq$$
$$\leqq kC \|z^h - w\|_1 \, \|f\|_{L^2(0,T)} \to 0 .$$

Thus, according to (74), the left-hand side of (69) tends to the left-hand side of (72). Similarly we can prove that the left-hand side of (70) tends to the left-hand side of (73).

Let us now consider the right-hand side of (69). According to (68), (38), (71) and (67), we have

(75) $$\int_0^T (Q_{\Delta t}, z^h) f_{\Delta t}\,\mathrm{d}t = \sum_{i=1}^r (Q^i, z^h) f^i \Delta t = \sum_{i=1}^r \left(\int_{t_{i-1}}^{t_i} Q\,\mathrm{d}t, z^h \right) f^i =$$
$$= \int_0^T (Q, z^h) f_{\Delta t}\,\mathrm{d}t = \int_0^T (Q, w) f\,\mathrm{d}t + \int_0^T (Q, z^h - w) f\,\mathrm{d}t +$$
$$+ \int_0^T (Q, z^h)(f_{\Delta t} - f)\,\mathrm{d}t \to \int_0^T (Q, w) f\,\mathrm{d}t\,.$$

Similarly (using in addition the trace theorem) we obtain

(76) $$\int_0^T (b_{\Delta t}, z^h)_p f_{\Delta t}\,\mathrm{d}t \to \int_0^T (b, w)_p f\,\mathrm{d}t\,.$$

According to (46), (71) and (64), we have

(77) $$(\operatorname{div} \boldsymbol{u}_0^h, z^h) \to (\alpha, w)\,,$$

(78) $$(\operatorname{div} \boldsymbol{U}^\delta(T), z^h) = (\operatorname{div} \boldsymbol{U}^\delta(T), z^h - w) + (\operatorname{div} \boldsymbol{U}^\delta(T), w) \to (\operatorname{div} \boldsymbol{g}, w)\,.$$

Relations (75)–(78) imply that the right-hand side of (69) tends to the right-hand side of (72).

As to the right-hand of (70), we have

(79) $$\int_0^T (\boldsymbol{X}_{\Delta t}, \boldsymbol{y}^h) f_{\Delta t}\,\mathrm{d}t = \left(\sum_{i=1}^r \boldsymbol{X}^i f^i \Delta t, \boldsymbol{y}^h \right),$$

(80) $$\int_0^T \|\boldsymbol{X} f\|_0\,\mathrm{d}t \leqq \|\boldsymbol{X}\|_{L^2(0,T;[L^2(\Omega)]^2)} \|f\|_{L^2(0,T)}\,.$$

According to (80) and [4, pp. 108–109], the Bochner integral of $\boldsymbol{X}f$ exists and

(81) $$\lim_{t\to 0} \left\| \sum_{i=1}^r \boldsymbol{X}^i f^i \Delta t - \int_0^T \boldsymbol{X} f\,\mathrm{d}t \right\|_0 = 0\,.$$

Relations (71), (79) and (81) imply

(82) $$\int_0^T (\boldsymbol{X}_{\Delta t}, \boldsymbol{y}^h) f_{\Delta t}\,\mathrm{d}t \to \int_0^T (\boldsymbol{X}, \boldsymbol{v}) f\,\mathrm{d}t\,.$$

In a similar way we can prove

(83) $$\int_0^T (\boldsymbol{q}_{\Delta t}, \boldsymbol{y}^h)_u f_{\Delta t}\,\mathrm{d}t \to \int_0^T (\boldsymbol{q}, \boldsymbol{v})_u f\,\mathrm{d}t\,.$$

It follows from (82), (83) that the right-hand side of (70) tends to the right-hand side of (73). Thus we have proved that the weak limits $p \in L^2(0, T; W)$, $\boldsymbol{u} \in L^2(0, T; [V]^2)$ satisfy relations (72) and (73). If we restrict f to $C_0^\infty((0, T))$ in (72), (73) we obtain relations (23), (24). Thus property a) of problem P is proved.

