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SOME EXAMPLES OF NON-MONOTONICITIES IN A TWO-UNIT REDUNDANT SYSTEM

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The present paper deals with the same cold-standby redundant system as [1], [2] and [3]. There are two identical units and a single repair facility in the system. Three states of units are considered: good (I), degraded (II), and failed (III). We suppose that only the following state-transitions of a unit are possible: $I \to II$, $II \to III$, $II \to I$, $III \to I$. Transition times of a unit between states I, II, and III (times of work of a unit in state I or II and times of repairs of a unit of the types $II \to I$ or $III \to I$ denoted respectively by $\mathscr{A}, \mathscr{B}, \mathscr{M}$, and $\mathscr{N}$ are supposed to be random variables positive with probability 1, mutually independent and generally distributed. Three starting situations of the system are considered:

$\mathscr{P}(P)$ – both units are new, i.e. in state I;

$\mathscr{P}(S)$ – a unit begins to operate in state I and a repair of the type $II \to I$ of the other one starts;

$\mathscr{P}(L)$ – a unit begins to operate in state I and a repair of the type $III \to I$ of the other one starts.

At moments when a unit deteriorates from I to II and the other one is in state I (i.e. in standby) we carry out a preventive maintenance, i.e. a repair of the type $II \to I$, of the former while the latter is switched into operation.

We use the stochastical ordering $\leqq^{(1)}$ between distribution functions (or, which is the same, between random variables) defined e.g. in [4] as follows:

$$R_1 \leqq^{(1)} R_2 \quad \text{if and only if} \quad R_1(x) \geqq R_2(x) \text{ for all real } x\,,$$

$$\mathscr{R}_1 \leqq^{(1)} \mathscr{R}_2 \quad \text{if and only if } R_1 \leqq^{(1)} R_2\,,$$

where $\mathscr{R}_i$ is a random variable with the distribution function R_i, $i = 1, 2$.

Let us denote the times to system failure (TSF) under the conditions $\mathscr{P}(P)$, $\mathscr{P}(S)$, and $\mathscr{P}(L)$ by $\mathscr{P}$, $\mathscr{S}$ and $\mathscr{L}$, respectively. The present paper shows on examples that the following seemingly true statements do not generally hold:

1) $\mathbf{E}\mathscr{L} \leqq \mathbf{E}\mathscr{S}$ and $\mathbf{E}\mathscr{L} \leqq \mathbf{E}\mathscr{P}$ (even if we suppose that $\mathscr{M} \leqq^{(1)} \mathscr{N}$).

2) If $\mathscr{A}$ is stochastically increased or if $\mathscr{M}$ or $\mathscr{N}$ are stochastically decreased then TSF becomes stochastically greater.

Let use note that it is proved in [3] that if solely the random variable $\mathscr{B}$ (time of work of a unit in state II) is stochastically increased then $\mathscr{P}$, $\mathscr{S}$, and $\mathscr{L}$ become stochastically greater.

1. A COMPARISON OF STARTING SITUATIONS OF THE SYSTEM

Three starting situations of the system indicated in the introduction are of particular importance because they concern the usual initial state of the system (see condition $\mathscr{P}(P)$) and its only regenerative states (see [1] for the random process $X(t)$ describing the behaviour of the system in question). Let us suppose that

$$\mathscr{M} \leqq^{(1)} \mathscr{N} . \tag{1.1}$$

By (1.1) and by the definition of $\mathscr{P}(P)$, $\mathscr{P}(S)$, and $\mathscr{P}(L)$ one can conjecture that

$$\mathscr{L} \leqq^{(1)} \mathscr{S} , \tag{1.2}$$

$$\mathscr{L} \leqq^{(1)} \mathscr{P} , \tag{1.3}$$

or at least that

$$\mathbf{E}\mathscr{L} \leqq \mathbf{E}\mathscr{S} , \tag{1.4}$$

$$\mathbf{E}\mathscr{L} \leqq \mathbf{E}\mathscr{P} . \tag{1.5}$$

The paper [1] shows that the distributions of $\mathscr{P}$, $\mathscr{S}$, and $\mathscr{L}$ are fully determined by an ordered 4-tuple of distribution functions of the random variables $\mathscr{A}$, $\mathscr{B}$, $\mathscr{M}$, and $\mathscr{N}$.

Example 1. Let the system be determined by the ordered 4-tuple

$$(A_0, B_0, M_0, N_0) , \tag{1.6}$$

where the values of $A_0(x)$, $B_0(x)$, $M_0(x)$, and $N_0(x)$, for every real x, are given in Table 1.