E) Now we prove property b) of problem P. Let us define functions

$$G(t) = (\operatorname{div} \boldsymbol{u}(t), w)\,, \tag{84}$$

$$H(t) = -D(p(t), w) + (Q(t), w) + (b(t), w)_p \tag{85}$$

where $w \in W$. As $p \in L^2(0, T; W)$, $\boldsymbol{u} \in L^2(0, T; [V]^2)$, $Q \in L^2(0, T; L^2(\Omega))$, $b \in \in L^2(0, T; L^2(\Gamma_{p2}))$ we see that $G(t) \in L^2(0, T)$, $H(t) \in L^2(0, T)$. Thus the function

$$F(t) = \int_0^t H(\tau)\,\mathrm{d}\tau \tag{86}$$

is absolutely continuous on $[0, T]$ and satisfies the relation

$$\dot{F}(t) = H(t) \quad \text{a.e. in} \quad (0, T)\,. \tag{87}$$

Using (23), (84)–(87) and integration by parts we obtain:

$$\int_0^T (G - F)\dot{f}\,\mathrm{d}t = 0 \quad \forall f \in C_0^\infty((0, T))\,. \tag{88}$$

According to [6, pp. 251–252], relation (88) implies

$$G(t) - F(t) = c_0 \quad \text{a.e. in} \quad (0, T)\,. \tag{89}$$

In order to determine the constant c_0 we choose in (72) a function $f(t) \in C^\infty([0, T])$ with $f(0) = 1$, $f(T) = 0$. Using (84) and (85) we can write (72) in the form:

$$-\int_0^T Hf\,\mathrm{d}t - \int_0^T G\dot{f}\,\mathrm{d}t = (\alpha, w)\,. \tag{90}$$

Integrating the second integral in (90) by parts and expressing $G(t)$ by means of (89) and (86) we come to $c_0 = (\alpha, w)$. Thus relations (84), (85), (86) and (89) imply

$$(\operatorname{div} \boldsymbol{u}(t), w) = (\alpha, w) - \int_0^t D(p(\tau), w)\,\mathrm{d}\tau + \int_0^t [(Q(\tau), w) + (b(\tau), w)_p]\,\mathrm{d}\tau \quad \forall w \in W \quad \text{a.e. in} \quad (0, T)\,. \tag{91}$$

This proves property b) of problem P.

F) Now we prove the uniqueness of the solution. As the problem is linear it suffices to prove that the corresponding homogeneous problem (92)–(94) has only the trivial solution:

$$\int_0^T D(p, w) f\,\mathrm{d}t - \int_0^T (\operatorname{div} \boldsymbol{u}, w)\dot{f}\,\mathrm{d}t = 0 \quad \forall w \in W, \quad \forall f \in C_0^\infty((0, T))\,, \tag{92}$$

$$\int_0^T a(\boldsymbol{u}, \boldsymbol{v}) f\,\mathrm{d}t - \int_0^T (p, \operatorname{div} \boldsymbol{v}) f\,\mathrm{d}t = 0 \quad \forall \boldsymbol{v} \in [V]^2\,, \quad \forall f \in C_0^\infty((0, T))\,, \tag{93}$$

$$(\operatorname{div} \boldsymbol{u}(0), w) = 0 \quad \forall w \in W\,. \tag{94}$$

It follows from (93) that

$$(95) \qquad a(\boldsymbol{u}(t), \boldsymbol{v}) - (p(t), \operatorname{div} \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in [V]^2, \quad \forall t \in E_1,$$

where $E_1 \subset [0, T]$, mes $E_1 = T$. Let us set $\boldsymbol{v} = \boldsymbol{u}(t)$ $(t \in E_1)$ and integrate relation (95) in $(0, t^*)$, where $t^* \in [0, T]$. We obtain

$$(96) \qquad \int_0^{t^*} a(\boldsymbol{u}(t), \boldsymbol{u}(t))\, \mathrm{d}t - \int_0^{t^*} (p(t), \operatorname{div} \boldsymbol{u}(t))\, \mathrm{d}t = 0\,.$$

In order to express the second integral in (96) in a suitable form let us integrate the second integral in (92) by parts. Then relation (92) implies

$$(97) \qquad D(p(\tau), w) + \frac{\mathrm{d}}{\mathrm{d}\tau}(\operatorname{div} \boldsymbol{u}(\tau), w) = 0 \quad \forall w \in W \quad \text{a.e. in} \quad (0, T)\,.$$

Integrating relation (97) in $(0, t)$ and using (94) we obtain a relation in which we set $w = p(t) \in W$. Then we have

$$(98) \qquad -(\operatorname{div} \boldsymbol{u}(t), p(t)) = \int_0^t D(p(\tau), p(t))\, \mathrm{d}\tau\,.$$

Let us define a vector $\boldsymbol{z}(t)$ by

$$(99) \qquad \boldsymbol{z}(t) = \int_0^t \operatorname{grad} p(\tau)\, \mathrm{d}\tau\,.$$

Then we can write, according to (18),

$$(100) \qquad \int_0^t D(p(\tau), p(t))\, \mathrm{d}\tau = k(\boldsymbol{z}(t), \dot{\boldsymbol{z}}(t)) \quad \text{a.e. in} \quad (0, T)$$

and relations (96), (98) and (100) imply

$$(101) \qquad \int_0^{t^*} a(\boldsymbol{u}(t), \boldsymbol{u}(t))\, \mathrm{d}t + k \int_0^{t^*} (\boldsymbol{z}(t). \dot{\boldsymbol{z}}(t))\, \mathrm{d}t = 0\,.$$

As the integrand of the second integral in (101) is equivalent to the derivative of the function $(\boldsymbol{z}(t), \boldsymbol{z}(t))/2$, relations (101) and (99) give:

$$(102) \qquad \int_0^{t^*} a(\boldsymbol{u}(t), \boldsymbol{u}(t))\, \mathrm{d}t + \frac{k}{2} \|\boldsymbol{z}(t^*)\|_0^2 = 0\,.$$

As $a(\boldsymbol{u}, \boldsymbol{u}) \geqq 0$ we get from (102) that

$$(103) \qquad \int_0^{t^*} a(\boldsymbol{u}(t), \boldsymbol{u}(t))\, \mathrm{d}t = 0\,.$$

Choosing $t^* = T$ in (103) and using Korn's inequality we obtain

$$(104) \qquad \|\boldsymbol{u}\|_{L^2(0,T;[V]^2)} = 0\,.$$

Relations (92) and (104) imply

$$(105) \qquad D(p(t), w) = 0 \quad \forall w \in W \quad \forall t \in E_2$$

where $E_2 \subset [0, T]$, mes $E_2 = T$. (Relation (105) follows also from (102), (103) and (100).) Choosing $t \in E_2$, setting $w = p(t)$, integrating (105) in $(0, T)$ and using

Friedrichs' inequality we get

$$\|p\|_{L^2(0,T;W)} = 0 . \tag{106}$$

Relations (104) and (106) imply that problem (92)–(94) has only the trivial solution. Theorem 1 is proved.

Using the same device as in [10, p. 215] we see that Theorem 1 implies following corollary:

Corollary. *Let the assumptions of Theorem* 1 *be satisfied. Then the sequences* $\{P^\delta\}$ *and* $\{\boldsymbol{U}^\delta\}$ *defined by* (57) *and* (58), *respectively, converge weakly* (*see* (62), (63)) *to the exact solution* p, $\boldsymbol{u}$ *of problem* P.

3. SOME ERROR ESTIMATES

For a greater simplicity we shall consider only the case of finite elements which cover the domain Ω exactly, i.e. in the case of curved boundary Γ we restrict our considerations to Zlámal's ideal curved triangular elements [7]. Nor is the effect of numerical integration taken into account.

The spaces V_h and W_h are now defined by

$$V_h = \{v \in Z_h^{(n)} \quad : v = 0 \text{ on } \Gamma_{u1}\} , \tag{107}$$

$$W_h = \{w \in Z_h^{(n-1)} : w = 0 \text{ on } \Gamma_{p1}\} \tag{108}$$

where $Z_h^{(k)} \subset C(\bar{\Omega})$ is a finite dimensional space constructed by means of triangular finite elements generated by polynomials of degree k.

The discrete approximate solution of the problem of consolidation of clay ($Q \equiv \equiv 0$, $b \equiv 0$, $\operatorname{div} \boldsymbol{u}(x, 0) \equiv 0$) is defined in the following way: Find $P^i \in W_h$ and $\boldsymbol{U}^i \in [V_h]^2$ ($i = 1, \ldots, r$) such that

$$\Delta t\, D(P^i, w) + (\operatorname{div}(\boldsymbol{U}^i - \boldsymbol{U}^{i-1}), w) = 0 \quad \forall w \in W_h , \tag{109}$$

$$a(\boldsymbol{U}^i, \boldsymbol{v}) - (P^i, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{X}^i, \boldsymbol{v}) + (\boldsymbol{q}^i, \boldsymbol{v})_u \quad \forall \boldsymbol{v} \in [V_h]^2 , \tag{110}$$

$$\operatorname{div} \boldsymbol{U}^0 \equiv 0 , \tag{111}$$

where $\boldsymbol{X}^i$, $\boldsymbol{q}^i$ are defined by (39).