Table 1

for x from the interval / d.f.	$(-\infty; 1)$	$[1; 2)$	$[2; 3)$	$[3; 5)$	$[5; 6)$	$[6; \infty)$
$A_0(x)$	0	0	1	1	1	1
$B_0(x)$	0	0	1	1	1	1
$M_0(x)$	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1
$N_0(x)$	0	0	0	$\frac{2}{3}$	$\frac{2}{3}$	1

The relation (1.1) is evidently fulfilled. Further, we have

$$\mathbf{P}(\mathscr{P} = 2 + 2k) = \mathbf{P}(\mathscr{S} = 2k) = \mathbf{P}(\mathscr{L} = 4k) = \tfrac{1}{3}(\tfrac{2}{3})^{k-1}\,, \tag{1.7}$$

for all natural k's. Thus, the inequalities

$$\sum_{k\in N:4k\leqq x} \frac{1}{3}\left(\frac{2}{3}\right)^{k-1} \leqq \sum_{k\in N:2+2k\leqq x} \frac{1}{3}\left(\frac{2}{3}\right)^{k-1} \leqq \sum_{k\in N:2k\leqq x} \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$$

hold for all real x so that

$$\mathscr{S} \leqq^{(1)} \mathscr{L}$$

and

$$\mathscr{P} \leqq^{(1)} \mathscr{L}\,,$$

and by [4] (Consequence of Theorem 1.2.2) we obtain

$$\mathbf{E}\mathscr{S} \leqq \mathbf{E}\mathscr{L}$$

and

$$\mathbf{E}\mathscr{P} \leqq \mathbf{E}\mathscr{L}\,.$$

So we find that it may happen that even reverse inequalities to (1.2), (1.3), (1.4), and (1.5) are true. Let us note that the validity of (1.4) and (1.5) is proved in [3] under the condition that the state-dependent preventive maintenance is convenient in the sense of the mean time to system failure (MTSF).

2. STOCHASTICAL CHANGES OF DISTRIBUTION FUNCTIONS A, M OR N

Let us introduce four ordered 4-tuples of distribution functions of random variables $\mathscr{A}$, $\mathscr{B}$, $\mathscr{M}$, and $\mathscr{N}$:

$$(A_1, B_1, M_1, N_1)\,, \tag{2.1}$$

$$(A_1, B_1, M_1, N_2)\,, \tag{2.2}$$

$$(A_1, B_1, M_2, N_2)\,, \tag{2.3}$$

$$(A_2, B_1, M_2, N_2)\,, \tag{2.4}$$

where the values of $A_i(x)$, $M_i(x)$, $N_i(x)$, $i = 1, 2$, and $B_1(x)$ for all real x are given in Table 2.

One can easily see that

$$A_1 \leqq^{(1)} A_2\,, \tag{2.5}$$

$$M_2 \leqq^{(1)} M_1\,, \tag{2.6}$$

$$N_2 \leqq^{(1)} N_1 \tag{2.7}$$

and

$$M_i \leqq^{(1)} N_j \quad \text{for} \quad i, j \in \{1; 2\}\,. \tag{2.8}$$

Table 2

d.f. \ for x from the interval	$(-\infty; 1)$	$[1; 1.9;)$	$[1.9; 2)$	$[2; 2.1)$	$[2.1; 3)$	$[3; 4)$	$[4; 6)$	$[6; 7)$	$[7; \infty)$
$A_1(x)$	0	0	1	1	1	1	1	1	1
$A_2(x)$	0	0	0	0	1	1	1	1	1
$B_1(x)$	0	0	0	0	0	1	1	1	1
$M_1(x)$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	1	1
$M_2(x)$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	1	1
$N_1(x)$	0	0	0	0	0	0	$\frac{3}{4}$	$\frac{3}{4}$	1
$N_2(x)$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	1

The relations (2.5) to (2.7) imply that each of the ordered 4-tuples (2.1) to (2.4) is stochastically "better" than the preceding ordered 4-tuple in one of its component (and evidently remains the same in all the others). For example, the only difference between (2.1) and (2.2) has the form of a stochastical decrease of the repair time of a unit of the type $III \to I$. It means that going successively from (2.1) to (2.4) we find still better and better characteristics of the individual units. Let us remark that the ordered 4-tuples (2.1) to (2.4) can be regarded as realistic ones because the time of the preventive maintenance $II \to I$ of a unit is according to (2.8) stochastically smaller than the time of the repair $III \to I$ of a unit.