In order to obtain the maximum rate of convergence we shall consider the case $\Gamma_{p1} = \Gamma$.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied and let the exact solution* p, $\boldsymbol{u}$ *have the properties*

$$p \in AC([0, T]; H^n(\Omega)) , \quad \dot{p} \in L^2(0, T; H^n(\Omega)) , \tag{112}$$

$$\operatorname{div} \dot{\boldsymbol{u}} \in AC([0, T]; C(\bar{\Omega})) , \quad \operatorname{div} \ddot{\boldsymbol{u}} \in L^2(0, T; L^2(\Omega)) , \tag{113}$$

$$\boldsymbol{u} \in AC([0, T]; [H^{n+1}(\Omega)]^2) , \quad \dot{\boldsymbol{u}} \in L^2(0, T; [H^{n+1}(\Omega)]^2) . \tag{114}$$

Moreover, if $n \leqq 3$ *let* $p \in C([0, T]; C^2(\bar{\Omega}))$ *and if* $n \leqq 2$ *let also* $\boldsymbol{u} \in C([0, T]; C^2(\bar{\Omega}))$. *Then*

$$(115)\qquad \|p - P\|_{l_2} + \|\boldsymbol{u}^i - \boldsymbol{U}^i\|_1 \leqq C\{h^n \Delta t^{-1/2}\|\boldsymbol{u}^0\|_{n+1} + h^n + \Delta t\}$$
$$(i = 1, \ldots, r)$$

where the constant C does not depend on h and Δt, $\boldsymbol{u}^i = \boldsymbol{u}(x, t_i)$, $p^i = p(x, t_i)$ *and the norm* $\|\cdot\|_{l_2}$ *is defined by*

$$(116)\qquad \|f\|_{l_2}^2 = \Delta t \sum_{i=1}^{r} \|f^i\|_0^2 .$$

Proof. The existence and uniqueness of the approximate solution P^i, $\boldsymbol{U}^i$ were proved in Section 2.

The function $\eta \in W_h$ and the vector $\boldsymbol{r} \in [V_h]^2$ satisfying

$$(117)\qquad D(p - \eta, w) = 0 \quad \forall w \in W_h$$

and

$$(118)\qquad a(\boldsymbol{u} - \boldsymbol{r}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in [V_h]^2$$

are called the Ritz approximations of p and $\boldsymbol{u}$, respectively. The following estimates are immediate consequences of interpolation theorems and Friedrichs' and Korn's inequalities:

$$(119)\qquad \|p - \eta\|_1 \leqq Ch^{n-1}\|p\|_n ,$$

$$(120)\qquad \|\boldsymbol{u} - \boldsymbol{r}\|_1 \leqq Ch^n\|\boldsymbol{u}\|_{n+1} .$$

As we assume $\Gamma_{p1} = \Gamma$ we can prove using (119) and Nitsche's trick:

$$(121)\qquad \|p - \eta\|_0 \leqq Ch^n\|p\|_n .$$

Let us set

$$(122)\qquad \boldsymbol{e}^i = \boldsymbol{r}^i - \boldsymbol{U}^i , \quad \varepsilon^i = \eta^i - P^i ,$$

$$(123)\qquad \boldsymbol{s} = \boldsymbol{u} - \boldsymbol{r} , \quad \xi = p - \eta .$$

Then

$$(124)\qquad \|p - P\|_{l_2} + \|\boldsymbol{u}^i - \boldsymbol{U}^i\|_1 \leqq \|\xi\|_{l_2} + \|\boldsymbol{s}^i\|_1 + \|\varepsilon\|_{l_2} + \|\boldsymbol{e}^i\|_1 \leqq$$
$$\leqq Ch^n + \|\varepsilon\|_{l_2} + \|\boldsymbol{e}^i\|_1$$

where the second inequality follows from (116), (120), (121) and (123). It remains to estimate the last two terms on the right-hand side of (124).

Let us multiply (109) by -1 and to the both sides let us add the expression

$$\Delta t\, D(\eta^i, w) + (\operatorname{div}(\boldsymbol{r}^i - \varrho_i \boldsymbol{r}^{i-1}), w)$$

where $\varrho_1 = 0$ and $\varrho_i = 1$ $(i \geqq 2)$. After simple calculations, in which we use (122), (123), (1), (3), (18) and Green's theorem, we obtain

$$(125)\qquad \Delta t\, D(\varepsilon^1, w) + (\operatorname{div} \boldsymbol{e}^1, w) = -(\operatorname{div} \boldsymbol{s}^1, w) +$$
$$+ (\operatorname{div} \boldsymbol{u}^1 - \Delta t \operatorname{div} \dot{\boldsymbol{u}}^1, w) \quad \forall w \in W_h ,$$

$$(126)\qquad \Delta t\, D(\varepsilon^i, w) + (\operatorname{div}(\Delta \boldsymbol{e}^i), w) = -(\operatorname{div}(\Delta \boldsymbol{s}^i), w) +$$
$$+ (\operatorname{div}(\Delta \boldsymbol{u}^i) - \Delta t \operatorname{div} \dot{\boldsymbol{u}}^i, w) \quad \forall w \in W_h \quad (i \geqq 2) .$$

Let us multiply (115) by -1 and to the both sides let us add the expression $a(\boldsymbol{r}^i, \boldsymbol{v}) - (\eta^i, \operatorname{div} \boldsymbol{v})$. After simple calculations, in which we use (2), (5), (6), (9)–(11), (21), (122), (123) and Green's theorem, we obtain

$$a(\boldsymbol{e}^i, \boldsymbol{v}) - (\varepsilon^i, \operatorname{div} \boldsymbol{v}) = (\xi^i, \operatorname{div} \boldsymbol{v}) \quad \forall \boldsymbol{v} \in [V_h]^2 . \tag{127}$$

In the case $i = 1$ let us set $w = \varepsilon^1$ in (125) and $\boldsymbol{v} = \boldsymbol{e}^1$ in (127) and sum up the resulting relations. We get

$$\Delta t\, D(\varepsilon^1, \varepsilon^1) + a(\boldsymbol{e}^1, \boldsymbol{e}^1) = (\xi^1, \operatorname{div} \boldsymbol{e}^1) + (\operatorname{div}(\Delta \boldsymbol{u}^1) - \Delta t \operatorname{div} \dot{\boldsymbol{u}}^1, \varepsilon^1) - \\ - (\operatorname{div} \boldsymbol{s}^1, \varepsilon^1) \tag{128}$$

because $\operatorname{div} \boldsymbol{u}^1 = \operatorname{div}(\Delta \boldsymbol{u}^1)$ if $\operatorname{div} \boldsymbol{u}^0 = 0$. In the case $i \geqq 2$ let us set $w = \varepsilon^i$ in (126) and $\boldsymbol{v} = \Delta \boldsymbol{e}^i \equiv \boldsymbol{e}^i - \boldsymbol{e}^{i-1}$ in (127) and add up the obtained relations. After summing the result from $i = 2$ to $i = j$ we obtain

$$\Delta t \sum_{i=2}^{j} D(\varepsilon^i, \varepsilon^i) + \sum_{i=2}^{j} a(\boldsymbol{e}^i, \Delta \boldsymbol{e}^i) = \sum_{i=2}^{j} (\xi^i, \operatorname{div}(\Delta \boldsymbol{e}^i)) + \\ + \sum_{i=2}^{j} (\operatorname{div}(\Delta \boldsymbol{u}^i) - \Delta t \operatorname{div} \dot{\boldsymbol{u}}^i, \varepsilon^i) - \sum_{i=2}^{j} (\operatorname{div}(\Delta \boldsymbol{s}^i), \varepsilon^i) . \tag{129}$$

We have

$$\sum_{i=2}^{j} (\xi^i, \operatorname{div}(\Delta \boldsymbol{e}^i)) = (\xi^j, \operatorname{div} \boldsymbol{e}^j) - (\xi^2, \operatorname{div} \boldsymbol{e}^1) - \sum_{i=2}^{j-1} (\Delta \xi^{i+1}, \operatorname{div} \boldsymbol{e}^i) ,$$

$$\sum_{i=2}^{j} a(\boldsymbol{e}^i, \Delta \boldsymbol{e}^i) \geqq \tfrac{1}{2} a(\boldsymbol{e}^j, \boldsymbol{e}^j) - \tfrac{1}{2} a(\boldsymbol{e}^1, \boldsymbol{e}^1) .$$

Thus, summing up (128), (129) and using Friedrich's and Korn's inequalities, we obtain

$$\Delta t \sum_{i=1}^{j} \|\varepsilon^i\|_0^2 + \|\boldsymbol{e}^j\|_1^2 \leqq C\{(\xi^j, \operatorname{div} \boldsymbol{e}^j) - \sum_{i=1}^{j-1} (\Delta \xi^{i+1}, \operatorname{div} \boldsymbol{e}^i) + \\ + \sum_{i=1}^{j} (\operatorname{div}(\Delta \boldsymbol{u}^i) - \Delta t \operatorname{div} \dot{\boldsymbol{u}}^i, \varepsilon^i) - (\operatorname{div} \boldsymbol{s}^1, \varepsilon^1) - \sum_{i=2}^{j} (\operatorname{div}(\Delta \boldsymbol{s}^i), \varepsilon^i)\} . \tag{130}$$