The paper [1] gives the following formulas for the mean time to system failure (MTSF) under the conditions $\mathscr{P}(P)$, $\mathscr{P}(S)$, and $\mathscr{P}(L)$:

$$(2.9) \qquad \mathbf{E}\mathscr{P} = \mathbf{E}\mathscr{A} + \mathbf{E}\mathscr{B} + \frac{(1 - c + d + e - f) \,.\, \mathbf{E}\mathscr{A} + (d - c) \,.\, \mathbf{E}\mathscr{B}}{(1 - c)(1 - f) + e(1 - d)},$$

$$(2.10) \qquad \mathbf{E}\mathscr{S} = \mathbf{E}\mathscr{B} + \frac{(1 - c + d + e - f) \,.\, \mathbf{E}\mathscr{A} + (d - c) \,.\, \mathbf{E}\mathscr{B}}{(1 - c)(1 - f) + e(1 - d)},$$

$$(2.11) \qquad \mathbf{E}\mathscr{L} = \frac{(1 - c + e) \,.\, \mathbf{E}\mathscr{A} + (1 - c) \,.\, \mathbf{E}\mathscr{B}}{(1 - c)(1 - f) + e(1 - d)},$$

where

$$c = \mathbf{P}(\mathscr{A} \geqq \mathscr{M}),$$
$$d = \mathbf{P}(\mathscr{A} + \mathscr{B} \geqq \mathscr{M}),$$
$$e = \mathbf{P}(\mathscr{A} \geqq \mathscr{N}),$$
$$f = \mathbf{P}(\mathscr{A} + \mathscr{B} \geqq \mathscr{N}).$$

The values of c, d, e, f, $\mathbf{E}\mathscr{A}$, $\mathbf{E}\mathscr{B}$, $\mathbf{E}\mathscr{P}$, $\mathbf{E}\mathscr{S}$, and $\mathbf{E}\mathscr{L}$ corresponding to the systems determined by the ordered 4-tuples (2.1) to (2.4) are given in Table 3.

We see that going successively from (2.1) to (2.4), MTSF of the corresponding systems decrease under each of the conditions $\mathscr{P}(P)$, $\mathscr{P}(S)$, and $\mathscr{P}(L)$. It has been prooved in [4] (Consequence of Theorem 1.2.2) that the relation "stochastically smaller" of distribution functions implies the inequality "less or equal" between the corresponding mathematical expectations. Therefore we find that the random variables $\mathscr{P}$ (or $\mathscr{S}$ or $\mathscr{L}$) concerning the systems determined by (2.i) and (2.j), for $1 \leqq i < j \leqq 4$, respectively, cannot be in the relation "stochastically smaller".

Table 3

the value of / the system determined by	c	d	e	f	$\mathbf{E}\mathscr{A}$	$\mathbf{E}\mathscr{B}$	$\mathbf{E}\mathscr{P}$	$\mathbf{E}\mathscr{S}$	$\mathbf{E}\mathscr{L}$
(2.1)	$\frac{1}{4}$	$\frac{3}{4}$	0	$\frac{3}{4}$	1.9	3	20.5	18.6	19.6
(2.2)	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	1.9	3	18.5	16.6	16.6
(2.3)	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	1.9	3	16.5	14.6	15.6
(2.4)	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	2.1	3	13.5	11.4	12.4

So we come to the general conclusion: If the units of the system are improved in such a way that d.f. A is stochastically increased or d.f. M or N are stochastically decreased then TSF need not become stochastically greater and even MTSF can decrease (under each of the conditions on the starting situation of the system, namely, $\mathscr{P}(P)$, $\mathscr{P}(S)$, or $\mathscr{P}(L)$). Let us note that an example given in [3] shows that such a situation can also arise that not only MTSF becomes less but even TSF itself becomes stochastically smaller with an improvement of the individual units.

References

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Souhrn

PŘÍKLADY NARUŠENÍ MONOTONIE V JEDNOM DVOUPRVKOVÉM SYSTÉMU SE ZÁLOHOU

ANTONÍN LEŠANOVSKÝ

Uvažujeme jistý systém s nezatíženou zálohou složený ze dvou prvků a jednoho zařízení pro jejich opravy. Prvky mohou být ve třech stavech: bezvadném (I), zhoršeném (II) a poruchovém (III). Předpokládáme, že možné jsou pouze následující změny stavu prvků: $I \to II, II \to III, II \to I, III \to I$. Článek je věnován srovnání některých důležitých počátečních situací systému a stochastickému zlepšení prvků (stochastickému zvětšení doby provozu prvků ve stavu I a/nebo stochastickému zmenšení dob jejich oprav typu $II \to I$ a $III \to I$). Na příkladech se ukazuje, že doba do poruchy systému obecně nemusí vzrůst při zlepšení počáteční situace systému, resp. charakteristik jednotlivých prvků.

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