Using the Cauchy inequality, Taylor's theorem with an integral remainder, estimates (120), (121) and inequality (51) with various γ's we can find

$$|(\xi^j, \operatorname{div} \boldsymbol{e}^j)| \leqq K h^{2n} \|p^j\|_n^2 + \tfrac{1}{2} \|\boldsymbol{e}^j\|_1^2 , \tag{131}$$

$$\Big|\sum_{i=1}^{j-1} (\Delta \xi^{i+1}, \operatorname{div} \boldsymbol{e}^i)\Big| \leqq \Delta t \sum_{i=1}^{j-1} \|\boldsymbol{e}^i\|_1^2 + K h^{2n} \|\dot{p}\|_{L^2(0,T;H^n(\Omega))}^2 , \tag{132}$$

$$\Big|\sum_{i=1}^{j} (\operatorname{div}(\Delta \boldsymbol{u}^i) - \Delta t \operatorname{div} \dot{\boldsymbol{u}}^i, \varepsilon^i)\Big| \leqq K \Delta t^2 \|\operatorname{div} \ddot{\boldsymbol{u}}\|_{L^2(0,T;L^2(\Omega))}^2 + \tfrac{1}{4} \Delta t \sum_{i=1}^{j} \|\varepsilon^i\|_0^2, \tag{133}$$

$$|(\operatorname{div} \boldsymbol{s}^1, \varepsilon^1)| \leqq \tfrac{1}{2} K h^{2n} \Delta t^{-1} \|\boldsymbol{u}^1\|_{n+1}^2 + \tfrac{1}{4} \Delta t \|\varepsilon^1\|_0^2 \leqq \\ \leqq K h^{2n} \int_0^{t_1} \|\dot{\boldsymbol{u}}\|_{n+1}^2 \, dt + K h^{2n} \Delta t^{-1} \|\boldsymbol{u}^0\|_{n+1}^2 + \tfrac{1}{4} \Delta t \|\varepsilon^1\|_0^2 \tag{134}$$

$$(135) \qquad \left|\sum_{i=2}^{j} (\operatorname{div}(\Delta s^i), \varepsilon^i)\right| \leqq \tfrac{1}{4}\,\Delta t \sum_{i=2}^{j} \|\varepsilon^i\|_0^2 + K h^{2n} \|\dot{\boldsymbol{u}}\|^2_{L^2(0,T;H^{n+1}(\Omega))}$$

where the constant K does not depend on h, Δt, $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, div $\ddot{\boldsymbol{u}}$, p and $\dot{p}$. Relations (130)–(135) and the discrete form of Gronwall's inequality imply

$$(136) \qquad \Delta t \sum_{i=1}^{j} \|\varepsilon^i\|_0^2 + \|\boldsymbol{e}^j\|_1^2 \leqq C\{h^{2n}\,\Delta t^{-1} \|\boldsymbol{u}^0\|^2_{n+1} + h^{2n} + \Delta t^2\}$$

where the constant C does not depend on h and Δt. As $1 \leqq j \leqq r$ relations (116), (124) and (136) imply inequality (115). Theorem 2 is proved.

Remark 3. In [9] curved elements, numerical integration and two-step A-stable difference methods are also considered.

Remark 4. It should be noted that the first term on the right-hand side of (115) is a consequence of estimating in the first step ($i = 1$) only. Moreover, if $h \leqq C\,\Delta t$ then $h^n\,\Delta t^{-1/2} \leqq C\,\Delta t$ because $n \geqq 2$.

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Souhrn

VĚTA O EXISTENCI A JEDNOZNAČNOSTI ŘEŠENÍ BIOTOVA MODELU KONSOLIDACE ZEMIN

ALEXANDER ŽENÍŠEK

V článku je dokázána věta o existenci a jednoznačnosti řešení variačního problému (23)–(25), jehož speciálním případem je variační formulace lineárního Biotova modelu konsolidace zemin. Důkaz existence řešení má konstruktivní povahu a je proveden kompaktnostní metodou. V druhé části článku jsou uvedeny odhady chyb při přibližném řešení problému konsolidace zemin kombinací metody konečných prvků a Eulerovy zpětné diferenční formule.

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